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The Palais-Smale Condition Versus Coercivity

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THE PALAIS-SMALE CONDITION VERSUS COERCIVITY

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1. Introduction

Given a functional $\varphi : X \longrightarrow \mathbb{R}$ on a Banach space X, φ is said to be coercive if $\varphi(u) \longrightarrow +\infty$ as $\|u\| \longrightarrow \infty$. This is equivalent to saying that, for every $d \in \mathbb{R}$ the set

$$\varphi^d = \{u \in X \mid \varphi(u) \leq d\}$$

is bounded. On the other hand, a (Fréchet) differentiable functional $\varphi : X \longrightarrow \mathbb{R}$ is said to satisfy the Palais-Smale condition at the level $d \in \mathbb{R}$, $(PS)_d$, if any sequence $\{u_n\} \subset X$ such that

$$\begin{cases} \varphi(u_n) \longrightarrow d \\ \| \varphi'(u_n) \|_{X^*} \longrightarrow 0 \end{cases}$$

possesses a convergent subsequence; and φ is said to satisfy (PS) if it satisfies $(PS)_d$ for every $d \in \mathbb{R}$.

Recently, among other results in critical point theory, Shujie [11] showed that if a C^1 functional $\varphi : X \longrightarrow \mathbb{R}$ is bounded from below and satisfies the condition (PS) then φ is coercive. Shujie's proof uses a "gradient flow" approach, through the so-called "deformation theorem" (cf. [3,10]) and, for that, he needs the notion of a pseudo-gradient vector field v associated with the functional φ (whose existence is guaranteed for C^1 functionals by Palais [8]).

In this note we present some new results which relate the Palais- Smale condition and the notion of coercivity, and are all based on the well-known Variational Principle due to Ekeland

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[5,6]. In particular, a new proof of the above mentioned result of Shujie is given. It should be pointed out that, throughout the paper, the given functional φ could be assumed to be only Gateaux differentiable, rather than C^1. And, in addition to being conceptually simpler, this approach could be used in more general situations where the functional is not even differentiable (cf. [4]). The strong form of Ekeland's Variational Principle, to be repeatedly used in the sequel, is the following

Theorem 0. Let M be a complete metric space and $\theta : M \longrightarrow \mathbb{R} \cup \{+\infty\}$, $\theta \not\equiv +\infty$, a lower semicontinuous function which is bounded from below, say $a = inf_M \theta$. Let $\varepsilon > 0$ be given and $\hat{u} \in M$ be such that

$$\theta(\hat{u}) \leq a + \varepsilon.$$

Then, for any $\lambda > 0$, there exists $u_\lambda \in M$ such that

$$(i) \qquad \theta(u_\lambda) \leq \theta(\hat{u})$$

$$(ii) \qquad \theta(u_\lambda) < \theta(u) + \frac{\varepsilon}{\lambda} d(u, u_\lambda) \quad \forall u \neq u_\lambda$$

$$(iii) \qquad d(u_\lambda, \hat{u}) \leq \lambda$$

Remark. Note that the special choice $\lambda = \sqrt{\varepsilon}$ gives $d(u_\lambda, \hat{u}) \leq \sqrt{\varepsilon}$ and $\theta(u_\lambda) < \theta(u) + \sqrt{\varepsilon} d(u, u_\lambda) \quad \forall u \neq u_\lambda$. Also note that, when $M = X$ is a Banach space and $\theta : X \longrightarrow \mathbb{R}$ is Gateaux differentiable, by taking $u = u_\lambda + th$, $h \in X$, in (ii) and letting $t \longrightarrow 0$, one obtains $\| \theta'(u_\lambda) \|_{X^*} \leq \varepsilon / \lambda$.

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2. Main Results

We start with a preliminary result which, although not directly needed for the main theorem, it typically illustrates our approach and shows how Ekeland's Principle comes naturall into scene.

Proposition 1. Let $\varphi \in C^1(X, \mathbb{R})$ be bounded from below, say $a = inf_X \varphi$. If φ satisfie $(PS)_a$ then the set $\varphi^{a+\alpha}$ is bounded, for some $\alpha > 0$.

Proof. Suppose, by contradiction, that $\varphi^{a+\alpha}$ is unbounded for all $\alpha > 0$. Then, there exist $\{\hat{u}_n\} \subset X$ such that

$$\begin{cases} a \leq \varphi(\hat{u}_n) \leq a + \dfrac{1}{n} \\ \| \hat{u}_n \| \geq n, \end{cases}$$

and Theorem 0 (with $\varepsilon = 1/n, \lambda = 1/\sqrt{n}$) implies the existence of $\{u_n\} \subset X$ satisfying

$$(1) \qquad \begin{array}{l} \text{(i)} \quad a \leq \varphi(u_n) \leq \varphi(\hat{u}_n) \leq a + \frac{1}{n} \\ \text{(ii)} \quad \varphi(u_n) \leq \varphi(u) + \frac{1}{\sqrt{n}} \| u - u_n \| \quad \forall u \in X \\ \text{(iii)} \quad \| u_n - \hat{u}_n \| \leq \frac{1}{\sqrt{n}} . \end{array}$$

We reach a contradiction with $(PS)_a$, since (1)(i)-(iii) give, respectively,

$$\begin{cases} \varphi(u_n) \longrightarrow a \\ \| \varphi'(u_n) \|_{X^*} \leq \dfrac{1}{\sqrt{n}} \longrightarrow 0 \\ \| u_n \| \geq n - \dfrac{1}{\sqrt{n}} \longrightarrow \infty. \end{cases} \qquad \blacksquare$$

A similar result, which is slightly more general is the following.

Proposition 2. Let $\varphi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$ be such that φ^d is unbounded for $d > c$ and φ^d is bounded for $d < c$. Then, there exists $\{u_n\} \subset X$ such that

$$(2) \qquad \begin{array}{l} \text{(i)} \quad \varphi(u_n) \longrightarrow c \\ \text{(ii)} \quad \| \varphi'(u_n) \|_{X^*} \longrightarrow 0 \\ \text{(iii)} \quad \| u_n \| \longrightarrow \infty . \end{array}$$

Proof. In view of the hypotheses, for any given $n \in \mathbb{N}$, there exists $R_n \geq n$ such that

$$\varphi^{c-\frac{1}{n}} \subset B_{R_n}(0). \tag{3}$$

Define $M_n = X \backslash B_{R_n}, \theta_n = \varphi \mid M_n : M_n \longrightarrow \mathbb{R}$, and note that

$$c_n \equiv inf_{M_n} \theta_n \geq c - \frac{1}{n} \tag{4}$$

in view of (3). Now, from the unboundedness of $\varphi^{c+\frac{1}{n}}$, we can pick $\hat{u}_n \in X$ satisfying

$$\varphi(\hat{u}_n) \leq c + \frac{1}{n}, \tag{5}$$

$$||\hat{u}_n|| \geq R_n + 1 + \frac{1}{\sqrt{n}}, \tag{6}$$

so that, in fact, we have $\hat{u}_n \in M_n$ and, from (4), (5),

$$\varphi(\hat{u}_n) \leq c + \frac{1}{n} \leq c_n + \frac{2}{n}. \tag{7}$$

Now, applying Theorem 0 (with $\varepsilon = 2/n, \lambda = 1/\sqrt{n}$)we obtain $u_n \in M_n$ satisfying

(8)
(i) $c - \frac{1}{n} \leq c_n \leq \varphi(u_n) \leq \varphi(\hat{u}_n) \leq c + \frac{1}{n} \leq c_n + \frac{2}{n}$
(ii) $\varphi(u_n) \leq \varphi(u) + \frac{2}{\sqrt{n}}||u - u_n|| \forall u \in M_n$
(iii) $||u_n - \hat{u}_n|| \leq \frac{1}{\sqrt{n}}$

In particular, (6) and (8) (iii) imply

$$\| u_n \| \geq R_n + 1, \tag{9}$$

so that u_n belongs to the interior of M_n and (8) (ii) gives

$$\| \varphi'(u_n) \|_{X^*} \leq \frac{2}{\sqrt{n}} \tag{10}$$

Therefore, (8)(i), (10) and (9) provide (2)(i), (ii) and (iii) respectively. The proof is complete. ■

Corollary 3. Let $\varphi \in C^1(X, \mathbb{R})$ satisfy $(PS)_c$. If φ^d is bounded for every $d < c$ then $\varphi^{c+\gamma}$ is also bounded, for some $\gamma > 0$.

Remarks 1) In particular, it follows from Corollary 3 that φ^c is bounded.

2) The conclusion of Corollary 3 holds whenever φ^c is bounded.

3) Notice that the above Corollary generalizes Proposition 1. Also, it is clear from Proposition 2 that this Corollary holds for φ satisfying a condition weaker than $(PS)_c$, namely $(\widehat{PS})_c$: whenever $\{u_n\} \subset X$ is a sequence verifying (2)(i), (ii), then $\{u_n\}$ must have a bounded subsequence

As another consequence of Proposition 2 we have the following.

Theorem 4. Let $\varphi \in C^1(X, \mathbb{R})$ be bounded from below. If φ is not coercive then φ does not satisfy $(PS)_{c_0}$, where

$$c_0 = sup\{d \in \mathbb{R} \mid \varphi^d \text{ is bounded}\}.$$

Proof. Let $C = \{d \in \mathbb{R} \mid \varphi^d \text{ is bounded}\}$. Since φ is bounded from below, we have $C \supset (-\infty, a)$ where $a = inf_X \varphi$, hence C is nonempty. If we define

$$c_0 = supC$$

then $c_0 < +\infty$ since φ is not coercive. And, by definition, it follows that φ^d is unbounded for $d > c_0$. So, Proposition 2 implies the result. ■

Corollary 5. [11] If $\varphi \in C^1(X, \mathbb{R})$ is bounded from below and satisfies (PS) (that is, $(PS)_c$ for every $c \in \mathbb{R}$) then φ is coercive.

Remarks 4) Notice that we could also characterize $c_0 = supC$ defined above as

$$c_0 = inf\{d \in \mathbb{R} \mid \varphi^d \text{ is unbounded}\}.$$

5) In general, for any functional $\varphi : X \longrightarrow \mathbb{R}$, the set $C = \{d \in \mathbb{R} \mid \varphi^d \text{ is bounded}\}$ is a (left) half-line, either open or closed, and we may have $C = \phi$ or $C = \mathbb{R}$ the latter case occurring if and only if φ is coercive. And the assumption of φ being bounded from below in Theorem 4 was used only to show that $C \neq \phi$. Therefore, in that theorem (cf. also next section), one could assume more generally that φ^d is bounded, for some $d \in \mathbb{R}$. Of course, for typical situations in differential equations, where φ takes bounded sets into bounded sets, the assumption $C \neq \phi$ is equivalent to φ being bounded from below.

6) Suppose that the set $D = \{d \in \mathbb{R} \mid \varphi \text{ satisfies } (PS)_d\}$ is nonempty. Unlike the set C, it is easy to see that D is not necessarily a half-line, even in the case that φ is bounded from below, for which $D \supset (-\infty, inf_X \varphi)$. However, if X is a reflexive Banach space, X^* is strictly convex and $\varphi \in C^1(X, \mathbb{R})$ is such that $\varphi' = J - K$, with $J : X \longrightarrow X^*$ the duality mapping and $K : X \longrightarrow X^*$ a compact mapping, then it is not hard to show that $c_0 = supC$ coincides with the number

$$\hat{c}_0 = sup\{d \in \mathbb{R} \mid \varphi \quad \text{satisfies} \quad (PS)_c \quad \text{for every} \quad c \leq d\}$$
$$= inf\{d \in \mathbb{R} \mid \varphi \quad \text{does not satisfy} \quad (PS)_d\}$$

The following example of a $\varphi \in C^1(\mathbb{R}, \mathbb{R})$ is illustrative, where $C = (-\infty, 0) = \{d \in \mathbb{R} \mid \varphi \text{ satisfies } (PS)_c \ \forall c \leq d\}$, $\{d \in \mathbb{R} \mid \varphi \text{ does not satisfy } (PS)_d\} = \{0, \frac{1}{2}\}$ and $c_0 = \hat{c}_0 = 0$:

$$\varphi(t) = \begin{cases} sint, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \\ \frac{1}{2}\left[1 + e^{-(t-\frac{\pi}{2})^2}\right], \ t \geq \frac{\pi}{2} \\ -e^{-(t+\frac{\pi}{2})^2}, \ t \leq -\frac{\pi}{2}. \end{cases}$$

7) Finally, it should be remarked that, in fact, Proposition 2 suggests the weaker Palais-Smale type condition $(\widehat{PS})_c$ defined in Remark 3) as the natural one to relate to

coercivity, in the sense that a converse to Corollary 3 also holds (trivially) true. More precisely, let $\varphi \in C^1(X, \mathbb{R})$ be such that φ^d is bounded for every $d < c$. Then, φ satisfies $(\widehat{PS})_c$ if and only if $\varphi^{c+\gamma}$ is bounded, for some $\gamma > 0$.

In particular, considering the set $\hat{D} = \{d \in \mathbb{R} \mid \varphi \text{ satisfies } (\widehat{PS})_d\}$, it is easy to see that: (i) $C = \mathbb{R} \Longrightarrow \hat{D} = \mathbb{R}$; (ii) $C = (-\infty, c_o)$ or $C = (-\infty, c_o] \Longrightarrow \hat{D} = (-\infty, c_o) \cup \hat{\hat{D}}$, where $(-\infty, c_o)$ is a component of $\hat{D}$ (that is, φ does not satisfy $(\widehat{PS})_{c_o}$.)

3. Some Extensions and a Resonant Problem

Given a vector $e \in \partial B_1(0)$ and a decomposition $X = \langle e \rangle \oplus W$, we shall hereafter write $u \in X$ as $u = te + w$, where $w \in W$. And a *set* $S \subset X$ will be said to be *e-bounded* if

$$S \subset \{u = te + w \mid t < R, w \in W\} \equiv H_R$$

for some $R \in \mathbb{R}$. Also, a functional $\varphi : X \longrightarrow \mathbb{R}$ will be called *e-coercive* if

$$\varphi(te + w) \longrightarrow +\infty \qquad \text{as} \qquad t \longrightarrow +\infty\ ,$$

uniformly for $w \in W$. And φ will be called ***e-bounded from below*** if there exists $a \in \mathbb{R}$ such that φ^a is e-bounded.

In this section we shall extend our previous results to include situations where φ is not necessarily bounded from below. In fact, it is not hard to check that results which are similar to Proposition 1 through Corrolary 5 hold true in this more general setting, with the words "bounded", "bounded from below" and "coercive" being replaced by "e-bounded", "e-bounded from below" and "e-coercive", respectively. As illustrations, we shall state the analogues of Proposition 2 and Corollary 5 and prove the former.

Proposition 2e. Let $\varphi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$ be such that φ^d is not e-bounded for $d > c$ and φ^d is e-bounded for $d < c$. Then, there exists $\{u_n\} = \{t_n e + w_n\} \subset X$ such that

$$(11)\qquad \begin{aligned}&\text{(i) } \varphi(u_n) \longrightarrow c\\ &\text{(ii) } \|\varphi'(u_n)\|_{X^*} \longrightarrow 0\\ &\text{(iii) } t_n \longrightarrow +\infty\ .\end{aligned}$$

Corollary 5e. If $\varphi \in C^1(X,\mathbb{R})$ is e-bounded from below and satisties (PS) then φ is e-coercive.

Proof of Proposition 2e. By the hypotheses, given $n \in \mathbb{N}$, there exists $\mathbb{R}_n \geq n$ such that

$$(12)\qquad \varphi^{c-\frac{1}{n}} \subset H_{R_n} = \{u = te + w | t < R_n, w \in W\}\ .$$

Define $M_n = X \backslash H_{R_n}, \theta_n = \varphi | M_n$ and note that (12) implies

$$(13)\qquad c_n \equiv inf_{M_n}\theta_n \geq c - \frac{1}{n}\ .$$

Since $\varphi^{c+\frac{1}{n}}$ is not e-bounded, there exists $\hat{u}_n = \hat{t}_n e + \hat{w}_n$ such that

$$(14)\qquad \varphi(\hat{u}_n) \leq c + \frac{1}{n}$$

$$(15)\qquad \hat{t}_n \geq R_n + 1 + \frac{1}{\sqrt{n}}\ ,$$

hence $\hat{u}_n \in M_n$ and, from (13), (14), we obtain

$$(16)\qquad \varphi(\hat{u}_n) \leq c + \frac{1}{n} \leq c_n + \frac{2}{n}\ .$$

Now, Theorem 0 (with $\epsilon = 2/n$, $\lambda = 1/\sqrt{n}$) gives $u_n = t_n e + w_n \in M_n$ satisfying

$$(17)\qquad \begin{aligned}&\text{(i) } c - \tfrac{1}{n} \leq c_n \leq \varphi(u_n) \leq \varphi(\hat{u}_n) \leq c + \tfrac{1}{n} \leq c_n + \tfrac{2}{n}\\ &\text{(ii) } \varphi(u_n) \leq \varphi(u) + \tfrac{2}{\sqrt{n}}\|u - u_n\| \qquad \forall u \in M_n\\ &\text{(iii) } \|u_n - \hat{u}_n\| \leq \tfrac{1}{\sqrt{n}}\ .\end{aligned}$$

From the continuity of the projection $P : X \longrightarrow < e >$ along W (say $\|P\| = 1$ without loss of generality), we obtain $|t_n - \hat{t}_n| \leq 1/\sqrt{n}$ and then, in view of (15),

$$(18)\qquad t_n \geq R_n + 1\ .$$

This shows that $u_n = t_n e + w_n$ belongs to the interior of M_n, hence

$$(19)\qquad \|\varphi'(u_n)\|_{X^*} \leq \frac{2}{\sqrt{n}}$$

in view of (17)(ii). The proof is complete since (17)(i), (19), (18) imply (11)(i), (ii), (iii), respectively. ■

We now present an example of a resonant problem whose corresponding functional φ is e-bounded from below but is not bounded. Moreover, φ will be shown to satisfy the hypotheses of Proposition 2e with $c = 1$ so that, in particular, it will not satisfy $(PS)_1$ in view of (11). Consider the Dirichlet problem

$$(*)\qquad \begin{aligned}-\Delta u &= \lambda_1 u + g(u) \quad \text{in} \quad \Omega\\ u &= 0 \quad \text{on} \quad \partial\Omega\ ,\end{aligned}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, λ_1 is the first eigenvalue of $-\Delta$ with zero Dirichlet boundary condition, and $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuos function satisfying the following conditions:

(g_1) g is bounded on $\mathbb{R}$, say $|g(s)| \leq M$ for every $s \in \mathbb{R}$ (and some $M > 0$);

(g_2) $\lim_{s \to +\infty} G(s) = -\frac{1}{|\Omega|}$ (where $G(s) = \int_0^s g(\sigma)d\sigma$);

(g_3) $\lim_{s \to -\infty} G(s) = +\infty$.

From (g_1) the corresponding functional

$$\varphi(u) = \int_\Omega \frac{1}{2}(|\nabla u|^2 - \lambda_1 u^2)dx - \int_\Omega G(u)dx = q(u) - \psi(u)$$

is well-defined and of class C^1 on the Sobolev space $X = H_0^1(\Omega)$, which we decompose as $X = < e > \oplus W$, with $e = \phi_1 > 0$ being the first (normalized) eigenfunction of $-\Delta$ on $H_0^1(\Omega)$ and $W = < \phi_1 >^\perp$.

Lemma 6. (i) There exists $R_0 > 0$ such that $\varphi(t\phi_1 + w) \geq -\psi(t\phi_1)$ for all $\|w\| \geq R_0$ and $t \in \mathbb{R}$

(ii) For any $R > 0$, we have $\lim_{t\to+\infty} \varphi(t\phi_1 + w) = q(w) + 1$ uniformly for $||w|| \le R$.

Proof. (i) By (g_1) and the mean value theorem applied to $G(t\phi_1 + w) - G(t\phi_1)$, we have

$$|\psi(t\phi_1 + w) - \psi(t\phi_1)| = |\int_\Omega g(t\phi_1 + zw)w\,dx| \le M_0||w|| \tag{20}$$

for some $M_0 > 0$. Therefore, we obtain

$$\varphi(t\phi_1 + w) = q(w) - \psi(t\phi_1 + w) \ge q(w) - M_0||w|| - \psi(t\phi_1)\ ,$$

and it is enough to take $R_0 > 0$ such that $q(w) - M_0||w|| \ge 0$ for all $||w|| \ge R_0$.

In order to prove (ii), we only need to show that

$$\lim_{t\to+\infty} \psi(t\phi_1 + w) = -1\ , \tag{21}$$

uniformly for $||w|| \le R$. Indeed, if we suppose that (21) does not hold. then we can find $t_n \longrightarrow +\infty$ and $w_n \in W, ||w_n|| \le R$, such that

$$\psi(t_n\varphi_1 + w_n) \not\longrightarrow -1\ , \quad \text{as} \quad n \to \infty\ . \tag{22}$$

Without loss of generality, we may assume that there exist $\hat{w} \in H_0^1(\Omega)$ and $h \in L^1(\Omega)$ such that

(23)
(i) $w_n \rightharpoonup \hat{w}$ weakly in H_0^1
(ii) $w_n \longrightarrow \hat{w}$ strongly in $L^p, 1 \le p < \frac{2N}{N-2}$ if $N \ge 3$ $[1 \le p < \infty$ if $N = 1, 2]$
(iii) $w_n(x) \longrightarrow \hat{w}(x)$ a.e. in Ω
(iv) $|w_n(x)| \le h(x)$ a. e. in Ω .

For each $n \in \mathbb{N}$, consider the set

$$A_n = \{x \in \Omega | t_n\phi_1(x) + w_n(x) < 0\}$$

and the function

$$f_n = G(t_n\varphi_1 + w_n)\chi_n$$

in Ω, where $\chi_n = \chi_{A_n}$ is the characteristic function of A_n. From (23)(iii) we obtain that $f_n(x) \longrightarrow 0$ a. e. in Ω. And, from (g_1), the mean value theorem and (23)(iv), we get that

$$|f_n(x)| \le M|t_n\phi_1(x) + w_n(x)|\chi_n(x) \le M|w_n(x)|\chi_n(x) \le M\ h(x) \text{ a.e. in } \Omega\ .$$

Therefore, by Lebesgue's Dominated Convergence Theorem,

$$\int_{A_n} G(t_n\varphi_1 + w_n)dx \longrightarrow 0, \text{ as } n \to \infty\ . \tag{24}$$

On the other hand, from (g_2), (23) and the fact that $G(s)$ is bounded on $\mathbb{R}^+ = \{s \in \mathbb{R} | s \ge 0\}$, we obtain that

$$\int_{\Omega/A_n} G(t_n\varphi_1 + w_n)dx \longrightarrow -1, \text{ as } n \to \infty\ . \tag{25}$$

Hence, (24) and (25) give $\psi(t_n\phi_1 + w_n) \longrightarrow -1$, which contradicts (22) and thus concludes the proof of Lemma 6. ■

Now, it follows from Lemma 6 that

$$\lim_{t\to+\infty} \varphi(t\phi_1) = 1\ ,$$

$$\lim_{t\to+\infty} \inf_{w\in W} \varphi(t\phi_1 + w) \ge 1\ ,$$

from which we obtain, respectively, that φ^d is not ϕ_1-bounded for $d > 1$ and is ϕ_1-bounded for $d < 1$. In particular, the functional φ is ϕ_1-bounded from below. However, φ is not bounded from below since (g_3) implies

$$\lim_{t\to-\infty} \varphi(t\phi_1) = -\lim_{t\to-\infty} \psi(t\phi_1) = -\infty. \tag{26}$$

Applying Proposition 2 e to the functional φ with $c = 1$, we conclude that there exists a sequence $\{u_n\} = \{t_n\phi_1 + \omega_n\} \subset H_0^1(\Omega)$ such that

$$\begin{cases} \varphi(u_n) \longrightarrow 1 \\ \| \varphi'(u_n) \|_{H^{-1}} \longrightarrow 0 \\ t_n \longrightarrow +\infty. \end{cases}$$

In particular, φ does not satisfy $(PS)_1$.

Note that there may exist $d \neq 1$ such that φ does not satisfy $(PS)_d$. However, this is not the case if we assume

$$(g_4) \qquad \lim_{s \to +\infty} g(s) = 0,$$

as the following lemma shows.

Lemma 7. If $(g_1) - (g_4)$ hold, then

$$D = \{d \in \mathbb{R} \mid \varphi \quad \text{satisfies} \quad (PS)_d\} = \mathbb{R}\setminus\{1\}.$$

Proof. Suppose that φ does not satisfy $(PS)_d$. Then, there exists $u_n = t_n\varphi_1 + \omega_n \in H_0^1$ such that

$$(27) \qquad \varphi(u_n) \longrightarrow d,$$

$$(28) \qquad \| \varphi'(u_n) \|_{H^{-1}} \longrightarrow 0,$$

but $\{u_n\}$ does not possess a convergent subsequence. From (28) we obtain

$$(29) \qquad | \varphi'(u_n).\omega_n | = | \| \omega_n \|^2 - \lambda_1 | \omega_n |_{L^2}^2 - \int_\Omega g(t_n\phi_1 + \omega_n)\omega_n dx | \leq \varepsilon_n \| \omega_n \|,$$

where $\varepsilon_n = \| \varphi'(u_n) \|_{H^{-1}} \longrightarrow 0$, so that

$$\| \omega_n \|^2 - \lambda_1 | \omega_n |_{L^2}^2 \leq (M_0 + \varepsilon_n) \| \omega_n \|$$

in view of (g_1) and, hence, $\| \omega_n \| \leq R$ for some $R > 0$. Since $u_n = t_n\phi_1 + \omega_n$ and $\nabla\varphi(u_n) = u_n - K(u_n)$ with $K : H_0^1 \longrightarrow H_0^1$ a compact operator, it must be the case that $| t_n | \longrightarrow \infty$. In fact, $t_n \longrightarrow +\infty$ necessarily in view of (20), (26) and (27). Now, arguing as in Lemma 6 and using (g_4), we obtain that

$$\lim_{n \to \infty} \int_\Omega g(t_n\phi_1 + w_n)w_n dx = 0,$$

and, hence, that $\| \omega_n \| \longrightarrow 0$ in view of (29). But then Lemma 6 (ii) yields

$$\lim_{n \to \infty} \varphi(t_n\phi_1 + w_n) = \lim_{n \to \infty} [q(\omega_n) + 1] = 1,$$

so that $d = 1$ necessarily. ■

Finally, we should observe that, if we assumed

$$(\hat{g}_3) \qquad \lim_{s \to -\infty} G(s) = -\frac{a}{|\Omega|},$$

instead of (g_3) then by the same argument used above with $e = -\phi_1$, we would conclude that φ does not satisfy $(PS)_a$. On the other hand, for results concerning existence of solutions for problem (*) under hypotheses on G of the type $(g_2) - (g_3)$ we refer the reader to e.g. [1], [2], [7], [9], [12], [13] and references there in.

4. Final Remarks and Comments

A close look at the proof of Proposition 2e shows that, in fact, a more general result is true. In order to state it, we need the following definition: given a functional $F : X \longrightarrow \mathbb{R}$, a set X will be said to be F-bounded if $S \subset F^r$ for some $r \in \mathbb{R}$.

Proposition 2F. Let $\varphi : X \longrightarrow \mathbb{R}$ be of class C^1, $F : X \longrightarrow \mathbb{R}$ be uniformly continuous and $c \in \mathbb{R}$ be such that φ^d is not F-bounded for $d > c$ and φ^d is F-bounded for $d < c$. Then, there exists $\{u_n\} \subset X$ such that

$$\begin{cases} \varphi(u_n) \longrightarrow c \\ \| \varphi'(u_n) \|_{X^*} \longrightarrow 0 \\ F(u_n) \longrightarrow +\infty. \end{cases}$$

Of course, if we give the other suitable (and natural) definitions, all the corresponding results of section 2 will also hold true in this new framework. Also note that Proposition 2 and Proposition 2e correspond to the choices $F(u) = \| u \|$ and $F(te + \omega) = t$, respectively, in Proposition 2F above. Another interesting choice, which may prove to be useful in situations where φ is an indefinite functional, is $F(v + \omega) = \| v \| - \| \omega \|$ for a suitable decomposition $X = V \oplus W$.

Finally, we mention a further related result which extends Corollary 5 in another direction, namely that of the underlying space X.

Corollary $\hat{5}$. Let $\varphi \in C^1(X, \mathbb{R})$ satisfy (PS) and be such that $X \backslash \varphi^a$ is not bounded for some $a \in \mathbb{R}$. If either

(i) $\qquad \varphi^{-1}(a) = \{u \in X | \varphi(u) = a\}$ or

(ii) $\qquad \varphi^{-1}(a, b] = \{u \in X | a < \varphi(u) \leq b\}$ for some $b > a$

is a bounded set, then φ is coercive on $X \backslash \varphi^a$, that is, $\varphi(u) \longrightarrow +\infty$ as $\| u \| \longrightarrow \infty$, $u \in X \backslash \varphi^a$.

Corollary $\hat{5}$ follows from a corresponding Proposition $\hat{2}$, whose statement we presently omit. Details and proofs of these and other results will appear elsewhere.

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