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# "Analytical Solutions for Flow of Water in Unsaturated Soils: Transient One-Dimensional Flows & Steady Multi-Dimensional Flows"

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***Please note: These are preliminary notes intended for internal distribution only.***

# PHYSICAL-MATHEMATICAL MODEL

## Balance of mass

$$\frac{\partial \theta}{\partial t} = -\frac{\partial F}{\partial z} \qquad (1)$$

where $t$ is the time, $z$ is the spatial coordinate (positive downward), $\theta$ is the volumetric water content and $F$ is the volumetric flux.

In (1), $z$ and $t$ are the independent variables, and $\theta[z,t]$ and $F[z,t]$ are the dependent variables. Often it is more convenient to take $\theta$ and $t$ as the independent variables, and $z[\theta,t]$ and $F[\theta,t]$ as the dependent variables. In that case (1) is transformed to

$$\left.\frac{\partial z}{\partial t}\right|_{\theta} = -\left.\frac{\partial F}{\partial \theta}\right|_{t} \qquad (2)$$

## Darcy's law

$$F = -k\,\frac{\partial h}{\partial z} + k\cos\alpha \qquad (3)$$

$$= -D\,\frac{\partial \theta}{\partial z} + k\cos\alpha \qquad (4)$$

where $h$ is the pressure head, $k$ is the hydraulic conductivity, $\alpha$ is the angle between the $z$-direction and the vertical direction, and $D$ is the diffusivity $= k\,dh/d\theta$

## Soil properties

1. retentivity curve: $h[\theta]$ (hysteretic!)

2. conductivity curve: $k[\theta]$ (not hysteretic, but $k[h]$ is hysteretic!)

Initial conditions

1. uniform

$$h = h_\infty \qquad \theta = \theta_\infty \qquad x \geq 0 \qquad t = 0 \qquad (5)$$

2. equilibrium

$$h = h_0 + x = h_e \qquad \theta = \theta_e[h_e] \qquad x \geq 0 \qquad t = 0 \qquad (6)$$

The subscripts 0 and $\infty$ denote values, respectively, at the soil surface ($x = 0$) and at large depth ($x \to \infty$)

Boundary conditions

1. constant water content

$$\theta = \theta_0 \qquad x = 0 \qquad t = 0 \qquad (7)$$

2. constant flux

$$F = F_0 \qquad x = 0 \qquad t = 0 \qquad (8)$$

3. constant pressure head $h_s$ on top of a crust

$$F_0[t] = \gamma'(h_s - h_0) \qquad (9)$$

CLASSICAL SOLUTIONS

Four solutions of the form

$x = \hat{x}[\theta, t]$ or $x = \hat{x}[h, t]$

A. Steady downward flows: see
Raats, P.A.C. 1973. Steady upward and downward flows in a class of unsaturated soils. Soil Sci. 115: 409-413
(Especially Eqns 5, 8, and 10)

B. Time-invariant profiles moving at speed $u$ → wetting front of constant shape

$$x = ut - \int \left(1 + \frac{\theta}{k} u + \frac{1}{k} f_2\right)^{-1} dh \qquad (11)$$

with

$$u = \frac{k_0 - k_\infty}{\theta_0 - \theta_\infty} \quad \text{speed of wetting front}$$

$$f_2 = \frac{\theta_0 k_\infty - \theta_\infty k_0}{\theta_0 - \theta_\infty} \quad \text{flux relative to wetting front}$$

For a general discussion of time-invariant moving profiles see P.A.C. Raats and W. R. Gardner, Movement of water in the unsaturated zone near a water table. Amer. Soc. Agron. Drainage Monograph (1973 or 1974)

C. <u>One-dimensional</u> (<u>horizontal</u>) <u>absorption</u> with constant water content boundary condition

$$x = \lambda[\theta, \theta_0, \theta_\infty]\, t^{1/2} \qquad (12)$$

where $\lambda = x/t^{1/2}$ is the Boltzmann similarity variable

D <u>One-dimensional (vertical) infiltration</u> with constant water content boundary condition

$$x = \sum_{n=1}^{\infty} \lambda_n[\theta, \theta_0, \theta_\infty]\, t^{n/2} \qquad (13)$$

where $\lambda_1 = \lambda$. This is Philip's series solution.

Later we will see that recent quasi-analytical solutions are also of the form $x = \hat{x}[\theta \text{ or } h, t]$.

# QUASI-ANALYTICAL SOLUTIONS

Integral conditions are used to constrain approximate solutions. Integration of (2) between $\theta$ and $\theta_\infty$ gives

$$f - f_\infty = \frac{\partial}{\partial t} \int_{\theta_\infty}^{\theta} x \, d\theta \qquad (14)$$

Time-integral condition (used by Philip and Knight)

$$\int_0^t \{f_0[t] - f_\infty\} dt = \int_{\theta_\infty}^{\theta_0[t]} x[\theta, t] \, d\theta \qquad (15)$$

Space-integral condition (used by Parlange)

$$\int_0^\infty \{f_x - f_\infty\} dx = \frac{\partial}{\partial t} \int_{\theta_0}^{\theta_\infty} \tfrac{1}{2} x^2 \, d\theta$$

$$= \frac{\partial}{\partial t} \int_0^\infty x (\theta - \theta_\infty) \, dx \qquad (16)$$

## INTERMEZZO

Special cases of 15

1. One-dimensional absorption: introducing (12) into (15) gives

$$Q = S t^{1/2} \longrightarrow dQ/dt = f_0 = \tfrac{1}{2} S t^{-1/2} \qquad (17)$$

where $Q = \int_0^t f_0 \, dt$ is the cumulative absorption and $S = \int_{\theta_\infty}^{\theta_0} \lambda \, d\theta$ is the sorptivity

2. One-dimensional infiltration: introducing (13) into (15) gives

$$I = f_\infty t + \sum_{n=1}^{\infty} S_n t^{n/2} \longrightarrow dI/dt = f_0 = f_\infty + \sum_{n=1}^{\infty} \tfrac{n}{2} S_n t^{n/2-1}$$

where $I = \int_0^t f_0 \, dt$ and $S_n = \int_{\theta_\infty}^{\theta_0} \lambda_n \, d\theta$ (18)

$(S_1 = S)$

<u>Philip's 2-term infiltration eqn.</u>

$$I = S t^{1/2} + A t \rightarrow dI/dt = f_0 = \tfrac{1}{2} S t^{-1/2} + A \quad (19)$$

<u>Large time asymptote</u>

$$I = I_{capillary} + k_0 t \rightarrow dI/dt = f_0 = k_0 \quad (20)$$

Estimates of $A/k_s$

| | |
|---|---|
| Green and Ampt | 2/3 |
| linear equation | 1/2 |
| Burger's equation | 0.36 |
| Talsma and Parlange | 1/3 |

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<u>Integration of Darcy's law</u>

at $x$ $\quad f_x = -D\,\partial\theta/\partial x + k\cos\alpha$

at $x \rightarrow \infty$ $\quad f_\infty = k_\infty \cos\alpha$

Subtracting and dividing by $\{f_0[t] - k_\infty \cos\alpha\}$ gives

$$F[\Theta, t] = -\frac{D[\theta]}{f_0[t] - k_\infty \cos\alpha}\frac{\partial\theta}{\partial x} + K[\theta, t]\cos\alpha \quad (21)$$

where $$\Theta[\theta, t] = \frac{\theta - \theta_\infty}{\theta_0[t] - \theta_\infty} \quad (22)$$

$$F[\Theta, t] = \frac{f[\theta, t] - k_\infty \cos\alpha}{f_0[t] - k_\infty \cos\alpha} \quad (23)$$

$$K[\theta, t] = \frac{k[\theta] - k_\infty}{f_0[t] - k_\infty \cos\alpha} \quad (24)$$

Integration from $x = 0$ to $x$ gives (25)

$$x[\theta, t] = \{f_0[t] - k_\infty \cos\alpha\}^{-1} \int_\theta^{\theta_0[t]} \frac{D[\theta]}{F[\Theta, t] - \frac{k[\theta] - k_\infty}{f_0[t] - k_\infty \cos\alpha}\cos\alpha}\, d\theta$$

Introducing $x[\theta,t]$ into (115) gives

$$\{F_0[t] - k_\infty \cos\alpha\} \int_0^t \{F_0[t] - k_\infty \cos\alpha\} \, dt \quad \int_{\theta_\infty}^{\theta_0[t]} \frac{(\theta - \theta_\infty)\, D[\theta]}{F[\Theta,t] - \dfrac{k[\theta] - k_\infty}{F_0[t] - k_\infty \cos\alpha} \cos\alpha} \, d\theta \qquad (26)$$

This integral mass balance relates the surface water content $\theta_0[t]$ and the flux $F_0[t]$. If $\theta_0[t]$ is given, then $F_0[t]$ can be calculated, and vice versa. With $\theta_0[t]$ and $F_0[t]$ known, the water content profile can be calculated from the expression for $x[\theta,t]$ given earlier. If $F[\Theta,t]$ is chosen judiciously then no iterations are necessary (See papers by Philip and Knight)

## THREE APPLICATIONS (of 25 and 26)

(see literature for details)

A One-dimensional absorption

$$x[\theta,t] = F_0[t]^{-1} \int_\theta^{\theta_0} \frac{D[\theta]}{F[\Theta,t]} \, d\theta \qquad (27)$$

$$F_0[t] \int_0^t F_0[t] \, dt = \int_\theta^{\theta_0} \frac{(\theta - \theta_\infty)\, D(\theta)}{F[\Theta,t]} \, d\theta \qquad (28)$$

It follows that

$$S = \left\{ 2 \int_{\theta_\infty}^{\theta_0} \frac{(\theta - \theta_\infty)\, D[\theta]}{F[\Theta,t]} \, d\theta \right\}^{1/2} \qquad (29)$$

$$\lambda = 2 S^{-1} \int_\theta^{\theta_0} \frac{D[\theta]}{F[\Theta,t]} \, d\theta \qquad (30)$$

B <u>Constant flux absorption</u>

$$Z = F_0 z[\theta, t] = \int_{\theta}^{\theta_0[t]} \frac{D[\theta]}{F[\Theta, t]} d\theta \qquad (31)$$

$$T = F_0^2 t = \int_{\theta_\infty}^{\theta_0[t]} \frac{(\theta - \theta_\infty) D[\theta]}{F[\Theta, t]} d\theta \qquad (32)$$

Note that $F_0$ <u>only enters</u> via the reduced variables $Z$ and $T$: $Z[\theta, T]$ profiles do not explicitly depend on $F_0$

Ponding time: $\theta_0 \to \theta_s$

$$T_p = F_0^2 t_p = \int_{\theta_\infty}^{\theta_0 = \theta_s} \frac{(\theta - \theta_\infty) D[\theta]}{F[\Theta, t]} d\theta \qquad (33)$$

$$= 1/2 \; S^2 \qquad (34)$$

$$t_p = 1/2 (S/F_0)^2 \qquad (35)$$

C. <u>Constant flux infiltration</u>

Same as B, except that the terms resulting from setting $\cos\alpha = 1$ should be added. In particular the ponding time:

$$T_p = (F_0 - k_\infty)^2 t_p = \int_{\theta_\infty}^{\theta_0 = \theta_s} \frac{(\theta - \theta_\infty) D[\theta]}{F[\Theta, t] - \frac{k[\theta] - k_\infty}{F_0 - k_\infty \cos\alpha} \cos\alpha} d\theta, \qquad (36)$$

At ponding there is a switch from $F_0$ = constant to $\theta = \theta_s$ = constant. Note that $F_0$ enters not only via $Z$ and $T$ but also via the integrand.

# Literature on quasi-analytical solutions

1. Parlange
Series of 11 papers published in Soil Science in the period 1971-1975
See also his review on "Water transport in soils" in Ann. Rev. Fluid Mechanics 12: 77-102

2. Philip and Knight
Series of 3 papers published in Soil Science in the period 1973-1974
See also paper by Knight in "Advances in infiltration. ASAE conference Chicago 1983

3. Following are the abstracts of 5 papers using the approach of Philip and Knight

a) White, I., D. E. Smiles, and K. M. Perroux. 1979. Absorption of water by soil: the constant flux boundary condition. Soil Sci. Soc. Am. J. 43:659-664.

Absorption of water into soil as the result of a constant flux condition at the soil surface is examined.

Experiments for a fine sand show that the surface water content, movement of the wetting front, and the water content profiles may be predicted from the soil water diffusivity function using the notion of the flux-concentration relation of Philip (1973).

Reduced space and time variables $X = V_* x$ and $T = V_*^2 t$ are introduced. Use of these variables greatly simplifies treatment of the system and reveals that the surface water content and the reduced position of the wetting front are uniquely defined by $T$ while the water content profiles at any value of $T$ are unique in terms of $X$.

b) White, I. 1979. Measured and approximate flux-concentration relations for absorption of water by soil. Soil Sci. Soc. Am. J. 43:1074-1080.

The flux-concentration relation of Philip (1973) is calculated using moisture profiles measured during absorption of water by a fine sand when water was supplied at both constant hydraulic potential and constant rate. The general behavior of the measured flux-concentration relations was found to be consistent with those of the model soils calculated by Philip. Within the accuracy of the measurements, no significant time-dependence of the relation was found for the constant flux boundary condition over a large time span. Measurements of the time-averaged flux-concentration relation, however, suggest some significant time-dependence for short times. Simple approximations for the flux-concentration relation, based on these observations, are suggested. These approximations enable easily evaluated predictions to be made of the salient features of constant rate application of water to soil, which are sufficiently accurate for most practical purposes.

c) Perroux, K. M., D. E. Smiles, and I. White. 1981. Water movement in uniform soils during constant-flux infiltration. Soil Sci. Soc. Am. J. 45:237-240.

An analysis is presented for constant-flux infiltration of water in soil based on the flux-concentration relation. The analysis is compared with laboratory experiments on constant-flux infiltration into columns of fine sand and silty clay loam. It is shown that the effect of gravity is small for the early stage and that during this stage sufficiently accurate predictions of moisture profile development can be made by using the simpler absorption analysis of I. White, D. E. Smiles, and K. M. Perroux.

d) Smiles, D. E., K. M. Perroux, S. J. Zegelin, and P. A. C. Raats. 1981. Hydrodynamic dispersion during constant rate absorption of water by soil. Soil Sci. Soc. Am. J. 45:453-458.

An analysis of hydrodynamic dispersion accompanying constant flux absorption of KCl solution by an initially relatively dry soil, is developed for the case when the hydrodynamic dispersion coefficient is pore water velocity-independent. It is shown that in this process both the water content and the soil water salt concentration are uniquely defined by $\theta(X,T)$ and $C(X,T)$, where $X = v_o x$ and $T = v_o^2 t$ are space- and time-like coordinates, and $v_o$ is the constant surface flux of water.

Quasi-analytical methods based on the flux-concentration relation predict $\theta(X,T)$ while an error-function solution, based on a material coordinate $Q$ labeling parcels of water, predicts the salt profile.

The analysis is demonstrated using a chemically inert sandy soil. The results show that during transient, unsaturated flow a simple piston-flow model described the process over a range of water contents. The method may be extended to explore dispersion in structured and chemically reactive soils.

e) Boulier, J.F., J. Touma, and M. Vauclin. 1984. Flux-concentration relation-based solution of constant-flux infiltration equation: I. Infiltration into nonuniform initial moisture profiles. Soil Sci. Soc. Am. J. 48:245-251.

The quasi-analytical solution of the infiltration equation based on the flux-concentration relation for constant flux condition and initially uniform water content profile is extended here to nonuniform moisture profiles for fluxes either smaller or greater than the saturated hydraulic conductivity. In the latter case, the solution is developed for the postponding stage. It is shown that the measured flux-concentration relation is well-approximated by $F(\theta^*) = \theta^*$ and that no significant time dependence can be observed. The quasi-analytical solution is then successfully compared with both laboratory experiments performed on a sandy soil column and numerical solution of the Richards equation for the nonponding, preponding, and postponding stages of infiltration.

# INFILTRATION THROUGH CRUSTS

The infiltrability of soils is often strongly influenced by the resistance experienced by the water in crusts. The pressure drop in the crust implies that even under ponded conditions the largest pores in the subcrust soil remain air-filled.

Crust play a role in:

1. Judging the effects of various methods of tillage;
2. Judging the effects of relatively high ESP (= exchangeable sodium percentage);
3. Evaluation of the potential of "runoff farming";
4. The so-called "crust-method" for determining the hydraulic conductivity.

## Structure of crusts

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### SCANNING ELECTRON MICROSCOPE OBSERVATIONS ON SOIL CRUSTS AND THEIR FORMATION

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ABSTRACT

Scanning electron micrographs (SEM) of crusts of loessial soils are presented. SEM observations were performed on crusts formed by raindrop impact at various stages of their formation. The crust structure was compared to the natural undisturbed soil. During the crust formation, a middle-term stage developed at which coarse particles, stripped of the fine ones, composed the surface layer of the soil. At the final stage of the crust formation, the coarse particles were washed away, and a thin seal skin, about 0.1 millimeter thick, formed the uppermost layer of the soil. A depositional crust, which was formed mainly by the translocation of fine particles, was marked by the presence of a thin skin also about 0.1 millimeter thick, suggesting involvement of similar secondary mechanisms of formation. This work illustrates the use of SEM for the study of soil crust formation and structure.

## Steady flow through crusts

See the following paper

Transactions, 10th International
Congress of Soil Science, Vol. I
Physics and Technology of Soils.
"Nauka" Publishing House, Moscow. 1974.

# STEADY INFILTRATION INTO CRUSTED SOILS

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Consider steady infiltration into a homogeneous soil profile with a crust at its surface. In the crust, the flux $q$ is given by:

$$q = C(h_c - h) \qquad (1)$$

where $C$ is the conductance of the crust, and $h_c$ and $h$ are the pressure heads at the top and bottom of the crust. In writing (1), the difference between the gravitational potentials at the top and at the bottom of the crust has been neglected. For a profile without a crust $C = \infty$. For a profile with an impermeable crust $C = 0$. The steady flux $q$ in the soil is given by:

$$q = k[h] \qquad (2)$$

where $k[h]$ is the hydraulic conductivity of the soil at the pressure head $h$. In this paper I present a simple, graphical analysis of flows described by equations (1) and (2). References to earlier work are given elsewhere (Raats, 1973).

Plots of equation (1) for a given crust with a given pressure head at its surface and of equation (2) for a given soil are shown in figure 1. Equation (1) describes a straight line with slope $C$ and intercept $h_o$ on the h-axis. Equation (2) corresponds to a curve which is simply a plot of the hydraulic conductivity of the soil versus the pressure head. Any intersection of such a straight line and such a curve describes a solution of equation (1) and (2). The rate of infiltration and the corresponding subcrust pressure head are, respectively, the ordinate and the abscissa of the point of intersection.

For a given soil and a given crust, the maximum, steady rate of infiltration in the absence of poded water on the surface, $q_m$, is reached when $h_o = 0$. With $h_o = 0$, equation (1) describes a straight line through the origin. Two such straight lines and plots of equation (2) for two soils are shown in figure 2. The maximum, steady rates of infiltration and the corresponding subcrust pressure heads are, respectively, the ordinates and the abscissas of the intersections between the straight lines and the curves. Note that in soils with a large reduction in the hydraulic conductivity at pressure heads slightly below zero, a crust whose conductance is large may still have a significant effect. In other words, just a slight crust may cause the large pores to remain emply and thus considerably reduce the maximum rate of infiltration.

Let $R$ be the rate of rainfall. If $R < q_m$ the rain is classified as non-ponding and if $R > q_m$ as ponding. If $R > q_m$ one can distinguish further a pre-ponding phase and a post-ponding phase. This generalizes the classification of Rubin (1966) for soils without crusts. For non-ponding rains $h_o < 0$. Of course, if water is allowed to accumulate on top of the crust, then the rate of infiltration will be larger than $q_m$.

The dimensionless flux $q_*$ pressure head $h_*$ and hydraulic conductivity $k_*$ are defined by:

$$q_* = q/K \quad , \quad h_* = -h/h_c \quad , \quad k_* = k/K \qquad (3)$$

where $K$ is the hydraulic conductivity of the soil when $h = 0$, and $h_c$ is the critical pressure head defined by (cf., Raats and Gardner, 1971):

$$h_c = \frac{\int^{-\infty} k[h]\,dh}{K} \qquad (4)$$

In term of dimensionless variables, (1) and (2) become:

$$q_* = C_*(h_{*o} - h_*) \qquad (5)$$

$$q_* = k_*[h_*] \qquad (6)$$

where the dimensionless conductance $C_*$ is defined by:

$$C_* = -h_c C/K \qquad (7)$$

Values of $h_c$ may range from -10 cm for a coarse sand to about -100 cm for a clay. Typical values of $C$ and $K$ are such that $C_*$ can be of the order 1 but also several orders of magnitude smaller. In the following it will be shown that, if for a given combination of soil and crust $C_* < 1$, then, the crust will significantly reduce the maximum rate of infiltration.

A versatile, empirical relationship between $k$ and $h$ is given by (cf., Raats and Gardner, 1971):

$$k = \frac{K}{(h/h_{\frac{1}{2}K})^n + 1} \qquad (8)$$

where $h_{\frac{1}{2}K}$ is the pressure head at which $k = \frac{1}{2}K$. The critical pressure head $h_c$ corresponding to (8) is given by (Raats and Gardner, 1971):

$$h_c = \frac{\pi}{n \sin(\pi/n)} h_{\frac{1}{2}K} \qquad (9)$$

Introducing (9) into (7) gives

$$C_* = -\frac{\pi C/K}{n \sin(\pi/n)} h_{\frac{1}{2}K} \qquad (10)$$

Plots of equation (5) with $h_{*_0} = 0$ for several values of $C_*$ and of equation (6) with (8) for $n$ ranging from 2 to 16 are shown in figure 3. For $C_* = 1$ the dimensionless flux ranges from about .45 for a soil with $n = 2$ to about .85 for a soil with $n = 16$. For $C_* = 10$ the range is .9 to nearly 1, while for $C_* = .01$ the range is .007 to 0.013. Note that for $C_* = 1$ and 10 the soils with low values of $n$ are affected most by the crust. Between $C_* = 1$ and .01 this trend gradually reverses itself.

The same graphical procedure can be used with other empirical relationships between $k$ and $h$ (cf., Raats and Gardner, 1971), including the relationships used by Hillel and Gardner (1969). Of particular interest is the form

$$k = Ke^{\alpha h} \qquad (11)$$



The critical pressure head $h_c$ corresponding to (11) is given by (Raats and Gardner, 1971):

$$h_c = -1/\alpha \qquad (12)$$

Introducing (12) into (7) gives:

$$C_* = C/(\alpha K) \qquad (13)$$

Introducing (11) into (5) and (6) gives:

$$C_* = -\frac{e^{h_*}}{h_*} \quad , \qquad C_* = -\frac{q_*}{\ln q_*} \qquad (14)$$

Plots of $h_*$ and $q_*$ as functions of $C_*$ according to (14) are shown in figure 4.

## References


Hillel D. and W.R.Gardner. Soil Sci.108: 137-142, 1969.

Raats P.A.C. The role of the unsaturated zone in hydrology. J.Hydrol. (in preparation), 1973.

Raats P.A.C. and W.R.Gardner. Water Resources Res. 7: 921-928, 1971.

Rubin J. Water Resources Res. 2: 739-749, 1969.


## Summary


Crusts play a key role in determining the amounts of runoff and erosion and in the performance of septic tank drainage fields. In this paper a simple graphical method for analyzing steady infiltration into crusted soils has been developed. The results clearly show that the reduction in the rate of infiltration caused by a given crust depends on the hydraulic properties of the underlying soil. It is shown that any crust/soil combination may be characterized by a single parameter which is equal to minus the product of the crust resistance and the so-called critical pressure head of the soil, divided by the hydraulic conductivity of the soil when saturated.

## Résumé

Dans un sol la croûte de surface joue un rôle déterminant dans l'importance du ruissellement et de l'érosion, et dans le rendement du drainage des fosses septiques. Dans cet article, on présente une méthode graphique simple qui permet d'analyser l'infiltration en régime permanent dans un sol avec une croûte. Les résultats montrent clairement que la réduction de vitesse d'infiltration due à une croûte dépend des propriétés hydrauliques du sol en dessous. On montre de plus que toute combinaison croûte/sol peut être caractérisée par un simple paramètre qui est égal à la valeur négative du rapport du produit de la résistance de la croûte par la charge critique du sol sur la conductivité hydraulique saturée du sol.

## Zusammenfassung

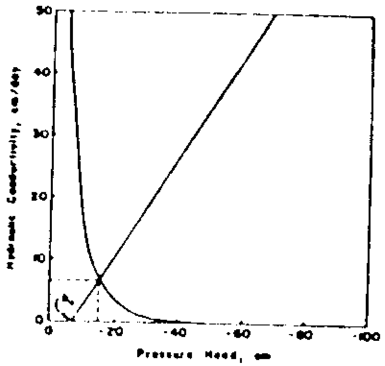
Der Ausmass des Ablaufs und der Erosion sowie das Funktionieren überhaupt von Versickerungsanlagen für Abwässer werden hauptsächlich von Bodenkrusten bestimmt. In der vorliegenden Arbeit wurde eine einfache graphische Methode zur Analyse der Dauerabsickerung in verkrüstete Böden entwickelt. Die Ergebnisse zeigen eindeutig, dass die jeweilige durch eine bestimmte Kruste verursachte Verminderung der Absickerungsgeschwindigkeit von den hydraulischen Eigenschaften des darunterliegenden Bodens abhängig ist. Es wird dargelegt, wie jede Kruste-Boden-Kombination durch einen einzelnen Parameter charakterisiert werden kann. Dieser entspricht dem negativen Produkt des Krustenwiderstands mit dem sogenannten kritischen Drucküberschuss des Bodens, geteilt durch den hydraulischen Leitfähigkeit des wassergesättigten Bodens.

## Резюме

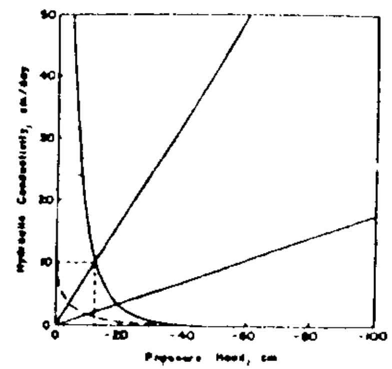
От поверхностных корковых образований значительно зависят объемы стока и эрозии, а также производительность полей орошения сточными водами. В данном докладе описан простой графический метод анализа установившейся инфильтрации в почвы, покрытые коркой. Полученные результаты четко показывают, что уменьшение скорости инфильтрации, вызванное данной коркой, зависит от гидравлических



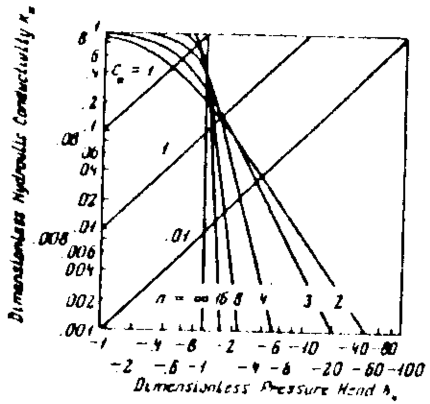

свойств подстилающей почвы. В докладе показано, что любое сочетание корка – почва может характеризоваться одним параметром, который равен отрицательной величине произведения сопротивления корки и так называемого критического напора почвы, деленного на коэффициент фильтрации насыщенной почвы.

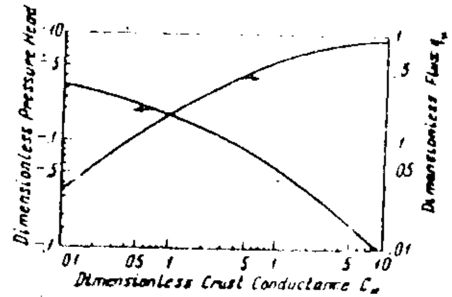



P i g. 1. Plots of equations (I) and (2) with arbitrary $h_o$.




P i g. 2. Plots of equations (1) and (2) for two crusts and two soils with $h_o$ = 0.




P i g. 3. The effect of certain crusts upon the rate of infiltration into a class of soils.

P i g. 4. Dimensionless pressure head and dimensionless flux as a function of the dimensionless crust conductance for soils with an exponential dependence of the hydraulic conductivity upon the pressure head.



12

Transient absorption through a crust

$$h = h_\infty, \quad \theta = \theta_\infty, \quad x > 0, \quad t = 0 \qquad (37)$$

$$F_0[t] = -k[h_0]\,\partial h/\partial x\big|_{x=0} = \gamma(h_s - h_0[t]) \qquad (38)$$

Reduced variables

$$Z = \gamma x \quad T = \gamma^2 t \quad \mathcal{F} = f/\gamma \qquad (39)$$

Flow problem in terms of reduced variables

$$\frac{\partial \theta}{\partial T} = -\frac{\partial \mathcal{F}}{\partial Z} \qquad (40)$$

$$\mathcal{F} = -k\,\partial h/\partial Z = -D\,\partial\theta/\partial Z \qquad (41)$$

$$h = h_\infty, \quad \theta = \theta_\infty, \quad Z > 0 \quad T = 0 \qquad (42)$$

$$\mathcal{F}_0[T] = -k[h_0]\,\partial h/\partial Z\big|_{Z=0} = (h_s - h_0[T]) \qquad (43)$$

The crust conductance $\gamma$ appears no longer explicitly. Unique dependence on $Z, T$

$$\Theta = \frac{\theta - \theta_\infty}{\theta_0[T] - \theta_\infty} \qquad (45)$$

$$F[\Theta, T] = \mathcal{F}/\mathcal{F}_0 \qquad (46)$$

Water content profiles

$$\mathcal{F}_0[T]\; Z[\theta, t] = \int_\theta^{\theta_0[T]} \frac{D[\theta]}{F[\Theta, t]}\,d\theta \qquad (47)$$

$$\mathcal{F}_0[T] \int_0^T \mathcal{F}_0[T]\,dT = \int_{\theta_\infty}^{\theta_0[T]} \frac{(\theta - \theta_\infty)\,D[\theta]}{F[\Theta, t]}\,d\theta \qquad (48)$$

This solution can be generalised to vertical flow and time-dependent $\gamma$

$$F_0[T] = dQ/dT \quad (49)$$

$$F_0 = h_s - h_0 \quad (50)$$

$$Q = \int_0^T F_0[T]\,dT$$ → function of $h_0$ (51) since both $\theta_0$ and $F_0$ in the integral mass balance are functions of $h_0$

Hence $T$ and $h_0$ are uniquely related

$$T = \int_0^Q F_0^{-1}\,dQ$$

$$= \int_{h_s-h_0}^{h_s-h_0} (h_s - h_0)^{-1} \frac{dQ}{d(h_s-h_0)}\, d(h_s-h_0)$$

$$= \frac{Q[h_s-h_0]}{h_s-h_0} + \int_{h_s-h_0}^{h_s-h_0} \frac{Q[h_s-h_0]}{(h_s-h_0)^2}\, d(h_s-h_0) \quad (52)$$

## Application of Philip/Knight approach

Smiles, D. E., J. H. Knight, and K. M. Perroux. 1982. Absorption of water by soil: the effect of a surface crust. Soil Sci. Soc. Am. J. 46:476–481.

The one-dimensional absorption of water by a uniform soil through a relatively impermeable surface layer is analyzed using conventional soil physics theory. The analysis permits calculation of the evolution of the water potential on the interface between the soil and the crust, the water content profiles, and the cumulative volume of water absorbed.

In the first instance the approach is tested for situations where the conductance of the crust is constant.

Extensions of the analysis to cases where the conductance is time-dependent are foreshadowed.

## Critique and improvement (using eqn (16))

Parlange, J.-Y., W.L. Hogarth, and M.B. Parlange. 1984. Optimal analysis of the effect of a surface crust. Soil Sci. Soc. Am. J. 48:494–497.

Recently, experiments have been presented for the horizontal flow of water by soil with a surface crust and compared to an analytical model. It was found that the water content at the surface was consistently less than predicted by about a 1% water content. Such a small discrepancy may seem to be more of theoretical rather than practical interest. In fact, due to the rapid variation of soil properties near saturation, a very small error in water content at the surface implies that the model is unreliable to predict the relationship between cumulative absorption and soil water content at the soil surface, especially in the early stages of infiltration. A new model is developed here, based on an optimal principle. The results from this model are closer to the experimental observations, but some discrepancy remains. To eliminate the possibility of experimental error a simple theoretical example is also considered, where an exact solution exists, which illustrates in detail the accuracy of the optimal model.

## Application to swelling material

Smiles, D.E., P.A.C. Raats, and J.H. Knight. 1982. Constant pressure filtration: the effect of a filter membrane. Chem. Eng. Sci. 37

**Abstract**—Theory developed to describe water movement and volume change in soils may be applied to many industrially important particulate liquid suspensions. The theory is used here to predict the important aspects of constant-pressure filtration where the filter membrane significantly impedes the escape of the liquid.

The method requires measured liquid content–liquid potential and liquid content–liquid diffusivity relations of the suspensions, and the conductance of the filter membrane.

Illustrative calculations for saturated bentonite slurry are presented. These calculations predict the evolution of the liquid and solid profiles in both material and physical space, and the cumulative volume of liquid expelled, as a function of time during filtration.

Experiments using bentonite at two different pressures, and with a range of values of membrane conductance, confirm integral predictions of the model.

## ANALYSIS OF STEADY MULTI-DIMENSIONAL FLOWS

1. For theoretical basis see attached paper on steady infiltration from line sources and furrows

2. A very useful solution is presented in: Wooding, R.A. 1968. Steady infiltration from a circular pond Water Resources Research 4:1259-127

Wooding showed that the flux $q$ from a circular pond with radius $R$ can be written as:

$$q = \pi R^2 k_0 + 2\pi R (2/\pi) k_0/\alpha$$

This result was used recently to determine $k_0$ and $\alpha$:

Scotter, D.R., B.E. Clothier, and E.R. Harper. 1982. Measuring saturated hydraulic conductivity and sorptivity using twin rings. Aust. J. Soil Res. 20:295-304.

*Abstract*

A method of measuring, with minimal soil disturbance, the saturated hydraulic conductivity and sorptivity of field soil is presented and discussed. It involves measuring the steady-state infiltration of ponded water from two rings, of different radii, that have been lightly pressed into the soil surface. The method is based on Wooding's solution for steady infiltration from a shallow, circular pond. Criteria for selecting ring radii are discussed. Results for three field soils are found to give consistent values for the conductivity and sorptivity.

3. Ponds with crusts still await analysis!

## TRANSPORT OF SOLUTES WITH INFILTRATING WATER

Some aspects are treated in

Raats, P.A.C. 1984. Tracing of parcels of water and solutes in unsaturated zones. In: B. Yaron, G. Dagan, and J. Goldschmid [Eds.] Pollutants in porous media: the unsaturated zone between soil surface and groundwater. Springer Berlin pp 4-16

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SOIL SCIENCE SOCIETY OF AMERICA

# PROCEEDINGS

VOL. 34 — SEPTEMBER-OCTOBER 1970 — No. 5

## DIVISION S-1—SOIL PHYSICS

### Steady Infiltration from Line Sources and Furrows[1]

P. A. C. RAATS[2]

ABSTRACT

Steady infiltration from an array of equally spaced line sources or furrows at the surface of a semi-infinite soil profile is analyzed. The discussion is based on the assumption that the hydraulic conductivity is an exponential function of the pressure head. It is shown that, under this assumption, the matric flux potential and the stream function for plane flows satisfy the same linear partial differential equation. Explicit expressions for the stream function, the flux, the matric flux potential, the pressure head, and the total head are obtained. Some implications with regard to furrow irrigation are discussed. The solution provides a rational basis for the discussion of leaching under furrow irrigation.

*Additional Key Words for Indexing:* partially saturated soil, plane flows, matric flux potential, stream function, furrow irrigation.

INTEREST in two- and three-dimensional movement of water in partially saturated soils is increasing rapidly. Analytical approaches to such problems have been reviewed by Philip (1969). Resistance network analog and digital computer solutions for two-dimensional steady flows have been reviewed briefly by Bouwer (1969). Whisler (1969) analyzed steady flow in an inclined soil slab with an electric analog. Digital computer solutions for some two-dimensional transient flows have been developed recently by Amerman (1969)[3], Hornberger et al. (1969), Rubin (1968), and Taylor and Luthin (1969). One may expect that large computers and improved numerical methods will enable future development of numerical solutions for transient flow problems of practical sizes. Advantages of analog and numerical solutions are that actual soil properties, and complex geometries and boundary conditions can be dealt with. However, it appears desirable that analytical and analog or numerical studies be pursued side by side. Analytical solutions, even if they are based upon simplifying assumptions, usually have the advantage of showing more clearly the structure of the flow and its dependence upon parameters characterizing the soil and the flow geometry.

The main purpose of this paper is to present an analytical solution for flow from an array of line sources at the surface for a soil whose hydraulic conductivity is an exponential function of the pressure head. The results show the main features of steady furrow irrigation.

## PLANE, STEADY FLOWS

Let $x, y, z$ be a rectangular Cartesian coordinate system, the $z$-direction being the vertical direction. Consider a steady flow, which is plane with respect to the $y$-direction. The balance of mass may then be written as:

$$\frac{\partial \theta u}{\partial x} + \frac{\partial \theta w}{\partial z} = 0 \qquad [1]$$

where $\theta$ is the volume of water per unit bulk volume, and $u$ and $w$ are the components of the velocity in the $x$- and $z$-directions, respectively. Expressions for the components of the volumetric flux are given by Darcy's law:

$$\theta u = -k\frac{\partial H}{\partial x} \qquad \theta w = -k\frac{\partial H}{\partial z} \qquad [2]$$

$$= -k[h]\frac{\partial h}{\partial x} \qquad = -k[h]\frac{\partial h}{\partial z} + k[h] \qquad [3]$$

$$= -D[\theta]\frac{\partial \theta}{\partial x} \qquad = -D[\theta]\frac{\partial \theta}{\partial z} + k[\theta] \qquad [4]$$

where $k$ is the hydraulic conductivity, $H$ is the total head, which is the sum of the pressure head $h$ and the gravitational head $-z$ (i.e., $H = h - z$), and $D = k dh/d\theta$ is the diffusivity. It is assumed that the steady flow is established through monotonic changes in pressure head, so that hysteresis need not be considered.

The forms of [3] and [4] suggest the introduction of a potential $\phi$ defined by:

$$\phi = \int_{h_o}^{h} k[h]\,dh = \int_{\theta_o}^{\theta} D[\theta]\,d\theta \qquad [5]$$

where $h_o$ and $\theta_o$ are reference values and $\theta_o = \theta[h_o]$. In terms of $\phi$, Darcy's law may be written as:

$$\theta u = -\frac{\partial \phi}{\partial x} \qquad \theta w = -\frac{\partial \phi}{\partial z} + k[\phi]. \qquad [6]$$

At any point the flux may be regarded as the sum of a matric component and of a gravitational component. The matric component is given by the gradient of $\phi$ and, therefore, it is appropriate to call $\phi$ the *matric flux potential*. Equation [5] implies that surfaces of equal pressure head, water content, and matric flux potential coincide. Some remarks concerning the history of the use of $\phi$ will be found at the end of this section.

Equation [1] is satisfied by a stream function $\psi$, defined as:

$$\theta u = -\frac{\partial \psi}{\partial z} \qquad \theta w = +\frac{\partial \psi}{\partial x}. \qquad [7]$$

By combining the pairs of expressions for the velocity components given in equations [6] and [7] one gets an analog of the Cauchy Riemann conditions:

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial z} \qquad \frac{\partial \phi}{\partial z} - k[\phi] = -\frac{\partial \psi}{\partial x}. \qquad [8]$$

Note the presence of the term $-k[\phi]$. Substitution of [6] into [1] gives the partial differential equation for $\phi$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{\partial k[\phi]}{\partial z}. \qquad [9]$$

Elimination of $\phi$ from [8] gives:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{\partial k[\phi]}{\partial x}. \qquad [10]$$

At this point it is convenient to assume that $k$ and $h$ are related by an expression of the form:

$$k = k_o \exp(\alpha h). \qquad [11]$$

The parameters $k_o$ and $\alpha$ characterize the soil. The parameter $k_o$ represents the hydraulic conductivity of the saturated soil. Introducing [11] into [5] and solving for $k$ gives (taking $\theta_o = 0$ and $h_o =$ [illegible] as reference values):

$$k = \alpha\phi. \qquad [12]$$

From [11] and [12] it follows that the pressure head $h$ is given by:

$$h = \frac{1}{\alpha}\ln\phi - \frac{1}{\alpha}\ln k_o/\alpha. \qquad [13]$$

The hydraulic head $H$ is given by:

$$H = \frac{1}{\alpha}\ln\phi - \frac{1}{\alpha}\ln k_o/\alpha - z. \qquad [14]$$

Substitution of [12] into [9] gives:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = \alpha\frac{\partial \phi}{\partial z}. \qquad [15]$$

Substitution of [12] into [10] and using the first of equations [8] gives:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = \alpha\frac{\partial \psi}{\partial z}. \qquad [16]$$

Surprisingly, the matric flux potential $\phi$ and the stream function $\psi$ satisfy the same partial differential equation. Therefore, *corresponding to any plane flow problem there exists a conjugate problem which is obtained by interchanging the matric flux potential $\phi$ and the stream function $\psi$.* This generalizes a well-known property of plane flows satisfying Laplace's equation.

A transformation of the type introduced in equation [5] was given around 1880 by Kirchhoff in his lectures on heat transfer (Kirchhoff, 1894). For this reason, the transformation is often referred to as the Kirchhoff transformation and the potential $\phi$ is sometimes called the Kirchhoff potential. The matric flux potential was introduced by Klute (1952) in an analysis of horizontal movement of water in a partially saturated soil. Gardner (1958) presented a partial differential equation for movement of water in a partially saturated soil, using the matric flux potential as the dependent variable and including the effect of gravity. To solve the equation for steady flows, he suggested the technique of separating the variables. He also noted that the introduction of the exponential relationship between $k$ and $h$, equation [11] above, leads to a further simplification, equation [15] above, making analytical solutions for certain steady flows possible. The first detailed example of such an analytical solution was given by Philip (1968) in his discussion of steady infiltration from buried point sources and spherical cavities. Wooding (1968) analyzed steady infiltration from a circular pond. Philip (1969) briefly discussed the solution for a single, horizontal line source in an infinite region.

## STEADY INFILTRATION FROM A LINE SOURCE

Consider an array of equally spaced line sources at the soil surface (Fig. 1). Taking the line sources to be infinitely

Handwritten notes in margin: "axi-symmetric Wooding's" / "linearizes steady flows" / "also used in problem" / "16"

[1] Contribution from the Corn Belt Branch, Soil & Water Conserv. Res. Div., ARS, USDA, Madison, Wis., in cooperation with the Wisconsin Agr. Exp. Sta. Approved for publication by the Director of the Research Division of the College of Agricultural & Life Sciences, Univ. of Wisconsin. Received May 8, 1970. Approved June 19, 1970.

[2] Research Soil Scientist (Physics), USDA, Madison, Wis. and Assistant Professor, The University of Wisconsin.

[3] Amerman, C. R. 1969. Finite difference solutions of unsteady, two-dimensional, partially saturated porous media flow. Ph.D. Thesis, Purdue Univ., Lafayette, Ind. 221 p.

long it is sufficient to describe the motion in the $x,z$-plane. Let $2L$ be the distance between the line sources and let $2S$ be the strength of the line sources per unit length in the $y$ direction. Before specifying the boundary conditions, it is convenient to introduce the following dimensionless variables

$$X = (1/L)x \qquad Z = (1/L)z \qquad [17]$$

$$\Theta U = (L/S)\theta u \qquad \Theta W = (L/S)\theta w \qquad [18]$$

$$\Phi = (\alpha L/S)\phi \qquad \Psi = (1/S)\psi \qquad [19]$$

$$\Gamma = (1/L)h \qquad \Gamma = (1/L)H. \qquad [20]$$

Note that

$$S/L = \theta_\infty w_\infty \qquad [21]$$

where $\theta_\infty w_\infty$ is the uniform volumetric flux for $z \to \infty$.

Introducing the dimensionless variables in [16] gives:

$$\frac{\partial^2\Psi}{\partial X^2} + \frac{\partial^2\Psi}{\partial Z^2} = \alpha L\frac{\partial\Psi}{\partial Z}. \qquad [22]$$

From the geometry of the problem it is evident that there are two bounding stream lines: (i) a stream line vertical downward from the line source, and (ii) a stream line running along the surface and vertical downward at the center between any two of the line sources. In terms of the dimensionless variables these conditions may be stated as:

$$X = 0 \qquad Z > 0 \qquad \Psi = 0 \qquad [23]$$

$$0 < X < 1 \qquad Z = 0 \qquad \Psi = 1 \qquad [24]$$

$$X = 1 \qquad Z > 0 \qquad \Psi = 1. \qquad [25]$$

An additional condition follows from the fact that for large $Z$ the flow will be uniform (cf. equation [21]):

$$0 \leq X \leq 1 \qquad Z \to \infty \qquad \Psi = X. \qquad [26]$$

## THE FLOW PATTERN

The solution of [22] subject to [23]-[26] may be written as

$$\Psi = X + \Psi_p. \qquad [27]$$

The first term on the right hand side of [27] may be regarded as a steady flow due to a source $S$ distributed uniformly over $0 < X < 1$ at $Z = 0$. The second term represents a perturbation of the uniform flow arising from the source being actually at $X = 0$, $Z = 0$. The form of $\Psi_p$ can be developed by separating the variables:

$$\Psi = X + \Psi_{pX}\Psi_{pZ} \qquad [28]$$

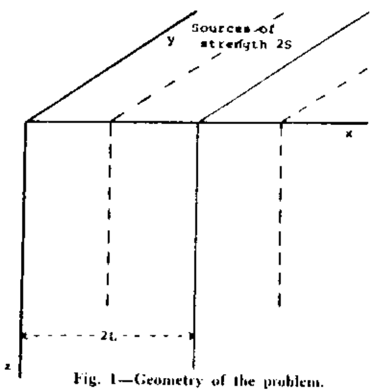



Fig. 1—Geometry of the problem.

where $\Psi_{pX}$ depends upon $X$ only, and $\Psi_{pZ}$ depends upon $Z$ only. Substitution of [28] into [22] and rearranging gives:

$$\Psi''_{pX}/\Psi_{pX} = -\Psi''_{pZ}/\Psi_{pZ} + \alpha L\,\Psi'_{pZ}/\Psi_{pZ} = -\lambda \qquad [29]$$

where $\lambda$ is a constant. It follows that

$$\Psi''_{pX} + \lambda\Psi_{pX} = 0 \qquad [30]$$

$$\Psi''_{pZ} - \alpha L\Psi'_{pZ} - \lambda\Psi_{pZ} = 0. \qquad [31]$$

For $\lambda > 0$, general solutions of [30] and [31] are, respectively:

$$\Psi_{pX} = C_1 \cos\sqrt{\lambda}X + C_2 \sin\sqrt{\lambda}X \qquad [32]$$

$$\Psi_{pZ} = C_3 \exp p_n Z \qquad [33]$$

where $C_1$, $C_2$, and $C_3$ are constants and $p_n$ is given by:

$$p_n = \alpha L/2 \pm \{\lambda + (\alpha L/2)^2\}^{1/2}. \qquad [34]$$

The expression for $p_n$ is found by substituting [33] into [31]. After substituting [32] and [33] into [28], it is a simple matter to show that the solution for the dimensionless stream function $\Psi$ which satisfies the conditions [23]-[26] is given by:

$$\Psi = X + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\exp(-p_n Z)}{n}\sin n\pi X \qquad [35]$$

with

$$p_n = -\alpha L/2 + \{(\alpha L/2)^2 + (n\pi)^2\}^{1/2}. \qquad [36]$$

Note that according to [35] and [36] the flow pattern is a unique function of $\alpha L$. In particular, the flow pattern is independent of the source strength $S$. For $\alpha L = 0$ equation [22] reduces to Laplace's equation and [36] reduces to:

$$p_n = n\pi \qquad [37]$$



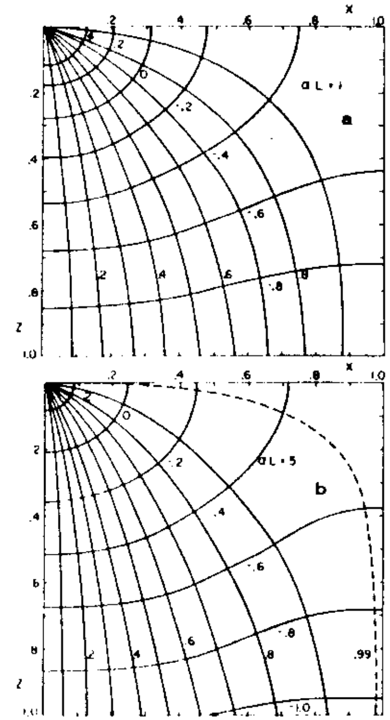



Fig. 2—Distribution of $\Psi$ and $\Gamma$ for $\alpha L = 1$ and 5.

In Fig. 2 the flow patterns for $\alpha L = 1$, and 5 have been plotted. (All calculations related to this study were performed by the Univac 1108 digital computer at the University of Wisconsin. The number of terms used in the various infinite series ranged from one to several hundred, depending on the form of the series and the $X$, $Z$ coordinates at which the series is to be evaluated). A comparison of Fig. 2a and 2b, indicates that with large $\alpha L$ gravity has a stronger effect resulting in a flow that is more concentrated around the $Z$-axis.

## THE DISTRIBUTION OF THE FLUX

Introduction of the dimensionless variables [17]-[19] into [7] gives:

$$\Theta U = -\frac{\partial\Psi}{\partial Z} \qquad \Theta W = \frac{\partial\Psi}{\partial X} \qquad [38]$$

Substituting [35] into [38] gives:

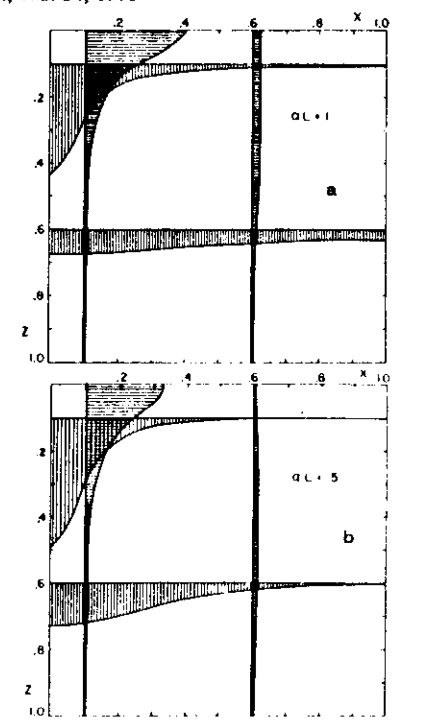



Fig. 3—Distribution of $\Theta U$ at $X = 0.1$ and 0.6, and of $\Theta W$ $Z = 0.1$ and 0.6 for $\alpha L = 1$ and 5.

$$\Theta U = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{p_n\exp(-p_n Z)}{n}\sin n\pi X \qquad [39]$$

$$\Theta W = 1 + 2\sum_{n=1}^{\infty}\exp(-p_n Z)\cos n\pi X. \qquad [40]$$

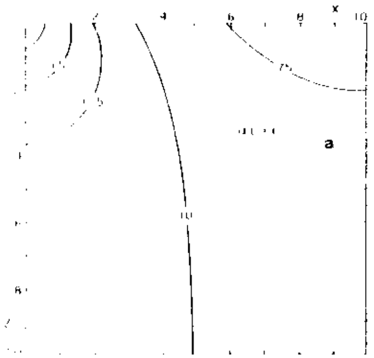
Some examples of calculated flux profiles for $\alpha L = 1$, and 5 are given in Fig. 3. For small depths the distribution of the flux is far from uniform, particularly if $\alpha L$ is large. For large depths $\Theta U$ approaches zero and $\Theta W$ approaches unity.

## THE DISTRIBUTION OF THE MATRIX FLUX POTENTIAL

Substituting [12] into [6] and introducing the dimensionless variables [17]-[19] gives:

$$\Theta U = -(1/\alpha L)\frac{\partial\Phi}{\partial X} \qquad [4$$

$$\Theta W = -(1/\alpha L)\frac{\partial\Phi}{\partial Z} + \Phi$$



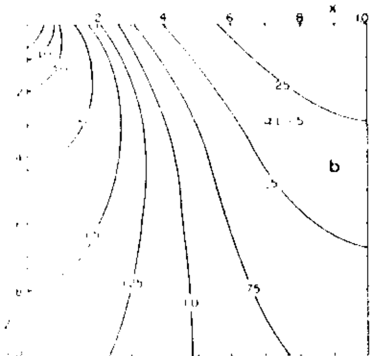


Fig. 4—Distribution of $\Phi$ for $\alpha L = 1$ and 5.

Substituting [39] into the first of these two equations and integrating gives the following explicit expression for the dimensionless matric flux potential:

$$\Phi = 1 + (2L\alpha/\pi^2) \sum_{n=1}^{\infty} \frac{p_n \exp(-p_n Z)}{n^2} \cos n\pi X . \quad [42]$$

In deriving [42], use was made of the condition that as $Z \to \infty$ the potential $\Phi \to 1$. The distributions of $\Phi$ for $\alpha L = 1$ and 5 are given in Fig. 4. Equation [5] implies that surfaces of equal matric flux potential $\phi$ are also surfaces of equal pressure head $h$ and water content $\theta$. From [13] and [19] it follows that the dimensionless pressure head is given by:

$$\Psi = (1/\alpha L)\{\ln\Phi - \ln k_0 L/S\} . \quad [43]$$

The dimensionless pressure head does not depend only upon $\alpha L$, but also upon $k_0$, $L$, and $S$. The matric flux potential $\Phi$ varies over a wider range when $\alpha L$ is larger, and, in view of the correspondence noted above, the same applies to the pressure head and the water content. In the region

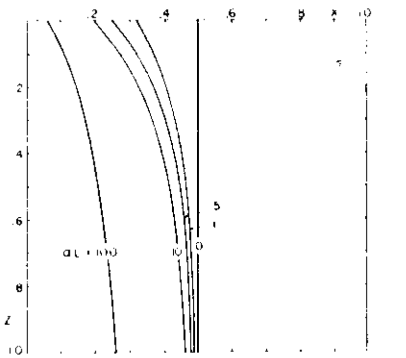


Fig. 5—Distribution of $\Phi = 1$ for $\alpha L = 0, 1, 5, 10$, and 100.

with $\Phi > 1$ the soil is wetter than it is for $Z \to \infty$. In the region with $\Phi < 1$ the soil is drier than it is for $Z \to \infty$. In Fig. 5 the loci of $\Phi = 1$ are given for several values of $\alpha L$.

## THE DISTRIBUTION OF THE HYDRAULIC HEAD

From [14], [17], [19], and [20] it follows that the dimensionless hydraulic head $\Gamma$ is given by:

$$\Gamma = (1/\alpha L)\{\ln\Phi - \ln k_0 L/S\} - Z . \quad [44]$$

The function $\Gamma + (1/\alpha L) \ln k_0 L/S = (1/\alpha L) \ln\Phi - Z$, which depends only on $X$, $Z$, and $\alpha L$, and differs from $\Gamma$ merely by the constant $(1/\alpha L) \ln k_0 L/S$, is plotted in Fig. 2. The stream lines and the lines of equal hydraulic head are orthogonal (cf., Raats, 1967; Sewell and van Schilfgaarde, 1963). The stream lines and the lines of equal matric flux potential are not orthogonal, except in the limits $R = (X^2 + Z^2) \to 0$ and $Z \to \infty$ (see also Philip, 1968, Wooding, 1968).

## DISCUSSION

It was pointed out already that the flow pattern is a unique function of $\alpha L$. The distributions of the dimensionless flux and the dimensionless matric flux potential are also unique functions of $\alpha L$. The parameter $L$ is a characteristic length of the flow region. The parameter $\alpha$ is roughly a measure of the coarseness or fineness of the soil. Taking [length] as the dimension of $h$, the dimension of $\alpha$ will be $[\text{length}]^{-1}$. According to Philip (1968), a typical value of $\alpha$ is 0.01 and the range 0.05 $cm^{-1}$ to 0.002 $cm^{-1}$ seems likely to cover most applications. Most of the data presented by Rijtema (1965) fall in this range. In this paper calculations for $\alpha L = 1$, and 5 have been presented. For a very fine-textured soil, with $\alpha = 0.002$ $cm^{-1}$, $\alpha L = 1$ and 5 correspond to line source spacings $2L = 10$ and 50 m, respectively. For a coarse-textured soil, with $\alpha = 0.05$ $cm^{-1}$, the values $\alpha L = 1$ and 5 correspond to line source spacings $2L = 0.4$ and 2 m, respectively.



A comparison of Fig. 2a and 2b, of 3a and 3b, and of 4a and 4b indicates that with large $\alpha L$ gravity has a stronger effect, resulting in a flow that is more concentrated around the Z-axis. This illustrates once again that with a finer textured soil the dimensions of the flow system must be larger for gravity to have a noticeable effect (cf., Corey et al., 1965; Kraijenhoff van de Leur, 1962; Miller and Miller, 1956). This basic principle is sometimes overlooked in discussions of numerical studies of two-dimensional flow problems.

The solution for flow from a line source is also exact for infiltration of water at a certain pressure head from furrows whose contours coincide with lines of equal hydraulic head $H$, or, in other words, from furrows whose contours are orthogonal to the stream lines. Examples of such contours have been drawn in Fig. 2. If the furrow is filled with water and if $h = 0$ at points in the furrow for which $Z = 0$, then everywhere along the furrow surface the pressure head will be positive, with a maximum at the point of intersection between the contour and the Z-axis. This implies that close to the furrow the pressure head in the soil will be positive and that the exponential relationship between $k$ and $h$, equation [11], will not be valid. Moreover, some of the data presented by Rijtema (1965) suggest that equation [11] gives a poor approximation in the range −10 to 0 cm pressure head. However, two factors offset this inadequacy of equation [11]. First, for small $R = (X^2 + Z^2)^{1/2}$ the flow pattern is relatively independent of the parameter $\alpha L$. Second, in practice the furrow surface will often be clogged or consist of a thin, relatively impermeable crust, resulting in a significant pressure drop over a small distance (Bouwer, 1969; Hillel and Gardner, 1969).

The flux profiles in Fig. 3 show that for small depths the distribution of the flux is far from uniform. The rapid downward movement under furrows in sandy soils was already noted by Loughridge (1908). The nonuniformity of the flux has important implications with regard to leaching of salts, as was pointed out by Bernstein and Fireman (1957), Haise (1948), and King (1902). Figure 4 shows that for small depths the distribution of the matric flux potential, and thus of the pressure head $h$ or the water content $\theta$, is also far from uniform.

It was pointed out before that corresponding to any plane problem there exists a conjugate problem which is obtained by interchanging the matric flux potential $\phi$ and the stream function $\psi$. Consider the flow resulting from keeping the points along the Z-axis at one value of $\Phi$, and the points along the X-axis and along $(X,Z) = (1,Z)$ at another value of $\Phi$. For such a flow the stream function is given by the right-hand side of [42] (Fig. 4), while the matric flux potential is given by the right-hand side of [35] (Fig. 2). The practical significance of this type of a flow is not apparent.

For large $Z$ the flow becomes uniform and thus the same as a steady flow due to a source $S$ distributed uniformly over $0 < X < 1$ at $Z = 0$. In particular the one-dimensional largetime solution may be used, giving the shape of the wetting front moving steadily downward. The re[ader] is referred to Philip (1969) for a review of the rele[vant] literature.

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# Flow and Transport in the Natural Environment: Advances and Applications

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# QUASIANALYTIC AND ANALYTIC APPROACHES TO UNSATURATED FLOW: COMMENTARY

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Summary. Analysis of movement of water in unsaturated soils is a basis for water management and for understanding the environment in which soil mechanical, chemical, and biological processes occur. This commentary presents further examples of quasianalytic and analytic approaches to unsaturated flow. Three approaches to one-dimensional problems are discussed in some detail: (1) approximate methods subject to integral constraints, (2) methods based on transformations of nonlinear equations to linear equations, and (3) solutions for vertical flows dominated by gravity. Also, some further examples of quasilinear analysis of steady multidimensional flows are given, with emphasis on flows from shallow circular ponds.

## ON THE ROLE OF WATER IN UNSATURATED SOILS

Spatial and temporal distribution of soil water is the central theme of soil physics, not only in an arid country like Australia, but also in a humid country like The Netherlands. Even in humid climates temporary shortages of water occur. The planning and execution of supplemental irrigation requires detailed analysis of movement of water in the unsaturated zone, especially concerning capillary rise, as an alternative source of water supply to crops. A certain degree of unsaturation is necessary for a soil to be sufficiently aerated and trafficable. Poor aeration may lead to impaired root growth and functioning and to rapid denitrification. The trends of mechanization in agriculture have stimulated a worldwide interest in the timing of soil tillage crop care and harvest operations.

Soil water plays a key role with regard to the dynamics of dissolved plant nutrients and contaminants. For transport over distances of 0.1 m or more, convective transport with the water dominates, resulting in significant average displacement and dispersion. Diffusive transport of solutes can be the prime mode of transport over distances of 0.01 m or less, as is the case for solutes being absorbed or exuded by plant roots. Availability of nutrients for plants is strongly influenced by the water status of an unsaturated soil, primarily as a consequence of the low values of the solute diffusivity at low water contents.

The list of publications of the former Agricultural Physics Section of the CSIRO Division of Plant Industry and the current CSIRO Division of Environmental Mechanics is an amazingly rich source of information on movement of water in unsaturated soil (Anonymous 1987). A large fraction of the existing quasianalytic and analytic approaches were originated, or stimulated, or critically evaluated by J.R. Philip and his numerous past and present co-workers. I acknowledge the recurrent inspiration derived from the work of this group.

RICHARDS' EQUATION

Richards (1931) put into rigorous mathematical-physical form the concepts of water flow in unsaturated soil due primarily to Buckingham (1907), but also partially formulated by others seemingly unaware of Buckingham's work. By now, the general flow equation is firmly established by reasoning and experimental evidence. Of course, Richards' theory had to be extended to cover complications arising from vapour transport, lack of continuity of soil air, limited mobility of soil air, deformation of the solid phase, structure of the solid phase, and deviations from Darcy's law associated with thermal and osmotic pressure gradients. These extensions in many directions have helped to put Richards' theory in a proper perspective. The modern theory of mixtures provides clear guidance for rational extensions of Richards' theory (Raats 1984b).

In the context of Richards' theory, specific soils are characterized by the dependencies of the pressure head $\Psi$ and hydraulic conductivity K upon the volumetric water content $\theta$. Most quasianalytic and analytic approaches are restricted to classes of specific soils. Examples are (i) the quasilinear analysis of steady, multidimensional flows in soils with an exponential relationship between the hydraulic conductivity and the pressure head, and (ii) unsteady flows in the broad Broadbridge-White class bridging a whole range from the Knight class to the Green-Ampt class.

All quasianalytic and analytic methods for solving Richards' equation involve simplifications. Problems can first of all be classified in geometric categories of vertical versus nonvertical flows and in temporal categories of steady versus nonsteady flows. Further useful simplifications may arise from considering the relative importance of capillary and gravitational components of the flux of the water. Thus far analytic solutions in the presence of a nonzero distributed sink $\lambda$ have been obtained only for certain types of steady flows with specific spatial distributions of $\lambda$ (Raats and Warrick 1981).

Many processes of flow of water in unsaturated soils so far have been, and perhaps will always be, amenable only to numerical methods. This is especially true for processes resulting from boundary conditions operating outdoors. Generally these processes are affected by hysteresis of the soil water retentivity curve. Nevertheless, anyone equipped with insights derived from quasianalytic and analytic approaches will be in a good position to describe and explain the main features in results of detailed calculations.

ONE-DIMENSIONAL PROBLEMS

More than one-third of Philip's (1988) review is devoted to one-dimensional problems. In the following, I firstly comment on what Philip calls the flux-concentration method, secondly, expand on Philip's remarks about solutions based on transformations of nonlinear equations to linear equations, and, thirdly, present some results for vertical flow dominated by gravity. The last class of flows has been an important source of inspiration for field studies over the past 15 years (e.g., references given in Raats 1983).

Approximate Methods Subject to Integral Contraints

Several time-honoured solutions of the Richards' equation are of the form

$$z = z(\theta,t) \ , \quad z = z(\Psi,t) \ , \qquad (1)$$

where z is the depth of the soil, positive downward, and t is time. Examples of this are (i) solutions for steady upward and downward flows, (ii) solutions in the form of travelling, time-invariant waves, (iii) the Boltzmann solution with $t^{1/2}$ proportionality for horizontal absorption, and (iv) Philip's series expansion in $t^{1/2}$ for vertical infiltration.

The first of the two functional relationships expressed in (1) is consistent with regarding the water content $\theta$ and the time t as independent variables, while the flux F and the position z are regarded as the dependent variables. In this context the mass balance is expressed by (Philip 1969)

$$\frac{\partial z}{\partial t}\bigg|_{\theta} = \frac{\partial F}{\partial \theta}\bigg|_{t} \quad . \tag{2}$$

Integration of (2) between $\theta$ and $\theta_\infty$ gives

$$F - F_\infty = \frac{\partial}{\partial t}\int_{\theta_\infty}^{\theta} z \, d\theta \quad . \tag{3}$$

Integration of (3) with respect to time gives

$$\int_0^t [F_0(t) - F_\infty]dt = \int_{\theta_\infty}^{\theta_0(t)} z(\theta,t)d\theta \quad , \tag{4}$$

where the subscript 0 denotes values at the soil surface. The *integral mass balance* (4) is the constraint used in the "flux-concentration method" discussed by Philip (1988). Integration of (4) with respect to space gives

$$\int_0^\infty (F - F_\infty)dz = \frac{\partial}{\partial t}\int_{\theta_0}^{\theta_\infty} \tfrac{1}{2}z^2 d\theta$$
$$= \frac{\partial}{\partial t}\int_0^\infty z(\theta - \theta_\infty)dz \quad . \tag{5}$$

The *integral moment balance* (5) is the constraint used in numerous papers by Parlange and coworkers (Parlange 1980; Hornung et al. 1987). The integral moment balance was originally introduced in proposals for determining experimentally the hydraulic conductivity (Zaslavsky and Ravina 1965) and the sorptivity (Youngs 1968).

The approximate methods using either (4) or (5) as integral constraints have yielded solutions for horizontal absorption and vertical infiltration with a given water content or flux at the soil surface and with or without a surface crust. For constant flux, horizontal absorption and small-time vertical infiltration, there is the interesting feature that the surface flux $F_0$ enters the solution only via the reduced position Z and time T defined by (cf., Philip 1969, Eq. (199); White et al. 1979; Perroux et al. 1981):

$$Z = F_0 z \ , \quad T = F_0^2 t \ . \tag{6}$$

An analogous statement is true for horizontal absorption and small-time, vertical infiltration with constant potential boundary conditions via a crust with constant properties; in (6) the crust conductance $\gamma$ must then replace the flux $F_0$ (Ahuja and Swartzendruber 1973; Smiles et al. 1982):

$$Z = \gamma z \quad , \quad T = \gamma^2 t \ . \tag{7}$$

Transformations of Nonlinear Equations to Linear Equations

The approximate methods subject to integral constraints apply to any soil. Philip (1988) also reviewed some results for particular classes of soils, namely Green-Ampt, linear, Knight, and Broadbridge-White soils. For these particular classes of soils, solutions covering the whole range from very small to very large times can be derived for certain boundary conditions. The Green-Ampt and linear soils do not allow proper large-time travelling waves, but this is not too disturbing since these classes are not realistic models of real soils anyway.

For the Knight class, solutions for infiltration with a wide range of boundary conditions are known. Philip showed that the special solution for infiltration with constant potential boundary condition is suitable for filling the gap between the series expansion in $t^{1/2}$ for small/intermediate time and the travelling, time-invariant wave for large time.

For the Broadbridge-White class the dependence of soil-water diffusivity D on relative water content $\theta$ is given by (Broadbridge and White 1988):

$$D = b(C)C(C - 1)\{S[\Delta\theta(C - \Theta)]\}^2 \ , \tag{8}$$

and the dependence of hydraulic conductivity K on $\theta$ is described by:

$$(K - K_n)\Delta K = \Theta^2(C - 1)/(C - \Theta) \ . \tag{9}$$

Here $\Theta = (\theta - \theta_n)/\Delta\theta$, with $\theta_n$ the initial soil-water content, $\Delta\theta = \theta_s - \theta_n$, with $\theta_s$ the value at field saturation, $S = S(\theta_s, \theta_n)$ is the sorptivity, $K_n = K(\theta_n)$, $\Delta K = K(\theta_s) - K_n$, C is a single free parameter of the model, and b(C) is a constant which depends on C. The parameter C is constrained to the range $1 < C < \infty$ with corresponding range $1/2 < b(C) < \pi/4$ for the constant b(C). As $C \to 1$, the hydraulic properties (8) and (9) approach those of a highly nonlinear Green-Ampt-like model. At the other end of the parameter range, as $C \to \infty$, (8) and (9) approach the properties of the weakly nonlinear Knight soil. For most soils C is confined to the range $1.01 < C < 2$.

Figure 1 shows results of calculations based on an analytical solution for constant rate rainfall infiltration into Broadbridge-White soils. The solution is parameterized in terms of the dimensionless rainfall rate $R_* = (R - K_n)/\Delta K$, dimensionless time $t_* = t/[bS/(\Delta K)^2]$, $\theta$, and C. Figure 1 shows the evolution of soil saturation profiles for both nonponding, $R_* < 1$, and ponding, $R_* > 1$, rain and for two values of C. Figure 2 shows results for the time to incipient ponding.

Vertical Flows Dominated by Gravity

Solutions of the form (1) also arise for purely gravitational flows satisfying

$$\frac{\partial\theta}{\partial t} = - \frac{dK}{d\theta}\frac{\partial\theta}{\partial z} \tag{10}$$

(Raats 1983). Equation (10) is a kinematic wave equation with wave speed $dK/d\theta = \kappa(\theta)$. According to (10), surfaces of constant $\theta$ will propagate downward at speeds $dK/d\theta = \kappa(\theta)$. The ratio R of the wave speed $dK/d\theta$ to the speed $K/\theta$ of a parcel of water due to the gravitational force is given by

$$R = \frac{\theta}{K}\frac{dK}{d\theta} = \frac{d\ln K}{d\ln\theta} \ . \tag{11}$$

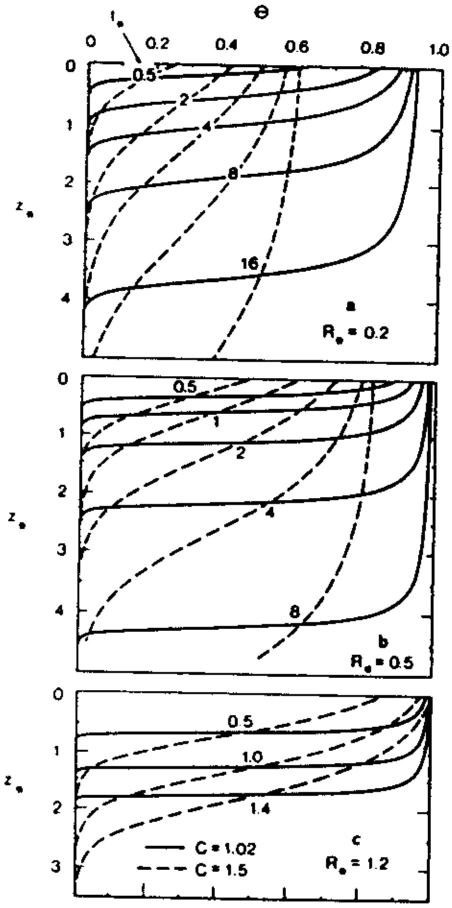


Fig. 1. Dimensionless saturation profiles at various dimensionless times $t_*$ during constant-rate rainfall for $C = 1.02$ and 1.5 with (a) $R_* = 0.2$, (b) $R_* = 0.5$, and (c) $R_* = 1.2$. (Broadbridge and White 1988).

Thus, R is the ratio of the relative rates of change of K and of $\theta$ (see Raats 1984a).

Consider the drainage of a uniform profile with a deep water table given the initial condition

$$\theta(z,0) = \theta_n \,, \quad K(z,0) = K_n \,, \quad \text{for } z > 0 \,, \tag{12}$$

where $\theta_n$ and $K_n$ are the uniform initial water content and hydraulic conductivity in the soil profile. Sisson et al. (1980) observed that if one supplements (12) with a fictitious condition

$$\theta(z,0) = \theta_m \,, \quad K(z,0) = K_m = 0 \,, \quad \text{for } z < 0 \,, \tag{13}$$

where $\theta_m$ is a fictitious, uniform water content in the region $z < 0$, chosen such that the hydraulic conductivity $K_m$ corresponding to $\theta_m$ is zero, then conditions (12) and (13) describe an initial shock, and solving (10) subject to these conditions describes the decay of this shock. The solution of this type of problem is due to Lax (1972). Values of the water content in the range

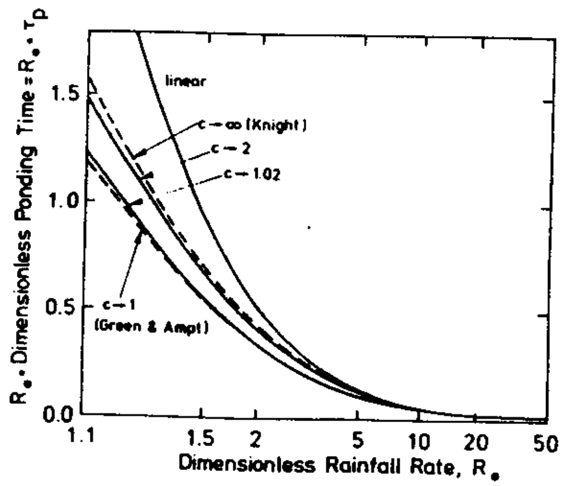


Fig. 2. Comparison of the dependence of the dimensionless ponding time upon the dimensionless rainfall rate for soils with the parameter C ranging from 1 (Green-Ampt soil) to ∞ (Knight soil), and for linear soils. (Broadbridge and White 1987).

$\theta_m < \theta < \theta_n$ will propagate downward from the soil surface at speeds $\kappa(\theta)$. This means that the depth z at which the water content will be $\theta$ at time t is given by

$$z(\theta,t) = \kappa(\theta)t \ . \tag{14}$$

For $z > \kappa(\theta)t$, the water content will be $\theta_n$. Note that (14) is yet another solution of the form (1).

It can be shown (Raats 1983) that the average water content *above* depth z is related to the water content *at* depth z by

$$\bar{\theta} = [1 - R^{-1}(\theta)]\theta \ . \tag{15}$$

It is remarkable that this relationship does not involve the depth z and time t explicitly.

Explicit expressions for $\theta(z,t)$ can be easily derived for various classes of soils, for example, for the class of mildly nonlinear soils with a linear retention curve and an exponential dependence of the hydraulic conductivity upon the water content. For this particular class of soils, it is possible also to evaluate the influence of capillarity upon the drainage process (Raats 1983).

Wooding (1975) analyzed flows induced by a variable rate of infiltration by the method of matched asymptotic expansions. In relatively uniform regions the distribution of the water content is represented asymptotically by an outer expansion, the first term of which takes into account the effect of gravity only and represents a kinematic wave. Higher order terms of the outer expansion include the small effect of capillarity. However, in regions where the gradient of water content is large, the effects of gravity and capillarity are of the same order of magnitude. Wooding shows that in such regions an appropriate inner expansion can be found.

QUASILINEAR ANALYSIS OF STEADY MULTIDIMENSIONAL FLOWS

The last third of Philip (1988) deals with quasilinear analysis of steady flows. Within this class of flows a wide variety of problems has been solved in the last two decades. This is not too surprising. In discussing Fourier's linear theory of heat conduction, Truesdell (1980) wrote

> By paring the physical model to the bone, the theorist may extract from it incredibly precise predictions about the most intimate detail of bodies of the most complicated shapes. In Fourier's theory the student must pay as the price for such detail and such precision a willingness to admit that differences of temperature propagate at infinite speed through material bodies and that the temperature within a body can be determined without taking into account the changes which the flow of heat into it may effect upon its size and shape.

Like Fourier's linear theory, the quasilinear analysis has its limitation: for a given geometry of the unsaturated region, a change of the boundary conditions by a constant factor changes the rate of flow but not the flow pattern. If $\alpha = K^{-1}dK/d\Psi$ is not constant, then the flow pattern is not subject to such invariance.

If a change of the boundary conditions causes a shift of the boundary between the saturated and unsaturated regions, then the flow pattern changes even if $\alpha$ in the unsaturated region is constant. Interest in flow in such adjoining saturated and unsaturated regions has been stimulated by the borehole permeameter (Philip 1985; Reynolds et al. 1985; Pullan 1988). For flow from cavities, the saturated-unsaturated system which results from allowing for the variation of hydrostatic pressure over the cavity surface is more physically realistic than the purely unsaturated system (Pullan 1988). The presence of the saturated region about a surface cavity removes the singularity in the flow velocity that occurs at the edges of the cavity when the head is ignored. Another effect of including the head is to increase the steady state flux from the cavity. This is illustrated in Fig. 3 for the case of a circular cylindrical cavity buried at an infinite depth, where the flux from the coupled system $F_c$ divided by the flux from the purely unsaturated system $F_u$ is plotted as a function of the reduced cavity radius $s$. For $s > 1$, it is difficult to evaluate $F_c$ accurately due to the increasing size of the saturated bulb (for $s = 1$ the bulb already extends to a depth of 7.9 times the radius of the cavity). However, these results are still sufficient to indicate the large errors that can arise when the head of water is neglected.

Another problem of great practical and current interest is infiltration from a shallow circular pond. The surface of the pond is an equipotential surface, while the remainder of the soil surface is a stream surface. Wooding (1968) derived a formal solution of the linearized flow equation subject to these mixed boundary conditions using a Hankel transform. This solution involves an unknown function $f(p)$ to be determined from the system of dual integral equations arising from the mixed boundary conditions. Wooding uses Tranter's method involving a series expansion of the unknown function $f(p)$. A natural choice for this expansion follows from the fact that at large distance from the disc source the flow should be indistinguishable from the flow due to a point source. The resulting system of linear algebraic equations has been approximately inverted by an analytical/numerical method for $s < 6$ (Wooding 1968) and by a wholly analytic method for $s < 0.15$ (Weir 1987).

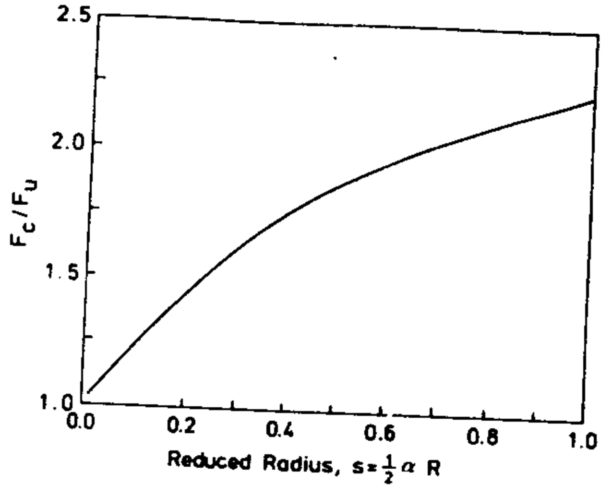


Fig. 3. Increase in flux from a coupled saturated-unsaturated system $F_c$ over that of a purely unsaturated system $F_u$ for the case of a circular cylindrical cavity buried at an infinite depth. (Pullan 1988).

Figure 4 shows the total flux from a disc source as a function of the dimensionless size of the disc. It is interesting that the total flux is larger than the sum of a purely capillary flux and a purely gravitational flux, although such a simple summation is a quite satisfactory approximation for large $s$. These results suggest that the results of Scotter et al. (1982) based on this approximation should be re-evaluated.

In an invited lecture at the 19th New Zealand Mathematics Colloquium, Philip (1984) described Wooding's analysis of 1968 as "an heroic piece of mathematics". Perhaps this assessment induced several New Zealand mathematicians to extend Wooding's analysis (Weir 1986, 1987; Pullan 1987, 1988; Pullan and Collins 1987).

Pullan (1987, 1988) used the boundary element method to calculate the transient quasilinearized infiltration with constant diffusivity from a shallow circular pond. Figure 5 shows the approach of the flux F to its steady value $F_\infty$ for several values of $s$. The dimensionless time in the horizontal axis involves only the properties $\alpha$ and D of the soil. Therefore, the family of curves compares the approach to steady state for ponds of different sizes on the same soil. A comparison of ponds of the same size on soils with different $\alpha$ but the same D is obtained by shifting the curves in Fig. 5 a distance $\log\alpha^2D^2$ along the time axis. If the soils being compared are similar in the sense of Miller and Miller (1956), then the diffusivity D should change by the same factor as $\alpha$ (Raats 1983). The required shift is then $\log\alpha^4$.

Finally, we note that the quasilinear analysis of steady flows can also be carried through for flow to sinks (Raats 1977; Warrick and Amoozegar-Fard 1977). Such flows are of interest in connection with flow to plant roots, and the operation of porous cup water samplers and suction lysimeters.

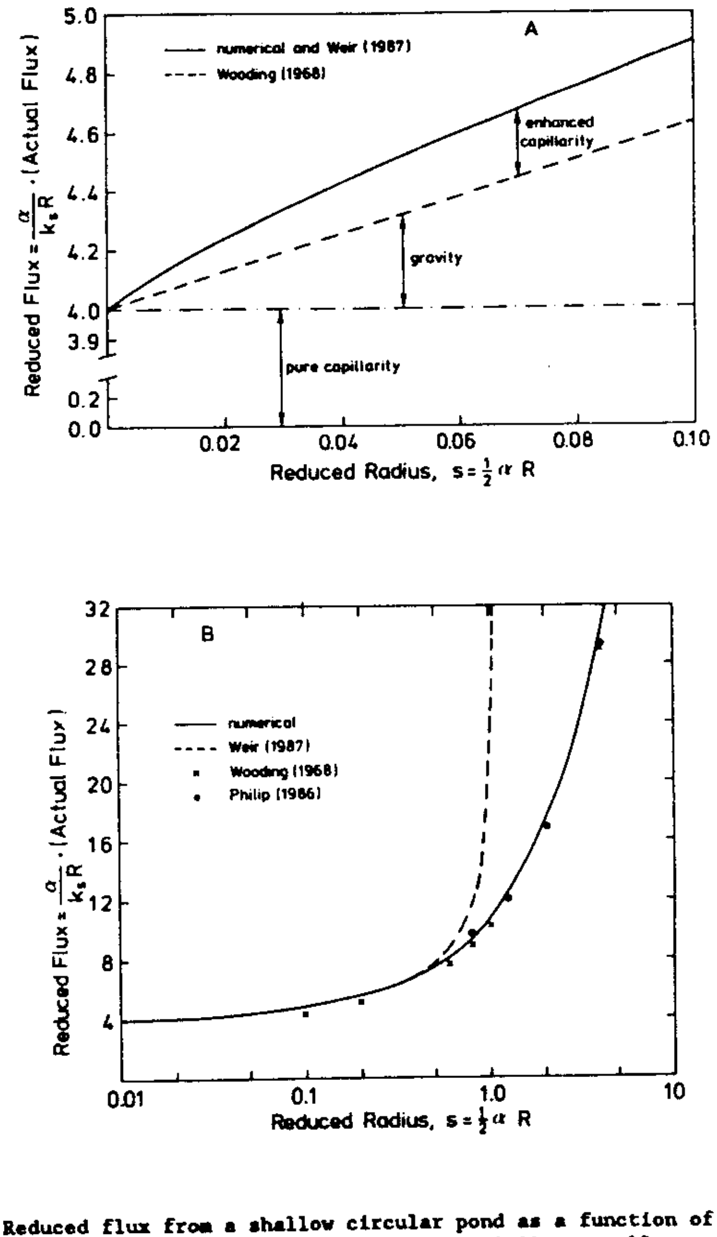


Fig. 4. Reduced flux from a shallow circular pond as a function of the reduced radius of the pond with (a) $0 < s < 0.1$, and (b) $0.01 < s < 10$.

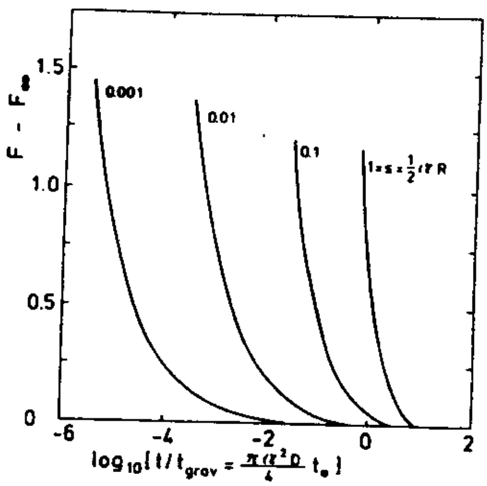


Fig. 5. Approach of the flux F from an axisymmetric flat pond to its steady value $F_\infty$ expressed as a function of $t/t_{grav}$, where $t/t_{grav}$ is the dimensionless time and $t_{grav} = (\tau s^2)^{-2}$. (Pullan 1987, 1988).

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