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"Unsaturated Flow of Water in Soil"

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UNSATURATED FLOW OF WATER IN SOIL

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.1 Introduction

Flow of water in soils can be described either using the Navier-Stokes' equations for an averaged system of pores or empirically as was done by Darcy (1856). The second approach which resulted in the formulation of the Darcy's law and which was confirmed by the theoretical approaches will be used preferentially here since it has the advantage in simple derivation and simple formulation for the solution of the engineering problems. However, at some instances, the results of the theoretical studies [redacted] should be recalled when the parameters of models and equations are to be explained physically. As the working method, the theory of the potential flow is applied in further chapters.

The basic flow equations are formulated first for the simplest case of flow in the saturated inert rigid soil. Further on the flow in the soil not fully saturated with water will be dealt with. This type of flow will be briefly denoted as the unsaturated flow while the first case will be called saturated flow and, to be more precise we should distinguish between the flow at only positive pressure and the flow at mainly negative pressure in soil water. If the flow of both, air and water in soil porous system is to be considered, we speak about the two phase flow. In the formulation of equations the solute concentration change is assumed to be negligibly affecting the described water flow.
Further on, the discussion is extended to the non-rigid, or no-inert soils, [redacted] i.e. to soils which swell when wetted and shrink when drained. ~~Up to this point, the~~

The isothermal conditions are assumed. ~~The flow at non-isothermal conditions and at temperatures below 0°C is discussed in the final subchapters.~~ All the derived equations are supposed to be the working tool not only of the analytical and approximate mathematical solutions of the components of the soil hydrological system but of all the deterministic models of the soil hydrology.

2 Saturated Flow

2.1 Explanation

We assume that at the positive pressure of soil water (or hydraulic head), water is flowing in all pores of the soil in spite of the fact that in the field situations the soil reaches "saturation", i.e. water content at the moisture potential equal zero when $\theta = m P$ where $m = 0.9$ to 0.95, the remaining part being occupied by the entrapped air. At this stage of discussion, the entrappement of the air is not considered and will be discussed later on.

In the Figure 1, a simple example of the saturated flow experiment is demonstrated. The soil is placed in a cylinder and its bottom part is inserted in a vessel containing water with constant elevation of its level, e.g. the vessel is provided by an overflow. Above the soil top surface, water level is kept at the constant elevation, too. The resulting discharge $Q = V/t$ is simply determined when the volume of water V either flowing in or flowing via overflow out of the system is measured in time t. The macroscopic flow rate v is

$$v = \frac{V}{A t} \qquad (1)$$

with A the crossectional area of the soil column perpendicular to the direction of flow. The term v is sometimes called the Darcian flow rate, the [redacted] flux concentration or flux density in order to emphasize the fact that the flux is not realized throughout all A, but only in the portion AP, i.e. in pores. The mean flow rate in pores v_p is then

$$v_p = v/P \qquad (2)$$

Darcy demonstrated in 1856 experimentally for columns of sand the linear relationship between the flow rate v and the hydraulic gradient I_h. In our experiment demonstrated in Fig. 1 is

$$v = K_S \frac{\Delta H}{L} = K_S \frac{\Delta H'}{L'} = K_S I_h \qquad (3)$$

where $\Delta H/L$ or $\Delta H'/L'$ is the hydraulic gradient I_h, ΔH is the difference of water levels on both ends of the soil column of the length L, $\Delta H'$ is the difference of water levels in the piezometers placed at the distance L' in the direction of flow, both being the hydraulic head drop along the soil. K_S is the hydraulic conductivity and as $\Delta H/L$ is dimensionless, it has the dimension of v $[LT^{-1}]$. When the potentials Φ are read by the piezometers at the elevations z_1 and z_2 we get

$$v = -K_S \frac{\Phi_2 - \Phi_1}{z_2 - z_1} \qquad (4)$$

with $\Phi = H + z$, the potential head, i.e. the potential related to the unit weight, ~~see the Chapter 4.2.2~~. Eq. (4) is transcribed in a more general way

$$v = -K_S \operatorname{grad} \Phi \qquad (5)$$

Eq. (5) states that the driving force of the flow, the gradient of the potential is proportional to the flow rate. As K_S = constant for the given soil, we write for $\varphi^* = K_S \Phi$

$$v = -\operatorname{grad} \varphi^* \qquad (6)$$

The negative sign in Eq. (4) and further on means that water flows against the positive direction of z in the Fig. 1. The value of K_S depends upon the nature of the soil. It can be demonstrated as the flow rate when the hydraulic gradient is equal to unity and its very low values are about 0.1 cm.day^{-1} ($\sim 10^{-8}$ m.s^{-1}) while its great values reach more than 10^2 cm.day^{-1}($\sim 10^{-5}$ m.s^{-1}), further extremes are not excluded, however.

Three examples of the proper formulation of the hydraulic gradient $\Delta H/L$ are in Fig. 2. In the examples (a) (c) we show that the soil column need not necessarily be oriented vertically and the value of L is measured in the direction of flow. In the example (b), water drops free from the column and the bottom of the column is identified with the water level after water has passed the soil as a very good approximation.

When we are dealing with the flow in layered soils and in soils with a distinct evolution of horizons differed by various saturated hydraulic conductivity, the introduction of the hydraulic resistance R is very useful. With $R = L/K_S$ we get for the flow in the series of horizons

$$v = \frac{\Delta H}{(R_1 + R_2 + R_3 + \cdots)} \qquad (7)$$

where ΔH is the total hydraulic head drop along the system consisting of layers 1, 2, 3,... with R_1, R_2, R_3, ... denoting their resistances, e.g. $R_1 = L_1/K_{S1}$ etc.

2.2 Saturated Hydraulic Conductivity

As the soil water potential Φ can be expressed in three modes, the dimension of the hydraulic conductivity is not necessarily in $[LT^{-1}]$. From the Eq. (5) we get for three various dimensions of Φ three various dimensions of K_S. Even if the expression of K_S in units of velocity (or flow rate) is the most convenient one, the units for all three modes of the potential should be mentioned as follows:

Units and Dimensions of

Φ	grad Φ	K_S
J.kg^{-1} $[L^2T^{-2}]$	J.kg^{-1}.m^{-1} $[LT^{-2}]$	s $[T]$
Pa $[ML^{-1}T^{-2}]$	Pa.m^{-1} $[ML^{-2}T^{-2}]$	m.s^{-1}/$\rho_W g$ $[M^{-1}L^3T]$
m $[L]$	dimensionless	m.s^{-1} $[LT^{-1}]$

In spite of the empirical, intuitive way of derivation of Eq. (5), the validity of the Darcy's law has been demonstrated theoretically from Navier-Stokes' equations transcribed for the porous media when scaling by the characteristic length and averaging over REV was used. In order to obtain the identity with Eq. (5), the terms expressing inertial effects were neglected and the density and viscosity of water were taken as constant (Bear, 1972, Whitaker, 1986). It has been also shown by Scheidegger (1957) that in vectorial transcription of Eq. (5), K_S is either a scalar, what is satisfactory for isotropic soils, or K_S is the tensor of the rank 2 and it should be considered so for anisotropic soils with the conductivity dependent upon the direction of flow. Since the tensor K_S is assumed to be symmetric, its principal axis are identical with the axis of the ellipsoid of conductivity and it is defined by six values. If the gradient of the potential is not parallel to the principal axis, the realized flow has the direction different from that one of the gradient.

From the theoretical treatment we obtain the physical interpretation of the hydraulic conductivity. The development discussed here will be a modified and simplified procedure of Kozeny (q.v. Scheidegger, 1957), where the model consisting of the bundle of parallel capillary tubes is used. In the soil and in the model, the equality of the porosity P , of the specific surface A_m $[L^{-1}]$, and of the macroscopic flow rate v $[LT^{-1}]$ is assumed. The mean flow rate v in one capillary of the radius r is according to the Hagen-Poiseuille's equation

$$v_p = \frac{\rho_W g r^2}{8\mu} I_h \qquad (\ 8)$$

with g the acceleration of gravity $[LT^{-2}]$, ρ_W the density of water $[L^{-3}M]$, μ the dynamic viscosity $[L^{-1}MT^{-1}]$, I_h the hydraulic gradient, dimensionless. The number of capillaries in the model is n, their length x = 1. The porosity of the model using a unit volume V_u is

$$P = n\pi r^2 x / V_u \qquad (\ .9)$$

and its specific surface

$$A_m = 2 n \pi r x / V_u \qquad (\ .10)$$

Since the values of r and of n are not known, we eliminate them from (5.9) and (5.10) and we get r = 2P/A_m. With Eq. (5.2) and (5.8) we get

$$v = \frac{1}{2} \frac{\rho_W g}{\mu} \frac{P^3}{A_m^2} I_h \qquad (\ 11)$$

Since the pores in the soil are mutually interconnected and their shape is irregular, ½ in Eq. (11) is replaced by a shape factor c and for

$$K_p = \frac{c P^3}{A_m^2} \qquad (\ 12)$$

we get

$$v = K_p \frac{\rho_W g}{\mu} I_h \qquad (\ 13)$$

which is identical with Eq. (3). The term K_p describes how the soil porous body enables the flow of any fluid and it is called therefore permeability $[L^2]$. The unusual dimension of K_p can be demonstrated as the crossectional area of the equivalent pore. Among other its units, 1 Darcy = 1μm^2 is sometimes used. For conversion of K_p to K_S we get for water at 20°C an approximate relationship 1μm$^2 \approx 1 \times 10^{-3}$ cm.s^{-1}. Since the flow channels in the soil are curved compared to the model of capillaries, the tortuosity factor τ has been introduced by Kozeny in (12) thus forming the final relation known as the Kozeny's equation:

$$K_p = \frac{c\,P^3}{\tau A_m^2} \qquad (14)$$

The tortuosity τ is the ratio between the real flowpath length L_e and straight distance L between the two points of the soil in consideration, $\tau = L_e/L$. Therefore we have $\tau > 1$, and in the monodispersed sand $\tau \approx 2$ as the flowpath forms approximately a sinusoidal curve (Corey, 1977). Equations identical or of similar type to (14) have been derived by many authors. If instead of capillary tubes the model of parallel oriented plates is used and when the slits are oriented in the direction of flow we obtain if the distance between the plates is 2d for the laminar flow

$$v_p = \frac{d^2 \rho_w g}{3\mu} I_\ell \qquad (15)$$

When B is the width of plates, we get $P = 2ndBx/V_u$ and $A_m = 2nx(2d + B)/V_u$. Taking again x = 1, and B = 1, we obtain $d = P\,(A_m - 2P)$ and finally

$$K_p = \frac{c\,P^3}{\tau (A_m - 2\Pi)^2} \qquad (16)$$

A certain type of compromise between Eq. (14) and Eq. (16) is the Kozeny - Carman equation (cf. Scheidegger, 1957)

$$K_p = \frac{P^3}{5 A_m^2 (1-P)^2} \qquad (17)$$

derived in a more complicated way than demonstrated here.

The relationship between K_S and K_p follows from the comparison of Eq.(.3) and (.13):

$$K_S = K_p \frac{\rho_w g}{\mu} \qquad (18)$$

5.2.3 Darcian and Non-Darcian Flow

It has been already mentioned that the linear form of the Darcy's law is valid only at the low values of the flow rate when the inertial terms of the Navier-Stokes' equations are negligible. The upper limit of the validity of the Darcy's law, Eq. (.3) to (6) is in the engineering practice indicated by the critical value of the Reynolds' number for porous media

$$Re = \frac{v\,d\,\rho}{\mu} \qquad (19)$$

where d denotes the length, in sands it is the effective diameter of the particle, or, with some corrections, the diameter of the effective pore. Sometimes the value of d is related to the permeability, e.g. $d = K_p^{1/2}$. However, in all other soils than sands, d is not definable at all and Eq. (19) is not applicable. The difficulty in defining d is reflected by the controversial critical values of Re in the literature (Slepička, 1961). Most frequently, the critical Re number is about 10^0. In this postlinear region, the flow is often described by the equation of Forchheimer (cf. Bear, 1972)

$$\frac{d\Phi}{dx} = a v + b v^2 \qquad (20)$$

where a is the material constant, analogical to K_S, b is functionally dependent upon the flow rate. This type of deviation from the linearity is due first to the action of inertia and only at very high values of the flow rate, the turbulence could start. More about the theoretical background see e.g. in Cvetkovič (1986)

The deviations from the Darcy's law have been found in laboratory experiments at very low rates, too and we can define therefore the prelinear region of flow, where v increases more than proportionally with I_h, see the

Fig. 3. This deviation from the Darcy's law has been observed mainly within pure clays with a very high specific surface of 10^2 m^2g^{-1} and it has been explained by the action of ~~two~~ these factors: (a) Shift of clay particles, mainly plates and consolidation of the clay paste owing to the imposed hydraulic gradient and the flow of water. (b) Different viscosity of water ~~in~~ close to the surfaces clay. Due to the great value of the specific surface, the volume of water influenced by the solid surface fields starts to be comparable to the volume of bulk water. The first molecular layers of thickness of two to four molecules of water have probably the viscosity differing from the bulk water, see the Eyring's molecular model where the viscosity depends upon the Gibbs free enthalpy (cf. Bird et all., 1956).

The flow in the prelinear region is described by the empirical equations on the $v(I_h)$ relation, e.g. by Swartzendruber (1962)

$$v = M\left\{I_h - J\left[1 - \exp\left(-CI_h\right)\right]\right\} \qquad (\ 21)$$

who supposed the existence of the threshold gradient J. Without the threshold gradient, the relation is (Kutílek, 1964)

$$v = M\left\{\frac{1}{B}\ln\left[A + \exp\left(BI_h\right)\right] - D\right\} \qquad (\ .22)$$

with $B = \frac{1}{D}\ln M/M'$ $\qquad A = M/M' - 1$

In both equations, M is the conductivity when the linear region $v(I_h)$ is reached, C,D are fitting parameters, M' is the initial conductivity at $v \to 0$ in Eq. (22). Big organic cations as well as the soil organic matter either reduce substantially the prelinear region, or, they do not allow its existence at all (Kutílek, Salingerová, 1966). A detailed discussion on the flow in the prelinear region has been given by Kutílek (1969).

2.5 Practical Conclusions

Saturated hydraulic conductivity depends upon the porous system of the soil and as this system depends highly upon the soil structure and compaction of the soil, the value of K_s is sensitive to these effects, too. Aggregation increases the value of K_s by orders, and, vice versa, desaggragation decreases it. For example, saturated hydraulic conductivity of the sealed surface after the heavy rain decreases by 3 to 4 orders compared to the original value of K_s in loessical soil owing to the action of two effects, mainly the desaggregation and the blockade of pores by the released clay particles (McIntyre, 1958). The compaction of the soil in A-horizon and in the bottom of the ploughed subhorizon results in the decrease in K_s in much greater extent than predicted from the simple Kozeny's equation because the compaction reduces mainly the content of big soil pores related to the values of the pressure head $H = 0$ up to $H = -100$ cm. This domain of pores influences mainly the value of K_s as it will be demonstrated in the next Chapter .3 in detail. When the textural classes of soils are considered, K_s values occur in wide ranges in each of the classes and any search for correlation between K_s and the textural composition of the soil is facing a failure. The lowest values of K_s in each of the main classes can be roughly indicated, only: In sandy soils, the minimum K_s is about 100 $cm.day^{-1}$, in silty loams it is 10 $cm.day^{-1}$ and in clays 0.1 $cm.day^{-1}$. In peats, K_s decreases with the increasing degree of the decomposition of the original organic substances and at the degree of decomposition of 40% to 50%, the K_s of the peat reaches the values of K_s in unconsolidated clays. In peats, the extreme drainage and drying causes the increase in bulk density and this compaction is accompanied by the decrease of K_s. The increased hydrophobism is referred in this connection, too and a greater extent of the entrappement of the air after wetting is assumed thus contributing to the decrease of K_S.

In loams and clays, the nature of the prevalent exchangeable cation plays an important role, Fig. 3a. In Vertisols, the increase of the percentage of the exchangeable sodium (ESP) is accompanied by the decrease in K_S and when ESP reaches 15% to 20%, the value of K_S sinks by two to three orders, provided that the content of soluble salts is kept at low value, i.e. with the electrical conductivity of the soil paste EC about 1 mS.cm^{-1} and less. If the concentration of salts is substantially increased up to EC about 8 mS.cm^{-1} and more, the K_S value is not significantly reduced even at high values of ESP (Kutílek, 1983). Great importance in the dynamics of the changes of K_S plays the value of the sodium adsorption ratio (SAR) in addition to EC, as Russo and Bresler (1977) have shown in their model study. The changes are closely related to the alteration in the degree of flocculation or peptization of the soil colloidical particles and to the dynamics of the value of the ξ-potential on the moving boundary of the double layer. Applying this theory, the decrease in K_S owing to the action of the rain water of very low salt content is simply predictable in soils of a high SAR value. If these results obtained on fine textured soils of arid zones are applied to soils pedologically different , precaution is recommended. For example, in Oxisols, the percolation by the solution of a high SAR value does not induce after the reduction of the solute concentration the decrease in K_S since the abundant free Fe-oxides prevent the soil particles from peptization and desaggregation. In clays, the value of K_S depends upon the composition of the clay fraction, too, and it decreases in the order kaolinite > illite > montmorillonite. The soil organic matter induces great increase in K_S, mainly by its cementing action upon the aggregate stability. Generally, there are many factors influencing the value of K_S that are usually not considered in the simplified models.

The soils may be classified according to the values of K_S as follows:

Very low permeable soils with K_S below 10^{-7} m.s^{-1}
low permeable, $K_S = 10^{-7}$ to 10^{-6} m.s^{-1}
medium permeable, $K_S = 10^{-6}$ to 10^{-5} m.s^{-1}
high permeable, $K_S = 10^{-5}$ to 10^{-4} m.s^{-1}
excessively permeable, K_S above 10^{-4} m.s^{-1}

Anyway, all the classification schemes are very schematical and for the classification of soils in a certain region, the analysis of the frequency distribution of K_S seems to be the most appropriate bases.

When the values of K_S in the soil profile are considered, the soils are groupped in 6 classes:

1. K_S of the A-horizon is substantially higher than K_S of the remaining part of the soil profile and there is no horizon of relatively extremely low K_S.
2. K_S gradually decreases with the depth without distinct minima or peaks.
3. K_S reaches a distinct minimum value in the illuvial horizon or in the compacted sub-horizon below the ploughed subhorizon.
4. Soil of a high permeability - with the development belonging to one of the first 3 classes - covers the underlying soil of a very low permeability (layering of the soil substrate).
5. Soil of a very low permeability - with the development belonging to one of the first 3 classes - covers the underlying soil of very high permeability (layering of the soil substrate).
6. K_S changes erratically in the profile due to the extreme heterogeneity in the soil substrate.

Since various units are used for the saturated conductivity even if the dimension is LT^{-1} and as sometimes confusion exists with the use of the term permeability K_p, the conversion graph in Fig. 4 can be convenient.

The influence of the temperature T upon the value of K_S is derived from the Eq. (.18). As ρ_W is negligibly influenced by the temperature, the change of $K_S(T)$ depends totally upon the viscosity change, $\mu(T)$. Some slight deviations ~~which~~ could occur. ~~if the non-Darcian prelinear effects were traceable are discussed in Chapter 5.1.2.~~

3 Unsaturated Flow in ~~Inert~~ Rigid Soils

3.1 Explanation

By the term ~~inert~~ rigid soil we understand soils which are not changing their volume with the change of the water content. The unsaturated flow in soils is governed by the same laws as the saturated flow. When the laws are further applied we have to consider the fact that a portion of pores is filled by the air and that the flow could result in the resaturation or in the drainage of the porous system. In some demonstrations of the phenomena, the capillarity is quoted here and the term capillary rise is quite frequently used in the literature. The general mathematical formulations of the physical phenomena should be, however, independent from the simplifying ideas on capillaries and when capillarity is mentioned, it is just for the sake of the approximative modeling of some effects occuring in the real soils.

A simple example of the unsaturated flow is demonstrated in Fig. 8 which is an analogy of the examples of experiments with the saturated flow in Fig. 2 . The cylinder containing the soil is provided by openings on the walls. The semipermeable membranes separate the soil from free water on both sides of the cylinder. The term semipermeable means ~~again~~ that the membrane is permeable to water but impermeable to air. The pools with water are connected by flexible tubes. First, the pools

are lifted to the position when the water level is at the same elevation in both pools as is the highest point of the soil and the openings on the side of the cylinder are closed. The full saturation of the soil is now assumed and at this instant, there is no flow in the system. Then the pool on the left side of the cylinder is lowered to the position H_1 and the pool on right side to the position H_2. In the mean time, the openings will be opened in order to enable the entry of the air since the soil starts to drain in a similar way like the soil placed in the tension plate apparatus. On the left side of the cylinder, the soil ~~is~~ would be drained to a lesser extent than on the right side, however, the decrease of the soil water content from the left to the right side is not linear. Simultaneously, the water flows from the left pool to the right pool similarly as in the syphon if its arms were submerged in our pools. Only the rate of flow is ~~in our example with~~ due to the soil significantly ~~depressed~~ reduced compared to the syphon. ~~due to the presence of the soil porous body.~~ When the water level is kept at the constant elevation, the steady flow will be reached after some time has elapsed from the installation of water levels at the given positions. Then, the soil water content will reach a certain constant value at each point of the soil, too. The flow rate v will depend again upon the hydraulic gradient and will be governed by the similar equation to Eq. (3):

$$v = K\frac{\Delta H}{L} \qquad (.23)$$

where K is the unsaturated conductivity $[LT^{1}]$ and it characterizes how the soil enables the flow. Since the soil is not fully saturated, the flow is realized only in pores filled by water and the value of K will be lower than was K_S for the same soil, in our figure $K(\bar{\theta})$ or $K(\bar{H})$, ~~too~~. Similarly, as when dealing with the saturated flow, we operate with the potential related to the weight, i.e. in the units of the pressure head. When we are solving the majority of practical problems, all components except of the gravitational and moisture potential will be neglected and the total soil water potential $\Phi = H + z$ with z being the vertical coordinate again and Eq. (23) will be rewritten into the form

$$v = -K(H)\frac{\Delta \Phi}{\Delta z} \qquad (24)$$

or

$$v = -K(H)\, \mathrm{grad}\, \Phi \qquad (25)$$

Since Eq. (25) is equivalent to ~~has been derived from~~ the Darcy's equation and Buckingham (1907) was the first who described the unsaturated flow as dependent upon the potential gradient, Eq. (24) and (25) are called Darcy-Buckingham's equations. The unsaturated conductivity K is physically dependent upon the soil water content since ~~here~~ in water saturating only ~~[illegible]~~ partly the pores ~~the~~ water flow is realized. Because ~~[illegible]~~ θ(H) relationship exists, we can write K(H), too, but this relationship is not direct, it has been derived and keeping in mind that θ(H) is influenced by hysteresis, the K(H) will be dependent upon the hysteresis, too.

The examples of the K(H) and K(θ) relationships are schematically demonstrated in Fig. 6 . It can be seen that at the saturation the soil more per~~vious~~meable than the other one does not necessarily keep its higher unsaturated permeability in the whole region of the unsaturation. When the hysteretic behavior of K(H) is studied, we can see that for the given H the value of K is higher for the drainage than for the wetting process.

The sole Darcy-Buckingham's equation is adequate for the description of the unsaturated flow if the soil water content is not changing in time at a certain point, only. However, this is a seldom case. When θ and v alter in time, we have to combine Eq. (25) with the equation of continuity which describes the rate of change of θ in time t related to the change of v in the considered small element of the soil. The final product is a non-linear differential equation (Chapter 3.3) and its solution even for the simple conditions is a very difficult task. Generally, Eq. (25) is in itself not satisfactory for the solution of such hydrologically important processes as the evaporation, rain infiltration, drainage and subsurface flow etc. Only very exceptional situations could be ~~[illegible]~~ solved by the single use of Eq. (25).

3.2 Unsaturated Hydraulic Conductivity

For the explanation of the physical meaning of K let us first apply the Kozeny's Eq. (14). Since only a portion of pores is filled with water we replace the porosity P by the soil water content θ. We have to increase the value of the tortuosity and according to Corey's (1954, also 1977) expression valid for sands we get the estimate of $\tau(\theta)$ as follows:

$$\frac{\tau_S}{\tau(\theta)} \approx \left(\frac{\theta - \theta_M}{\Pi - \theta_M}\right)^2 \qquad (26)$$

where τ_S is the tortuosity in the saturated soil, $\tau(\theta)$ is the tortuosity at the soil water content θ, θ_M is again the minimum or residual soil water content. ~~as introduced in the Chapter 4.3.2.~~ When the alteration of the tortuosity due to the change of the soil water content is not inserted in Eq. (14), the exponent should change and Leibenzon (1947) derived in this way

$$\frac{K}{K_S} = \left(\frac{\vartheta - \vartheta_M}{P - \vartheta_M}\right)^n$$

27 (~~5.33~~)

where n should be in ranges from 3.3 up to 4, while Averianov (1949) has proposed n = 3.5 as an average value, this being still a good rough estimate for the first approach to the problems. Childs and Collis-George (1950) obtained similarly an equation which can be transcribed to the form

$$K = \alpha \frac{\vartheta^3}{A_m^2}$$

28 (~~5.34~~)

This equation is comparable to the Deryaguin's et al.(1956) equation derived for the flow of water films in the soil:

$$K = \frac{\alpha A_m d^3}{2\mu}$$

29 (~~5.35~~)

Identity of both equations is reached when the average thickness of the water film d is taken as functionally dependent upon ϑ for a given specific surface A_m. In both equations, α is the empirical coefficient.

Since $\vartheta(H)$, the dependence of K upon H is deducible, too. Gardner (1958) has modified the Wind's (1955) empirical proposal

$$K = a H^{-m}$$

30 (~~5.36~~)

to the relationship applicable to H = 0:

$$K = \frac{a}{|H|^m + b}$$

31 (~~5.37~~)

where a, b, and m are empirical coefficients. In Eq. (31 ~~5.37~~) for H = 0 is $a/b = K_S$. Or, in a modified form, Eq. (31 ~~5.37~~) is transcribed in

$$K = K_S \frac{a}{|H|^m + a}$$

32 (~~5.38~~)

In the analytical solutions, the Gardner's (1958) exponential relationships are frequently used:

$$K = K_S \exp(cH)$$

33 (~~5.39~~)

$$K = K_S \exp[c(H - H_A)]$$

34 (~~5.40~~)

with the empirical coefficient c which can reach a physical meaning when the Green and Ampt's approach is ~~are~~ used, ~~see the Chapter 6~~. When once K_S is known, the constant c is closer related to the texture of the soil than the saturated hydraulic conductivity and the region of the most frequent occurence of c as related to the percentage of the silt and clay particles in soils of the mild zone of middle Europe is in Fig. 7 . Close to H = 0 or to $H = H_A$, respectively, the relationship K(H) is better described by $c = (H - H_A)^n$ and

$$K = K_S \exp(-|H - H_A|^n)$$

35 (~~5.41~~)

or, if $H_A = 0$ [redacted]

$$K = K_S \exp(-|H|^m)$$

36 (~~5.42~~)

The Eq. (32 ~~5.39~~), and (33 ~~5.40~~) are valid not far from the saturation, i.e. in the wet range. In the dry range, these equations are not well applicable and either Eq. (30 ~~5.36~~) or Eq. (36 ~~5.42~~) should be preferred.

From the studies on capillarity in sands, Brooks and Corey (1964) have obtained a freqently used relationship

$$\frac{K}{K_S} = \left(\frac{H_A}{H}\right)^m$$

37 (~~5.43~~)

where m depends upon the pore distribution. When Eq. (37) was used ~~in the~~ empirically ~~way~~ for fine textured soils, m was mainly in ranges demonstrated in Fig.7 . .

In some of the earlier models of soil water regime simulation, K(θ) was defined in specified ranges for $H_A < H < H_{lim}$ by Eq. (34), for $H < H_{lim}$ by Eq. (30) or (31).

The deri~~f~~vation of K(θ) from the retention curve which reflects the distribution function of pores f(r), ~~see Eq. (4.40) up to Eq. (4.43)~~ is the most atractive and most chalenging procedure and it has offered the best background for the modeling of the unsaturated flow. One of its part~~c~~ial results was the already mentioned Eq. (~~5.43~~ 37). In the general way of the derivation of the model we start with the mean flow rate v_p in pores of the radius r according to the Hagen-Poiseuille's equation

$$v_p(r) = a r^2 I_h \qquad \overset{38}{(5.\ \ )}$$

where a = $\rho_W g / 8\mu$, see Eq. (8). The crossectional area of the soil taken perpendiculary to the direction of flow is the sum of three areas, A = $A_S + A_W + A_A$, where the indexes denote S the solid phase, W water, and A air. We asume also

$$A_W(r) = (A - A_S)\, f(r) \qquad \overset{39}{(\cancel{5.45})}$$

Considering it together with Eq. (. 2) we get for the Darcian (macroscopioc) flow rate v

$$v = \frac{A - A_S}{A} \int_0^r v_p(r) \frac{df(r)}{dr}\, dr \qquad \overset{40}{(\cancel{5.46})}$$

and with Eq. (5.44) since v = K for I_h = 1

$$K(f(r)) = a \frac{A - A_S}{A} \int_0^{f(r)} r^2(f(r))\, df(r) \qquad \overset{41}{(\cancel{5.47})}$$

For $r = c_1/H$ (capillary height related to r) and $c = a c_1^2$ we obtain

$$K(f(r)) = c \frac{A - A_S}{A} \int_0^{f(r)} \frac{df(r)}{H^2(f(r))} \qquad \overset{42}{(\cancel{5.48})}$$

Approximating that $f(r) \approx \theta_E(H)$ (43)

and for $\theta_S = A - A_S$ when with θ_M, the residual (or minimum) θ the effective θ_E is used :

$\theta_E = (\theta - \theta_M)/(\theta_S - \theta_M)$ Eq. ($\overset{42}{\cancel{5.48}}$) is

$$K(H) = c\, \theta_S\, \theta_E^{\,b}(H) \int_{-\infty}^{H} \frac{d\theta_E(H)}{H^2} \qquad \overset{(44)}{(\cancel{5.49})}$$

This equation is valid for the model and it should be matched to the real soil. The simplest way to do it, even if not fully satisfactory as it will be shown later, is by comparing it to the measured saturated hydraulic conductivity. This is why the relative conductivity K_r is introduced as

$$K_r = \frac{K(H)}{K_S} \qquad \overset{(45)}{(\cancel{5.50})}$$

Burdine (1953) obtained after further developments which are not going to be rewritten here in detail, b = 2 and his equation has the final form

$$K_r = \theta_E^2 \left[\int_0^{\theta_E} \frac{d\theta_E}{H^2(\theta_E)} \Big/ \int_0^1 \frac{d\theta_E}{H^2(\theta_E)} \right] \qquad \overset{(46)}{(\cancel{5.51})}$$

Mualem's (1976) derivation was slightly modified and with b = 0.5 his equation is read as follows

$$K_r = \theta_E^{0.5} \left[\int_0^{\theta_E} \frac{d\theta_E}{H(\theta_E)} \Big/ \int_0^1 \frac{d\theta_E}{H(\theta_E)} \right]^2 \qquad \overset{(47)}{(\cancel{5.52})}$$

If van Genuchten's (1980) procedure of combination of the equation of the retention curve ~~(4.38) with the model~~

$$\theta_E = \frac{1}{\left(1 + (\alpha |H|)^n\right)^m} \qquad (48)$$

with $K_r(\theta_E)$ is generalized, we obtain

$$K_r(\theta_E) = \theta_E^{\,b} \left[1 - \left(1 - \theta_E^{1/m} \right)^m \right]^a \qquad \overset{49}{(\cancel{5.53})}$$

with m = 1 - c/n, n > 1

a = 1, b = 2, c = 2 for the model of Burdine

a = 2, b = 0.5, c = 1 for the model of Mualem.

~~If the modification of the van Genuchten's Eq. (4.39) is applied, we obtain a similar equation to (5.53), see Šír~~

If K(H) is ~~[illegible]~~ expressed in a similar way, the equation is for the Mualem's model

$$\frac{K(H)}{K_S} = \frac{\left\{1-(\alpha|H|)^{n-1}\left[1+(\alpha|H|)^{n}\right]^{-m}\right\}^{2}}{\left[1+(\alpha|H|)^{n}\right]^{m/2}} \qquad 50. \quad (\cancel{5.55})$$

In a similar way, for the retention curve of Brooks and Corey ~~(4.36) is~~ $\vartheta_E = (H_A/H)^{\lambda}$ is (51)

$$\frac{K(\vartheta_E)}{K_S} = \vartheta_E^{\,b+a/\lambda} \qquad 51a \quad (\cancel{5.56})$$

and

$$\frac{K(H)}{K_S} = \left(\frac{H_A}{H}\right)^{a+b/\lambda} \qquad 52b \quad (\cancel{5.57})$$

with a = 2, b = 2 in Childs and Collis-Georges' (1950) method,

a = 2, b = 3 in Burdine's method

a = 2, b = 2.5 in Mualem's method.

It should be remarked that in the derivation of the above equations, approximations were done at several steps. First, the soil porous system was modeled by the bundle of cylindrical capillary tubes, then the approximation in equalling the pore size distribution function to the retention curve followed and further on there was the derivation of the value of b. However, in spite of all the approximations adopted in the derivation of the $K(\vartheta_E)$ and K(H) equations, the most problematic is the proper interpretation of the soil water retention curve close to ϑ_S. ~~One example taken from the detailed study of Vogel et al. (1985)~~ 1988 ~~will illustrate the magnitude of errors~~

The formal sensitivity analysis of Wösten and van Genuchten (1988) shows that when K_S is properly measured, the differences in K_r increase with the decrease in H, i.e. with the drying of the soil, when one of the parameters is altered, see Fig. 8. The behavior of the exponent n is the only one exception in the very wet region. In the graphs, ϑ_E is replaced by the real ϑ and the soil water retention curve is

$$\vartheta = \vartheta_M + \frac{\vartheta_S - \vartheta_M}{1+(\alpha\,|H|^{n})^{m}} \qquad (53)$$

More applicable are the results of the genetical sensitivity analysis (Vogel et al. 1985, 1988). First, they show the role of the error $\delta H(\vartheta_E)$ in the experimental determination of $H(\vartheta_E)$. If $\delta H(\vartheta_E)$ is constant in the range $0 \leq \vartheta_E \leq 1$, the absolute error η of $K(\vartheta_E)$ rises steeply with the increase of ϑ_E, see Fig. 9. Secondly, our improper knowledge of the internal porous structure leads to substantial errors, see Fig. 10. The domain of ϑ close to ϑ_S is the dominant source of errors. Here, the monomodal pore size distribution is assumed and the van Genuchten's Eq. (48) is modified with ϑ_e which replaces ϑ_E:

$$\vartheta_e = \frac{\vartheta - \vartheta_b}{\vartheta_a - \vartheta_b} \qquad (54)$$

where ϑ_a, ϑ_b are fitting parameters, only, allowing $\vartheta_a \geq \vartheta_S$ and $\vartheta_b \leq 0$. Eq. (54) is treated as an empirical equation without physical meaning of parameters, The retention cúrve is transformed to a more flexible type, see Fig. 11. The Eq. (49) is then modified to

$$K_r(\vartheta_e) = \vartheta_E^{\,b}\left[\frac{d-(1-\vartheta_e^{1/m})^{m}}{d-(1-\vartheta_e^{1/m})^{m}\big|_{\vartheta_E=1}}\right]^{a} \qquad (55)$$

$$d = \left(1-\vartheta_e^{1/m}\big|_{\vartheta_e=0}\right)^{m}$$

The physically real part is in $0 \leq \vartheta \leq \vartheta_S$.

It follows from the sensitivity analysis that for a reliable determination of $K_r(\vartheta_e)$ we have to apply one of the feasible procedures:

1. $K_r(\theta_e)$ is computed from the $H(\theta_e)$ which is a directly measured relationship. As matching factors, we have to measure directly K_S, and K at θ not far from θ_S.
2. The measured $H(\theta_e)$ and K_S are taken as starting values in the optimisation procedure, where the measured flux process (e.g. infiltration, or redistribution) with simple boundary conditions is combined with the numerical simulation of the same flux process.

When the soil porous system is characterized by a bi-modal pore size distribution curve (Fig. 12), the relation K(H) shows distinctly two regions: In $0 > H > H_1$ is K(H) expressible by EQ. (33) or (34) and the parameter c can be detected only by direct measurement of K(H) in the field. In the region $H_1 > H$, the Mualem's model (48) is applicable, i.e. K(H) is computed from the experimental $H(\theta_e)$ or from $H(\theta_E)$, see Dickkrieger et al,(1989).

Up to this point we have assumed that the validity of the Darcy's law is fully applicable to the unsaturated flow. However, when the hesitation on the validity of Darcy's law exists for the saturated flow in clays, the non-Darcian prelinear flow domain should be even more pronounced in the unsaturated flow in clays. The earlier experiments indicate this possibility (Swartzendruber, 1962) with the opportunity to the theoretical explanation (Bolt and Groenevelt, 1969).

The influence of the temperature upon the $K(\theta)$ is usually expressed by

$$K(\theta) = K_r(\theta) K_p \rho_W g / \mu_W \qquad \overset{56}{(\cancel{5.58})}$$

However, there is the experimental evidence (Novák, 1975, Constanz, 1982) that in some instances Eq. (.56) could be taken as approximative, only. It is to be reminded here, that in the saturated clay paste, the influence

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of the temperature upon the change of the non-Darcian behavior of water has been found (Kutílek, 1967) thus offering one of the possible explanations for K(T) different from μ(T). While the deviations of the measured $K(\theta, T)$ data from that ones obtained from Eq. (56) is of less importance for θ_E near to 1, in the range of $\theta_E < 0.5$ the deviations should be considered even in the solution of practical problems if the experimental evidence was confirmed.

The measuring technics of the determination of $K(\theta)$ are mainly related to the solution of the specified unsteady flow processes. ~~The method will be therefore concisely discussed at the end of Chapter 6.~~

3.3 Richards' Equation

The equation ($\overset{15}{\cancel{5.31}}$) is fully applicable to the solution of the steady unsaturated flow when dv/dx = 0, dv/dt = 0 and $d\theta/dt = 0$. In practical situations we meet more frequently $d\theta/dt \neq 0$ with the unsteady type of flow. In this type of problems, the flow rate and the rate of change of θ in time are to be solved and two equations are therefore needed. While the flow rate is described by the Darcy-Buckingham's equation, the rate of filling or emptying of the soil pores is described by the equation of continuity. For its derivation, a prism element is considered, Fig. $\overset{13}{\cancel{5.11}}$. The difference of the volume of water flown into the element and flown out of the element is equal to the difference of water content in the element in time Δt. The length of edges of the element are Δx, Δy, Δz. The rate of the inflow (macroscopic) is v, the rate of the outflow is v_1. The change in v in the element is continuous.

The conditions in the direction of the x axis are:

The inflow rate is v_x.

The outflow rate is $v_{1x} = v_x + \Delta x \, \partial v_x / \partial x$

The inflow volume is $v_x \Delta y \Delta z \Delta t$

The outflow volume is $(v_x + \Delta x \, \partial v_x / \partial x) \Delta y \Delta z \Delta t$

The difference of the inflow and the outflow volume is

$$v_x \Delta y \Delta z \Delta t - \Delta t \, (v_x + \Delta x \, \partial v_x / \partial x) \Delta y \Delta z = \\ = -(\partial v_x / \partial x)(\Delta x \Delta y \Delta z \Delta t) \qquad (57)$$

Similarly in the direction of y axis, the difference of the inflow and the outflow volume is (58a)

$$-(\partial v_y / \partial y)(\Delta x \Delta y \Delta z \Delta t)$$

and in the direction of z axis

$$-(\partial v_z / \partial z)(\Delta x \Delta y \Delta z \Delta t) \qquad (58b)$$

The sum of the differences is equal to the change of the content of water in the element. If $\theta(t)$ has a continuous derivation for t > 0 then for Δt is

$$-\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) \Delta x \Delta y \Delta z \Delta t = \frac{\Delta \theta}{\Delta t} \Delta x \Delta y \Delta z \Delta t \qquad (.59)$$

For limits $\Delta x \to 0$, $\Delta y \to 0$, $\Delta z \to 0$, $\Delta t \to 0$ we get the equation of continuity

$$-\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) = \frac{\partial \theta}{\partial t} \qquad (.60)$$

If we insert for v_x, v_y, v_z from Eq. (~~5.31~~ 25) we obtain

$$\frac{\partial}{\partial x}\left[K(H)\frac{\partial \Phi}{\partial x}\right] + \frac{\partial}{\partial y}\left[K(H)\frac{\partial \Phi}{\partial y}\right] + \frac{\partial}{\partial z}\left[K(H)\frac{\partial \Phi}{\partial z}\right] = \\ = \frac{\partial \theta}{\partial t} \qquad (.61)$$

provided that the soil is isotropic. In one-dimensional form for $\Phi = H + z$ is

$$\frac{\partial}{\partial z}\left[K(H)\frac{\partial H}{\partial z}\right] + \frac{\partial K}{\partial z} = \frac{\partial \theta}{\partial t} \qquad (.62)$$

Eq. (.61) and (' 62) are called Richards' (1931) equations according to the author who derived them for the porous media.

If one direction of the process is supposed, i.e. either drainage or the wetting, θ will be uniquely dependent upon H only and

$$\frac{\partial \theta}{\partial t} = \frac{d\theta}{dH}\frac{\partial H}{\partial t}$$

From it, the capacitance form of the Richards' equation is obtained:

$$C_W(H)\frac{\partial H}{\partial t} = \frac{\partial}{\partial z}\left[K(H)\frac{\partial H}{\partial z}\right] + \frac{\partial K}{\partial z} \qquad (.63)$$

with the water capacity (or specific or differential water capacity) $C_W = d\theta/dH$ $[L^{-1}]$. Eq. (.61) is frequently used ~~in the numerical methods~~. The alternative type of the development leads to the diffusivity form of Richards' equation with

$$\frac{\partial H}{\partial z} = \frac{dH}{d\theta}\frac{\partial \theta}{\partial z}$$

and

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[D(\theta)\frac{\partial \theta}{\partial z}\right] + \frac{dK}{d\theta}\frac{\partial \theta}{\partial z} \qquad (.64)$$

where the soil water diffusivity D is the term derived from

$$D(\theta) = K(\theta)\frac{dH}{d\theta} \qquad [L^2T^{-1}] \qquad (.65)$$

The name of Eq. (. 64) was derived from the equation's resemblence to the equation of the molecular diffusion when the second term on the right hand side of the Eq. (. 64) was omitted. The units of our D are identical to the units of the diffusion coefficient, too. The Equations (61) to (64) are ~~strongly~~ non-linear and they are called Fokker-Planck's equations, too. The diffusivity form is frequently used in the analytical and semianalytical solutions. However, if in the soil is the domain of positive pressure, Equation (.64) is not applicable. Sometimes, the Kirchhoff's transformation is used with

$$U = \int_{H_0}^{H} K(H)\, dH \qquad (\ \ 66)$$

and Eq. (5.61) is then transformed to the form

$$\frac{C_W(H)}{K(H)} \frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial z^2} - \frac{1}{K(H)} \frac{dK}{dH} \frac{\partial U}{\partial z} \qquad (\ .67)$$

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 U}{\partial z^2} - \frac{\partial K}{\partial z}$$

Since the second term on the left hand side of Eq. (.60) or the last term on the right hand side of Eq. (.61) and (.62) originated due to the existence of the gravitational component z of the total potential Φ, it is frequently referred to as the gravitational term of the Richards' equations. The first term in the mentioned equations expresses the flow of water in the soil due to the gradient of the moisture (matric) potential component. In some instances, the gravitational term is neglected and the problems are solved approximately. The ~~classical~~ non-linear diffusion equation is then applied with the non-constant diffusivity

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x} \left[D(\theta) \frac{\partial \theta}{\partial x} \right] \qquad (\ .68)$$

Solutions of (.68) to various types of boundary conditions are analyzed in the literature dealing with the mathematics of diffusion or of the heat flow (Crank, 1956, Carslaw and Jaeger, 1959). If the flow is horizontal, the solutions according to Eq.(66) are exact.

3.4 Soil Water Diffusivity

The most frequent D(θ) relatioship is demonstrated in Fig. .1~~2~~4 by the curve (a). With the exception of the domain of a very low soil water content it is a curve steeply rising with θ. This monotoneous D(θ) relationship starts at a very low value of θ_H, obviously related to the moisture potential below H = - 10^{-5} cm. In the dry region between $\theta = 0$ and θ_H, the great portion of pores is ~~drained and~~ filled with air and they are thus free to the water vapor flow while the liquid water flow is limited to the flow of the only existing very thin water films on the soil solid surface. The strong dependence of the rate of flow of the film upon the thickness of the film has been demonstrated by Eq. (~~5.35~~ 29). Here, the vapor flux exceeds the liquid flux. A more detailed discussion on the water vapor flux will be given in the next Chapter 3.5. Now, we shall study in detail the monotoneously rising part of the D(θ), i.e. for $\theta > \theta_H$. Among the empirical equations, the exponential form of Gardner and Mayhugh (1958) is well known and frequently used:

$$D = D_0 \exp\left[\beta(\theta - \theta_0)\right] \qquad (\ \ 69)$$

where D_0 corresponds to θ_0. Or,

$$D = \alpha \exp(\beta \theta_E) \qquad (\ .70)$$

where θ_M is replaced by θ_H in θ_E and with D = α at θ_H. The empirical coefficient β is in ranges from units up to approximately 30. Swartzendruber and Ahuja (1972) have proposed a three-parametrical equation supposing the existence of H_A and therefore $dH/d\theta \to \infty$ at $\theta \to \theta_S$:

$$D = a \frac{\theta^b}{(\theta_S - \theta)^a} \qquad (\cancel{5}.71)$$

Physically more exact will be the form derived from the soil water retention curve and from the physically expressed K(θ). With Eq. ($\overset{49}{\cancel{5.53}}$) in Mualem's form and with Eq. ($\overset{48}{\cancel{4.38}}$) inserted in Eq. (5.65), van Genuchten (1980) obtained for D(θ_E)

$$D = \frac{(1-m)\,K_S}{\alpha\, m\,(\theta_S-\theta_M)}\,\theta_E^{1/2-1/m}\cdot\left[\left(1-\theta_E^{1/m}\right)^{-m}+\left(1-\theta_E^{1/m}\right)^{m}-2\right] \tag{$\cancel{5}$.72}$$

If a simpler Eq. ($\overset{51}{\cancel{4.36}}$) is applied instead of ($\overset{49}{\cancel{4.38}}$) we receive

$$D = \frac{H_A\,K_S}{\lambda\,(\theta_S-\theta_M)}\,\theta_E^{(b-1)+(a-1)/\lambda} \tag{$\cancel{5}$.73}$$

with the values of a and b derived earlier when K_r was discussed, see Eq. ($\overset{52a}{\cancel{5.56}}$) and ($\overset{52b}{\cancel{5.57}}$), i.e.

a = 2, b = 2 in Childs and Collis-Georges' method
a = 2, b = 3 in Burdine's method
a = 2, b = 2.5 in Mualem's method.

The derived equations starting from Eq. ($\cancel{5}$.72) are applicable mainly to the numerical solutions~~, as they are too complicated to the analytical solutions~~. Nevertheless, the comparison of Eq. ($\cancel{5}$.73) after its rearrangement to Eq. ($\cancel{5}$.71) could offer the approximative physical interpretation of the empirical coefficients.

In some clays, mainly alkali Vertisols, D decreases with the increase of θ, if the soil is prevented from swelling by confinement, see curve (b) in Fig. $\overset{14}{\cancel{5.12}}$ (Kutílek, 1984). In some undisturbed soils D does not vary as strongly with θ as above discussed and as mainly proved for the disturbed repacked soil columns in laboratories (Clothier and White, 1981). If $D_{max} - D_{min}$ is less less than half an order, the Eq. ($\cancel{5}$.68) could be used in the linearized form as a good approximation:

$$\frac{\partial\theta}{\partial t} = \bar{D}\,\frac{\partial^2\theta}{\partial x^2} \tag{$\cancel{5}$.74}$$

where the mean weighted diffusivity $\bar{D}$ is for the wetting process (Crank, 1957)

$$\bar{D} = \frac{5}{3(\theta_0-\theta_i)^{5/3}}\int_{\theta_i}^{\theta_0}(\theta-\theta_i)^{2/3}D(\theta)\,d\theta \tag{$\cancel{5}$.75}$$

and for the drainage process

$$\bar{D} = \frac{1.85}{(\theta_i-\theta_0)^{1.85}}\int_{\theta_i}^{\theta_0}(\theta_i-\theta)^{0.85}D(\theta)\,d\theta \tag{$\cancel{5}$.76}$$

where θ_i is the initial soil water content and θ_0 is θ at x = 0 for t > 0.

The soils where D = const. are called linear because the Eq. ($\cancel{5}$.74) is a linear equation with $\bar{D}$ replaced by D = const. as exact. If the Brooks and Corey's retention curve, Eq. ($\cancel{5}$.74) is applied, then $K_r(\theta_E)$ is described by Eq. ($\overset{52a}{\cancel{5.56}}$) and D($\theta_E$) by Eq. ($\cancel{5}$.73). The condition of D = const. is satisfied in these equations if

either $\lambda = -(a-1)/(b-1)$
or a = b = 1.

If the first condition is applied to Burdine's method we get $H = H_A\theta_E^{2}$ and $K_r = \theta_E^{-1}$. Neither of these conditions is physically real and similar results are obtained with other two approaches, i.e. of Childs and Collis-Georges and of Mualem. The first condition is therefore unreasonable. From the second condition we get for the linear soils (Kutílek et al., 1985):

$$H = H_A\,\theta_E^{-1/\lambda} \tag{$\cancel{5}$.77}$$

$$K_r = \theta_E^{1/\lambda+1} \tag{$\cancel{5}$.78}$$

$$D = -\frac{H_A}{\lambda}\,\frac{K_S}{(\theta_S-\theta_M)} \tag{$\cancel{5}$.79}$$

If $\lambda = 1$, the relative conductivity function is quadratic and meets the requirements of the solutions of the Burgers' equation (Clothier et al., 1981). ~~see the Chapter 6.~~ In general, there exist theoretically a family of linear soils described by the above equations. Eq. (~~5~~.78) of $K_r(\theta_E)$ looks different from the up to now described soils. We have to emphasize that the roots in this deviation lie in the transfer from the theoretical distribution function of the soil pores f(r) to the reality of the soil water retention curve θ (H).

Concluding, we should keep in mind that we apply the soil water diffusivity in the basic Richards' equation in order to reduce the number of variables and the physical meaning of D is indirect, D is strictly defined only mathematically, see Eq. (~~5~~.65). Further on, as $D(\theta)$ is dependent upon the derivative of the soil water retention curve, it should have different values for wetting and drainage processes. The influence of the temperature upon $D(\theta)$ is predictable in an approximative way, only, due to some not well defined effects acting upon both, the θ(H, T) and K(θ, T) in addition to the change of the surface tension and of the viscosity with T.

.3.5 Diffusion of Water Vapor

In the previous Chapter .3.4 we have already shown that the peak in $D(\theta)$ relationship in the dry region of soil is due to the flow of water vapor. It means that the soil water diffusivity D contains two components: D_L, the diffusivity of the liquid water, and D_G, the diffusivity of the water vapor, i.e. in the gaseous phase and $D = D_L + D_G$ (Philip, 1974). To derive D_G, we follow the procedure of Jackson (1964). The Fick's law of diffusion is

$$v_G = -D_p \frac{\partial \rho_{rG}}{\partial x} \qquad (\cancel{5}.80)$$

where v_G is the flow rate of the water vapor, ρ_{rG} is the relative density (concentration) of water vapor, D_p is the diffusion coefficient of water vapor in soil which is approximated by

$$D_p = D_a \alpha (P - \theta)^{\mu} \qquad (\cancel{5}.81)$$

where D_a is the diffusion coefficient of water vapor in free air, α is the correction factor of the porous media and μ ~~which should be identical with~~ takes into account the tortuosity. More about it and about Eq. (~~5~~.81) see Curie (1961) ~~and Rose (1964)~~, and Kowalik (1985) The analogy of the Richards' equation, i.e. the combination of Eq. (~~5~~.80) with the equation of continuity is

$$\frac{\partial \rho_{rG}}{\partial t} = \frac{\partial}{\partial x}\left(\frac{D_p}{\Pi - \theta} \frac{\partial \rho_{rG}}{\partial x}\right) - \frac{1}{P - \theta} \frac{\partial \theta}{\partial t} \qquad (\cancel{5}.82)$$

where the second term on the right hand side of Eq. (~~5~~.82) denotes the sink due to the adsorption by which $\partial \rho_{rG}/\partial t$ is altered, too, not only by $\partial v_G/\partial x$. Since the change in θ is small compared to $(P - \theta)$ in the relatively dry soil where the vapor flow is examined, we can approximate $P - \theta =$ = const., and

$$(\Pi - \theta)\frac{\partial \rho_{rG}}{\partial t} + \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(D_p \frac{\partial \rho_{rG}}{\partial x}\right) \qquad (\cancel{5}.83)$$

Taking the first term on the left hand side of Eq. (~~5~~.83) as negligible when compared to $\partial\theta/\partial t$ and since for the single valued $\rho_{rG}(\theta)$ is

$$\frac{\partial \rho_{rG}}{\partial x} = \frac{d \rho_{rG}}{d \theta} \frac{\partial \theta}{\partial x}$$

we get the final equation

$$\frac{\partial \vartheta}{\partial t} = \frac{\partial}{\partial x}\left(D_G \frac{\partial \vartheta}{\partial x}\right) \qquad (\cancel{5}.84)$$

where the water vapor diffusivity D_G is analogical to the earlier introduced soil water diffusivity and it is expressed by

$$D_G = D_p \frac{d\rho_{AG}}{d\vartheta} \qquad (\cancel{5}.85)$$

The term $d\rho_{AG}/d\vartheta$ is actually the slope of the adsorption isotherm and its inflection point corresponds to the peak in $D_G(\vartheta)$. The water vapor diffusivity rises to this maximum from $\vartheta = 0$ and reaches this peak at the relative water vapor pressure $p/p_o = 0.3$ up to 0.4 in majority of soils.The maximum value of D_G is in wide ranges between 10^{-6} up to 10^{-3} $cm^2 s^{-1}$ in dependence upon the texture, mineralogy of clay fraction and organic matter content, see the Fig. ~~5.13~~ 15. Further increase of ϑ results in the decrease of D_G while the component in D_L increases, exceeding D_G at $p/p_o \doteq 0.5$ up to 0.8. In terms of the average thickness of the adsorbed water films on the soil solid surface, the water vapor maximum in D_G is reached after the first molecular layer is completed and before or at least when the second molecular layer is formed. D_L exceeds D_G when about 4 up to 6 molecular layers of adsorbed water are formed.

Here, we can show that if the non-Darcian prelinear flow is expected in the first molecular layers of water, the effects can be overshadowed by the flux of the water vapor.

3.6 Two Phase Flow

Up to now we have supposed that the water flow does not meet the resistance when water is filling in the pores. The fluidity of the soil air is by order~~s~~ higher than that one of water so that the resistance due to the slow flow of air is implicitly neglected. The assumption on one phase flow, i.e. the flow of water only, is not sufficient in such instances when the free escape of air is blocked by small passages owing to the existence either of a layer of a higher water content, or of a less permeable layer. If such a case is met, the flow of both, of water and air should be solved as influencing mutually each the other.

In the theory of the two phase flow, the two phases, air and water are assumed to behave like two immiscible liquids and the basic flow equations of the Darcy-Buckingham's type will be rewritten in the form simply fitting to both phases and the pressure gradient is taken first as the driving force (Morel-Seytoux, 1969):

$$v_W = -\lambda_W\left(\frac{\partial p_W}{\partial x} - \rho_W g\right) \qquad (\cancel{5}.86)$$

$$v_A = -\lambda_A\left(\frac{\partial p_A}{\partial x} - \rho_A g\right) \qquad (\cancel{5}.87)$$

where the index W denote water and A air, p is the pressure $[L^{-1}MT^{-2}]$, ρ is the density $[L^{-3}M]$, g the acceleration of gravity $[LT^{-2}]$. The coefficent λ is the fluid mobility:

$$\lambda = \frac{K_p K_r}{\mu} \qquad (\cancel{5}.88)$$

where K_r is the relative conductivity of the given fluid,

K_p is permeability $[L^2]$ and μ is the dynamic viscosity. The dependence of both K_{rA} and K_{rW} upon the soil water content as determined experimentally by Touma and Vauclin (1986) in sand is in Fig. ~~5.14~~ 16. The continuity equation is for water

$$\frac{\partial(\rho_W \vartheta)}{\partial t} + \frac{\partial(\rho_W v_W)}{\partial z} = 0 \qquad (\cancel{5}.89)$$

and for air

$$\frac{\partial(\rho_A \vartheta_A)}{\partial t} + \frac{\partial(\rho_A v_A)}{\partial z} = 0 \qquad (\cancel{5}.90)$$

with $\vartheta_A = P - \vartheta$ the volumetric air content. The difference between the pressure of two fluids is the capillary pressure $p_c = p_A - p_W$ and for water taken as incompressible fluid we get

$$\frac{\rho_A}{\rho_{Ao}} = \frac{p_A}{p_{Ao}} \quad , \qquad \frac{\rho_W}{\rho_{Wo}} = 1 \qquad (\cancel{5}.91)$$

where the index o denotes the value at the reference atmospheric pressure. As the pressure head was more convenient in the one-phase flow formulations, we can use

$$H_W = \frac{p_W - p_{Ao}}{\rho_W\, g} \qquad H_A = \frac{p_A - p_{Ao}}{\rho_W\, g} \qquad (\cancel{5}.92)$$

$$H_c = H_A - H_W \qquad (\cancel{5}.93)$$

and Eq. ($\cancel{5}$.86) and ($\cancel{5}$.87) are transcribed in a more familiar form

$$v_W = -K_W(\vartheta)\left(\frac{\partial H_W}{\partial z} - 1\right) \qquad (\cancel{5}.94)$$

$$v_A = -K_A(\vartheta)\left(\frac{\partial H_A}{\partial z} - \frac{\rho_A}{\rho_W}\right) \qquad (\cancel{5}.95)$$

with $H_A - H_W = H_c$ ~~with H_c~~, K_W is identical to earlier notation $K(\vartheta)$ and $K_A = \rho_W g \lambda_A$ is the air conductivity, both in $[LT^{-1}]$. The analogy of Richards' equations is obtained (Touma and Vauclin, 1986) for water

$$\frac{\partial \vartheta}{\partial t} = \frac{\partial}{\partial z}\left[K_W(\vartheta)\left(\frac{\partial H_W}{\partial z} - 1\right)\right] \qquad (\cancel{5}.96)$$

and in the capacitance form

$$C_W\left(\frac{\partial H_A}{\partial t} - \frac{\partial H_W}{\partial t}\right) = \frac{\partial}{\partial z}\left[K_W(\vartheta)\left(\frac{\partial H_W}{\partial z} - 1\right)\right] \qquad (\cancel{5}.97)$$

with

$$C_W = d\vartheta / dH_c$$

For air is

$$\frac{\partial(\rho_A \vartheta_A)}{\partial t} = \frac{\partial}{\partial z}\left[\rho_A K_A(\vartheta)\left(\frac{\partial H_A}{\partial z} - \frac{\rho_A}{\rho_W}\right)\right] \qquad (\cancel{5}.98)$$

When we substitute $\vartheta_A = P - \vartheta$ with $\rho_A = \rho_{Ao}(1 + H_A/H_o)$ and using C_W we get again the analogy of the capacitance equations

$$\rho_A C_W \frac{\partial H_W}{\partial t} + \left[(P-\vartheta)\frac{\rho_{A0}}{H_0} - \rho_A C_W\right]\frac{\partial H_A}{\partial t} = \frac{\partial}{\partial z}\left[\rho_A K_A(\vartheta)\left(\frac{\partial H_A}{\partial z} - \frac{\rho_A}{\rho_W}\right)\right] \qquad (\cancel{5}.99)$$

When the air is continuously connected to the external atmospheric pressure which was taken as the reference, then $H_A = 0$ and $H_C = H_W$ and Eq. (~~5~~.97) is identical with Eq. (~~5~~.63).

The presented theory offers reliable results if the pressure (or potential) drop in water across the less permeable barrier is small, or, if the barrier is not reached by the wetting front in the case of infiltration and limits only the flow of air. If a substantially less permeable layer exists in the domain of the flow of water, a steep gradient of the water pressure (or of the water potential) develops usually across the less permeable layer. In these circumstances, air and water cannot be considered as immiscible fluids. The dissolution of the air in water should be formulated as the dissolution of individual gases of the mixture according to Henry's law

$$C_i = k_i\, p_i \qquad (\cancel{5}.100)$$

where C_i is the concentration of the i-th gas, k_i is a constant dependent upon the temperature and the nature of the gas and p_i is the partial pressure of the i-th gas. Considering the numerical values of k_i and of p_i of gases composing the soil air, the dissolution of the nitrogen and of the oxygen will play role. When the pressure in soil water varies in order, the same will be for the concentration of the dissolved N_2 and O_2 in soil water. With the abrupt decrease in soil water pressure when water passes the less permeable layer, the originally dissolved gases will be released. Small bubles of air are thus usually accumulated mainly on the bottom boundary of the less permeable layer. By this mechanism the hydraulic resistance of the less permeable layer increases and the increase is time dependent. For a detailed quantitative understanding, the experimental data are still unsufficient.

.3.7 Practical Conclusions

The unsaturated conductivity K is physically dependent upon ϑ and as ϑ(H) exists, K(H) is derived, too. From the mathematical treatment of the equations the characteristic soil water diffusivity D(ϑ) is derived. It is typical by the less steep relation to ϑ, when compared to K(ϑ). D changes usually up to two orders less with ϑ than K in the range of soil water content between dry and saturated soil. ~~An example of the basic hydraulic characteristics H(ϑ), K(ϑ) and D(ϑ) is in Fig. 5.16.~~ 17 Owing to the close relations of K(ϑ) and D(ϑ) to K_S and to H(ϑ) we can expect that the dependence of K(ϑ) and D(ϑ) upon the nature of the soil can be deduced from the earlier discussed influences. The change of the soil bulk density ρ_T owing to a simple compaction results in the decrease of the volume of pores of big radius and in extreme the domain of these pores is eliminated at all. This change is reflected by the change of H(ϑ), sometimes its original analytical expression according to Eq.(~~4.38~~ 48) has to be substituted by the model of Brooks and Corey, Eq.(~~4.36~~ 51). The consequence is not only in the shifting K(ϑ) to lower values but it means that K_r(ϑ) is changed.

The nature of the exchangeable cations influences the shape of K(ϑ) and of D(ϑ), too, mainly by the alteration of the aggregation and due to the coagulation and peptization processes. Practically, it is dominantly the value of the exchangeable sodium percentage ESP which plays the dominant role. The lower is the concentration of salts in the soil water, the more expressed is the influence of ESP upon the reduction of K(ϑ) values. ~~see Fig. 5.17 but $K_r(\vartheta)$ remains unaltered in loamy soils. In clays, $K_r(\vartheta)$ may change at low EC when ESP exceeds approximately 20%, see Fig. 18~~. When this type of drastic changes in K occurs, the swelling capacity of soils is altered, too and the decrease in ϑ does not necessarily mean~~s~~ the increase of the degree of unsaturation, ~~and the theoretical discussion of the Chapter 5.4 should be applied.~~ The change in K(ϑ) and in D(ϑ) brings about the alteration in availability of soil water to plants. If we assume e.g. for low gradients ~~and for~~ of potential that $D_c = 2 \times 10^{-3} cm^2 min^{-1}$ is a critical value for the flux to plant roots to prevent the permanent wilting, we obtain the critical value of ϑ_c which replaces merely empirical wilting point. This value of ϑ_c is extremely sensitive to the ESP in clays of the Vertisol type provided again that EC is kept at a low value of EC < 1 mmhos.cm^1, see Fig. ~~5.17~~ 17 which summarizes the data of Kutílek and Semotán (1975) and of Kutílek (1983). Further informations on the combined effects of EC and ESP upon the hydraulic characteristics of soils are available in the study of Shainberg (1984).

5.4 Flow in Non-~~Inert~~ Rigid Soils

When the soil swells or shrinks due to the change of the water content, the equations untill now developed should be modified. For the sake of simplicity, the one-dimensional flow of water will be considered, only, involving the one-dimensional volume change, too. A complete theory has not yet been developed on three-dimensional volume changes resulting in opening of cracks when the soil dries and in closing them when the soil is moistened. The volumetric change of the soil to the change of the soil water content depends upon the degree of the preconsolidation of the soil, especially in the wetting process, this being again a difficult task. The theory presented here is therefore just the starting point to the understanding of the swelling-shrinkage phenomena accompanying the flow of water in soil. It is more applicable to the artefacts of laboratory procedures then to the field hydrological problems and only some clearly specified terrain processes can be solved approximatively by these theories.

When the soil swells during wetting, the Darcy's law should be modified according to Gersevanov (1937) and the flow rate of water should be related to the solid phase. Instead of the Euler's coordinate system, the Lagrange's coordinates are considered. In the onedimensional treatment of swelling, the original proposal was redefined by Smiles and Rosenthal (1968) as follows:

$$\frac{dm}{dx} = \frac{1}{1+e} \qquad (\cancel{5}.101)$$

and integrating

$$m = \int_{-\infty}^{x} \frac{dx}{1+e} \qquad (\cancel{5}.102)$$

or

$$m = \int_{-\infty}^{x} (1-P)\,dx \qquad (\cancel{5}.103)$$

Equation (~~5~~.101) shows that the ratio of the material coordinate m to the Eulerian coordinate equals the ratio of the volume of the solid phase to the total volume of the soil. Fig. ~~5.19~~ 18 is instructive in demonstrating the material coordinate and it depicts the following type of experiment: The dry soil has been originally put in a tube up to a certain height denoted by x = 0. After that, the surface is flooded with water. As water infiltrates into the soil, the interface surface of soil water rises since the soil swells. After a certain time of infiltration we measure the depth of the wetting front from the original x = 0 and denote it by x_f. Simultaneously with the water content, the bulk density is measured and the porosity P is so determined and plotted against x. P will be plotted into the negative part of x, too, since the soil surface rose compared to the original position of the surface x = 0. The material coordinate m expresses the volume of the solid phase flown by the water. Since we can not predict to which elevation the soil will swell, we define the lower limit of integration as $-\infty$. Mind that the integral (5.103) has a zero value for P = 1 and therefore m = 0 in ranges $<-\infty$, recent surface>. If the soil is preconsolidated, we can assume that $P = e(1 + e)$ = const. for $x > x_f$. If the soil is not preconsolidated, the soil is compressed ahead of the wetting front by the swelling pressure. This peak of the minimum P_{min} is moving at the same rate as the wetting front and the value of P_{min} decreases, i.e. the value of $(P_i - P_{min})$ increases with time and with the surface load (Kutílek, 1984). All these effects are very distinct in montmorillonitic clays of Vertisols and they increase with the increase of ESP and with the decrease of EC.

Darcy's equation in the Lagrangeian coordinate system is

$$v_m = -K_m \frac{d\Phi}{dm} \qquad (5.104)$$

where Φ is the total potential where the envelope potential is included and K_m is the function of the soil water ratio ϑ which is replacing the volumetric soil water content in the equation of continuity which is modified,

to

$$\frac{\partial \vartheta}{\partial t} = \frac{\partial v_m}{\partial m} \qquad (5.105)$$

Since the element of water in the element of dm is

$$(1+e)\,\theta\, dm = \frac{\theta}{1-\Pi}\, dm \qquad (5.106)$$

and the change of this element in time dt is equal to the change of the flow rate over the dm, we obtain

$$\frac{\partial}{\partial t}\left[(1+e)\,\theta\right] = -\frac{\partial v_m}{\partial m} \qquad (5.107)$$

Combination of (5.104) with (5.105) leads to

$$\frac{\partial \vartheta}{\partial t} = \frac{\partial}{\partial m}\left[K_m(\vartheta)\,\frac{\partial \Phi}{\partial m}\right] \qquad (5.108)$$

and as dm = dx/(1 + e), we get for K_m

$$K_m = \frac{K}{1+e} \qquad (5.109)$$

where K should be practically K_S as the saturation is supposed, but in the swelling soils $K(\vartheta)$ or K(e) and even for the saturation the conductivity can not be taken as constant. The term K is determined in the Eulerian coordinate system. If the single valued relationship $\vartheta(H)$ is assumed, we get for the horizontal flow in the unloaded system

$$\frac{\partial \vartheta}{\partial t} = \frac{\partial}{\partial m}\left[D(\vartheta)\,\frac{\partial \vartheta}{\partial m}\right] \qquad (5.110)$$

with

$$\frac{\partial H}{\partial m} = \frac{dH}{d\vartheta}\,\frac{\partial \vartheta}{\partial m}$$

and $D(\vartheta) = \frac{K}{1+e}\,\frac{dH}{d\vartheta} = K_m\,\frac{dH}{d\vartheta}$

Further developments for the loaded surface and for vertical infiltration see Philip (1975)

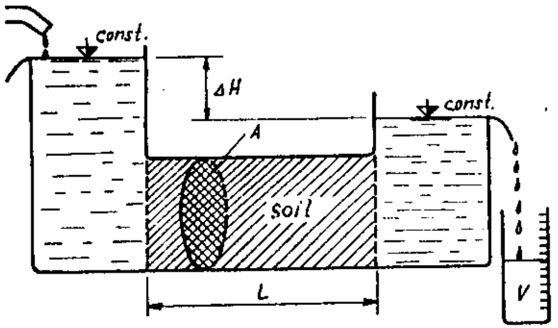


FIG. 1/u.

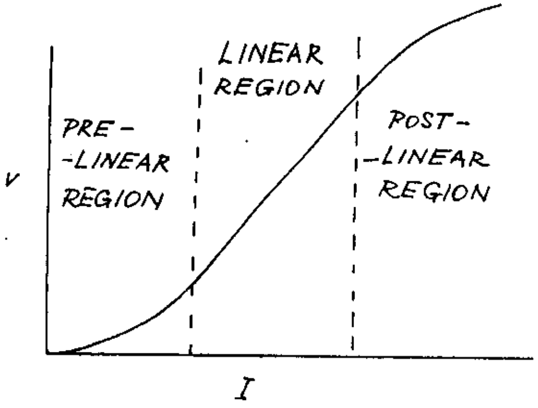


FIG. 3/u.

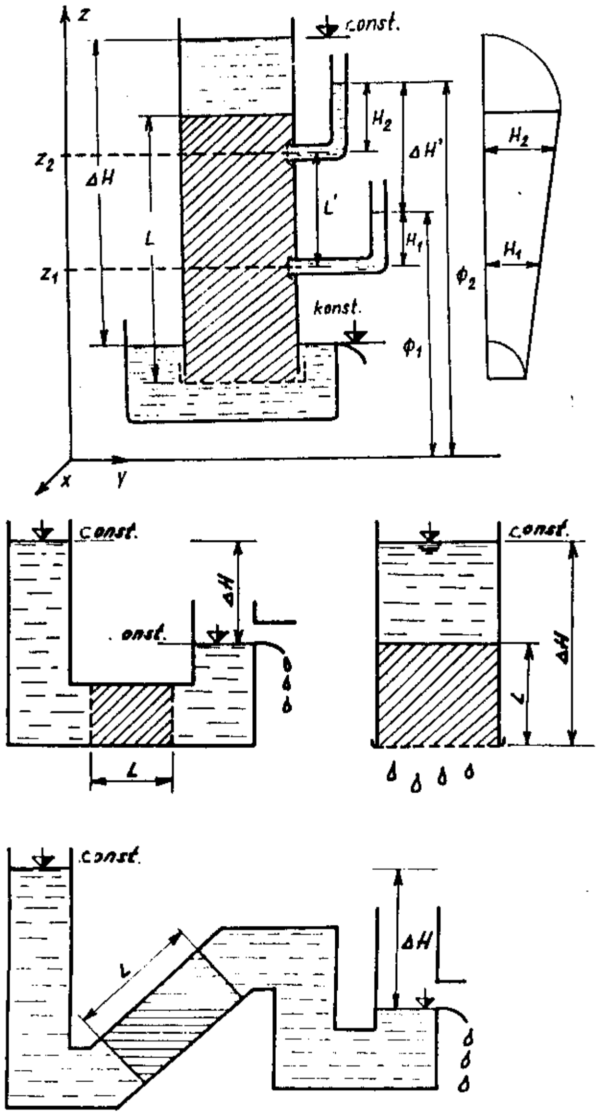


FIG. 2/u.

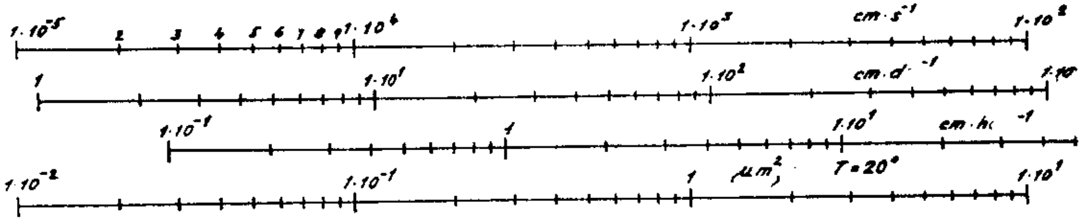


FIG. 4/u.

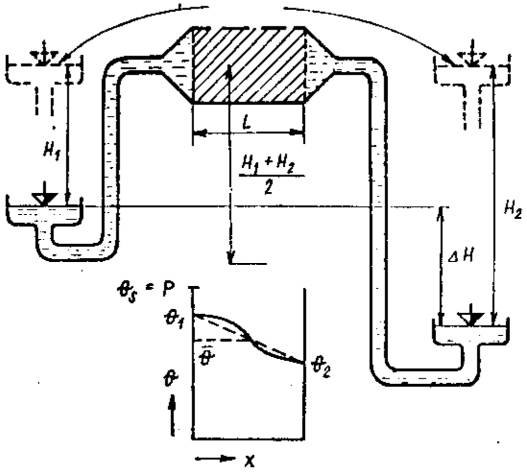


FIG. 5/u.

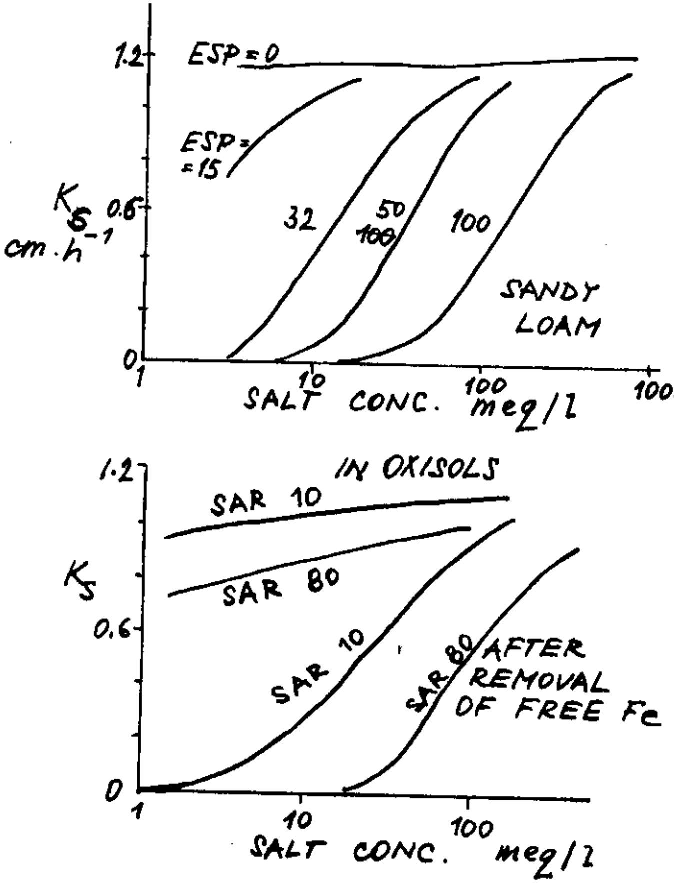


FIG. 3a/U

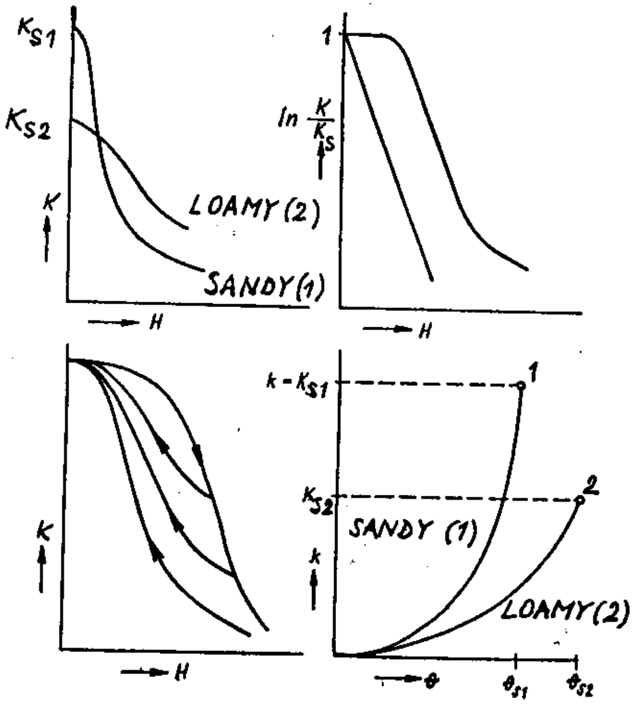


FIG. 6/U.

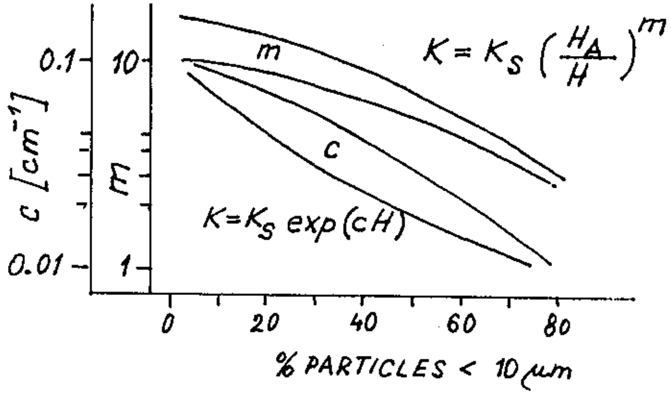


FIG. 7 /U.

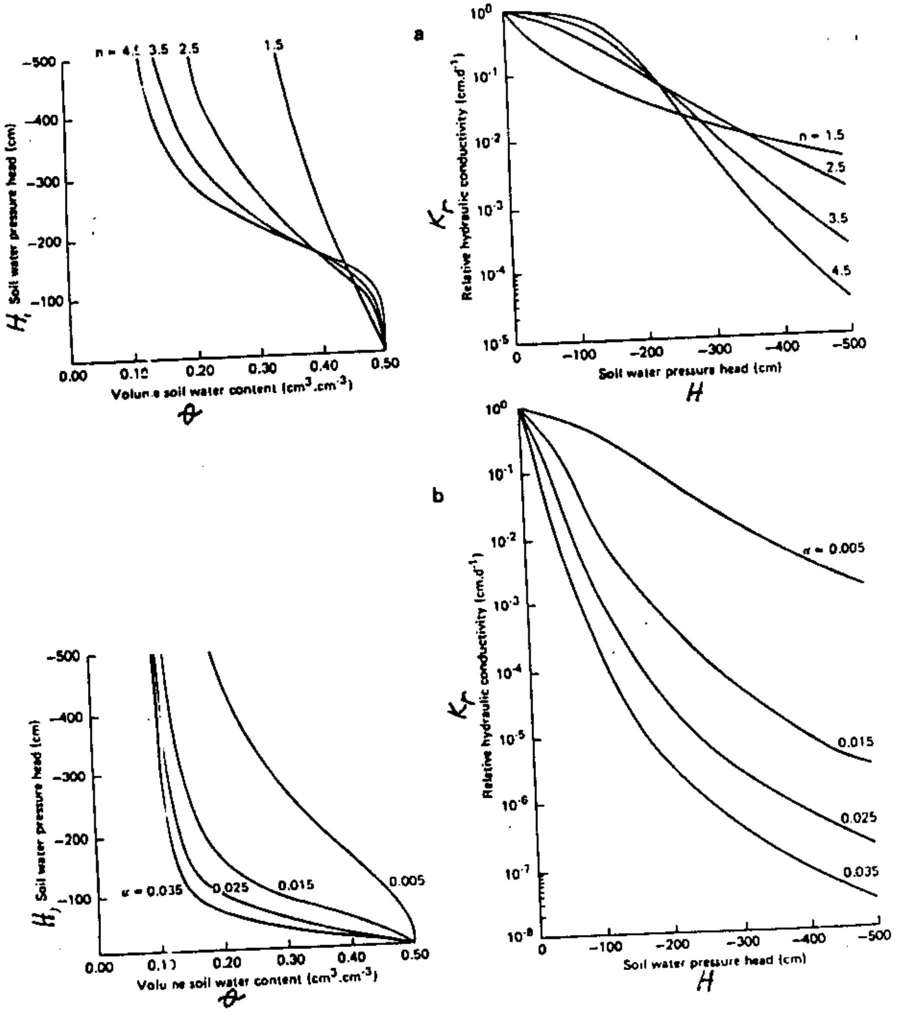


FIG. 8/U

FOR δH(θ_E) CONST.

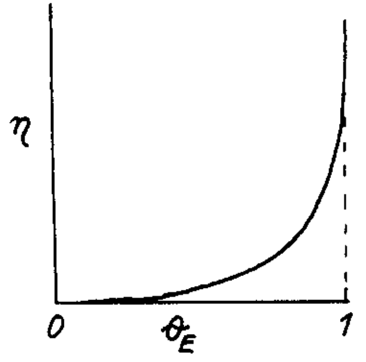
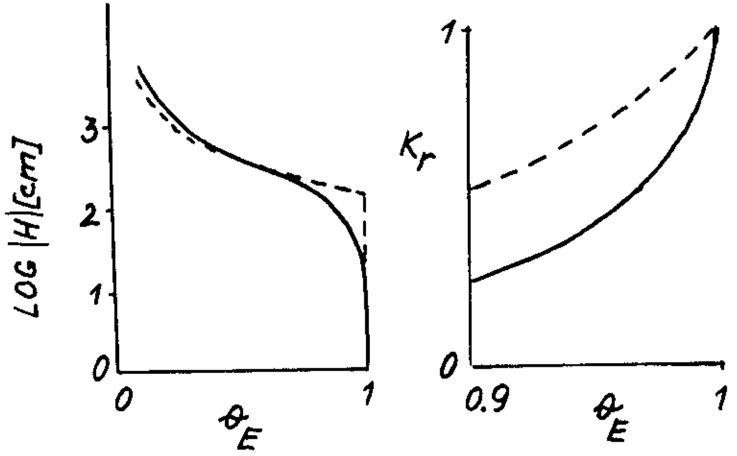


FIG. 9/U.



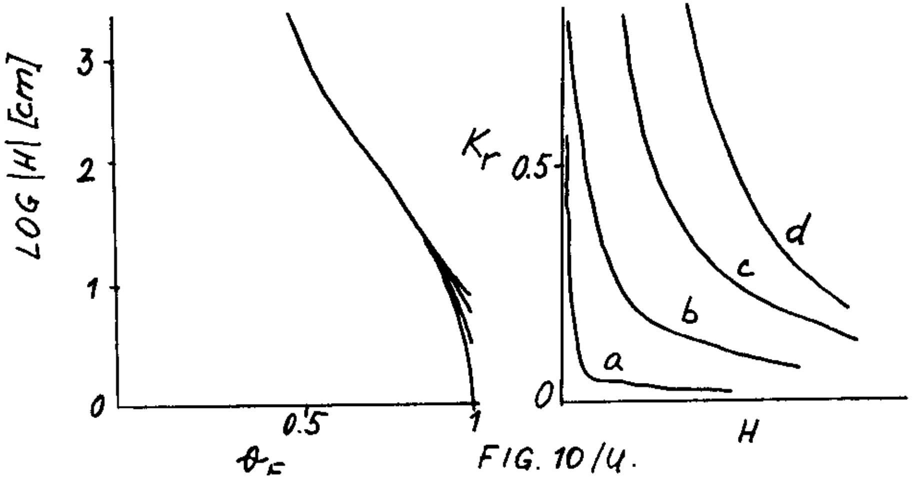


FIG. 10/U.

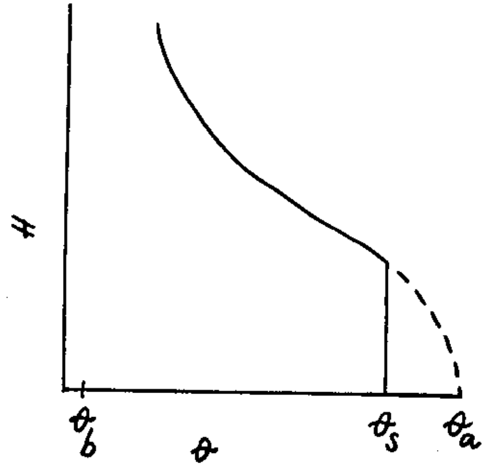


FIG. 11/U.

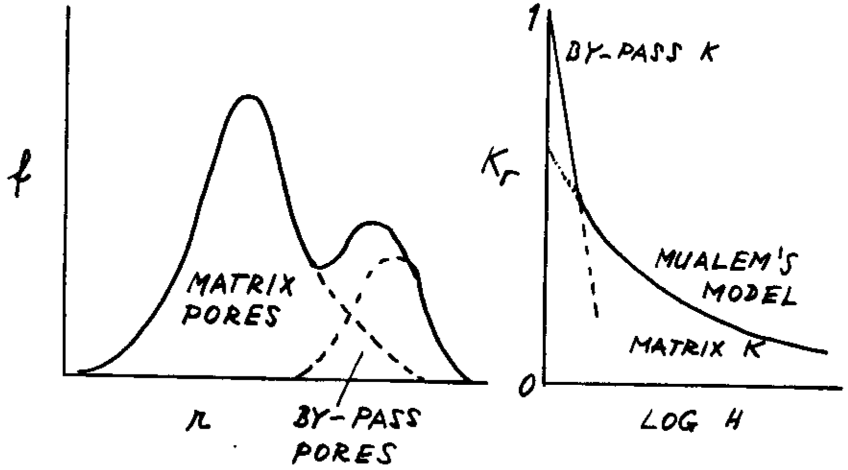


FIG. 12/U.

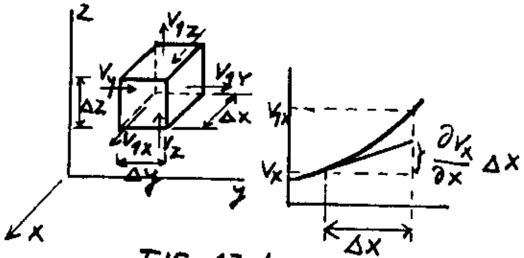


FIG. 13/U.

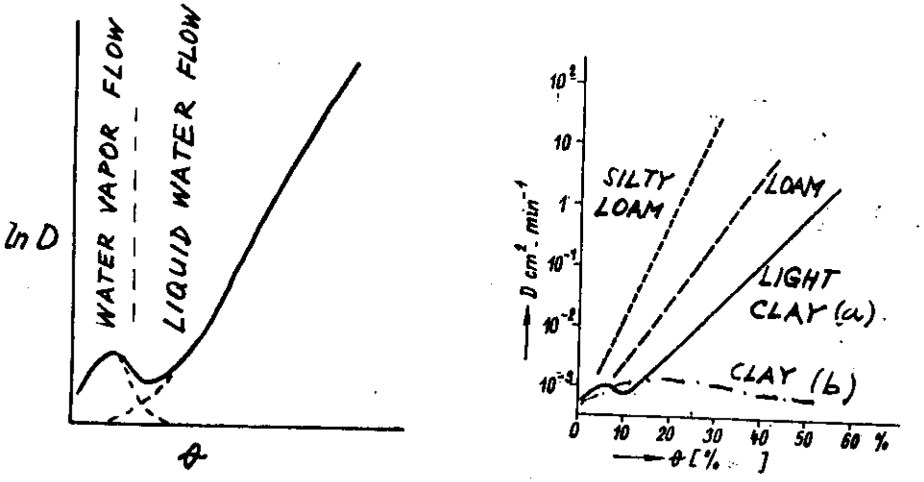


FIG. 14/U.

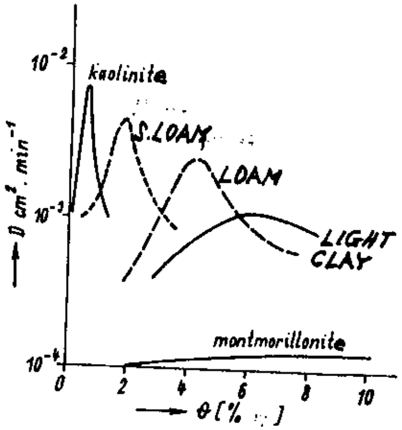


FIG. 15/U.

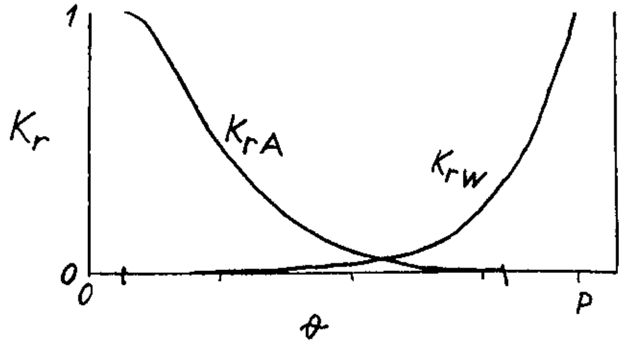


FIG. 16

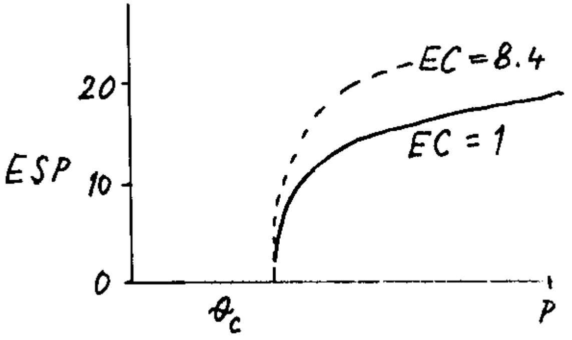


FIG. 17/u.

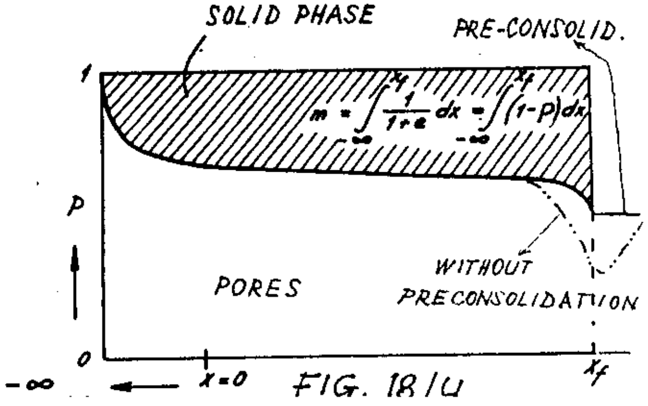


FIG. 18/u

INTRODUCTION TO THE INFILTRATION PROCESS

by

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Notes to the lecture at the College on Soil Physics, ICTP, Trieste, Italy.

1. DEFINITIONS

Infiltration is the process of entry of water into the soil through the soil surface.

Soil surface - plane: horizontal or inclined,
- concave or convex, combination,
- cavity: sferical or tubular.

Source of water - covers the whole surface,
- only a part /point, line.../

Solutions: one dimensional vertical
horizontal / = absorption/
two and three-dimensional

This introduction is restricted to the one-dimensional vertical infiltration through the horizontal plane. Soil will be considered as inert to water /no shrinkage, no swelling/.

2. NOTATION AND BASIC TERMS

A	parameter in Philip´s algebraic equation of infiltration
A_2, A_3..	parameters in Philip´s series solution
C_{mv}	empirical constant in empirical infiltration equations
C_w	soil water capacity, $C_w = d\Theta/dH$
D	soil water diffusivity
DBC	Dirichlet´s boundary condition
F	flux concentration relation, $F = q/q_o$
H	soil water pressure head
H_A	air entry value of H
H_i	initial value of H
H_o	pressure head at $z = 0$
H_f	pressure head at the wetting front in Green and Ampt´s approximate solution
I	cumulative infiltration
I_D	cumulative infiltration with DBC
K	hydraulic conductivity $K(\Theta)$
K_i	hydraulic conductivity at the initial soil water content Θ_i
K_s	saturated hydraulic conductivity
L_f	depth of the wetting front
S	sorptivity
q	flux
q_D	infiltration rate with DBC
q_e	effective rain intensity
q_o	flux at $z=0$, infiltration rate
q_{oc}	constant infiltration rate for DBC, theoretically at $t \to \infty$
K_A, K_B	**approximates of K_s**

q_{o1}	infiltration rate at $t=1$, usually at $t = 1$ min
q_r	rain intensity
t	time
t_p	ponding time
z	vertical coordinate
$\alpha, \beta, \gamma, \varepsilon$	empirical exponents in empirical infiltration equations
η	Boltzman´s variable
Θ	soil water content
Θ_i	initial soil water content
Θ_o	soil water content on the soil surface, at $z = 0$
Θ_s	saturated soil water content

3. ROLE OF INFILTRATION

A. In hydrology: Infiltration divides the precipitation in

- (a) portion of the surface water excess:
  - (a1) immediate surface runoff,
  - (a2) surface storage etc.
- (b) portion of water entering the soil:
  - (b1) storage of water in soil porous medium,
  - (b2) feeding the ground water.

(a1) and (b2) supply the river discharge.

B. In plant production: Continuous flux of water through the plant - Discontinuous source of water in precipitation Soil: storage reservoir with A.(b1).

C. In environmental production: Water = carrier of polutants. Rate of transport of polutants from the source of polutants: by A.(a1) $\gg$ A(b2).

D. In soil hydrology: One of the elementary processes.

A note: Soil hydrology = elementary processes (infiltration, redistribution, drainage to ground water, upward fluxes, evapotranspiration) + meteorologic situations and processes. The resulting storage of soil water is classified according to the degree of the excess or insufficiency and to the duration of such a period. The year´s and vegetational period is considered, too.

4. BOUNDARY CONDITIONS

Richards´ equation is solved either in the diffusive form (Klute, 1952)

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[D(\theta)\frac{\partial \theta}{\partial z}\right] - \frac{dK}{d\theta}\frac{\partial \theta}{\partial z} \qquad (I.1)$$

or in the capacitance form

$$\frac{d\theta}{dH}\frac{\partial H}{\partial t} = \frac{\partial}{\partial z}\left[K(H)\frac{\partial H}{\partial z}\right] - \frac{\partial K(H)}{\partial z} \qquad (I.2)$$

$d\theta/dH = c_w$ the soil water capacity.

Soil is considered as the semiinfinite column. Richards´ equation is solved for (a) the initial condition, in the semianalytic and approximative procedures:

$$\theta = \theta_i \qquad t = 0 \qquad z > 0 \qquad (I.3)$$

In numerical methods $H = H_i$ instead of $\theta = \theta_i$. In the field: $\theta = \theta_i(z)$.

(b) the boundary conditions. We distinguish Dirichlet´s boundary condition and Neuman´s boundary condition.

4.1. Dirichlet´s boundary condition (DBC, concentration condition)

Formulated as either soil water content θ_i or the soil water pressure head H on the boundary:

$$\theta = \theta_0 \qquad z = 0 \qquad t > 0 \qquad (I.4)$$

Usually: $\theta_0 = \theta_s$

$$H = H_0 \qquad z = 0 \qquad t > 0 \qquad (I.5)$$

(exact)

Field situations: Floods.

Irrigation: Check-basin irrigation,
border irrigation,
furrow irrigation.

Infiltration tests: Double ring infiltration test,
basin infiltration test.

Note: $H = H_0(t)$ is formulated in some numerical procedures, too, e.g. for infiltration with falling water table.

Results of the tests and of the theoretical treatment of the problem:

The soil water content profile $\theta(z,t)$ is similar to piston-like flow (step-like profile), especially in sands. Inclination of the wetting front in clay increases with time (Fig.1 data from Haverkamp et al., 1977). If θ_i increases, the rate of advance of the wetting front increases, too.

Cumulative infiltration in time $I(t)$ is steep at short time, after long time $I(t)$ is a straight line (theoretically at $t \to \infty$). Infiltration rate q_0 decreases first rapidly with time, at $t = 0$ is $q_0 \to \infty$. After long time, $dq_0/dt \to$ const., theoretically $\lim_{t \to \infty} q_0(t) = K_s$. Constant rate q_0 is denoted sometimes q_{0c} (Fig.2). The shape $I(t)$, $q_0(t)$ depends upon the hydraulic characteristics of the soil $K(\theta)$, $\theta(H)$ and upon θ_i, θ_s. With the increase of θ_i the rate q_0 decreases (Fig.3, light clay, Philip, 1957). When $H_0 > 0$, and the depth of water on the surface increases, the infiltration rate q_0 increases, too (Fig.4, light clay, Philip, 1958).

The process can be partitioned into matric and gravitational components, Fig.5 (Kunze, Nielsen, 1982). The matric forces have the dominant influence at the early stage of infiltration, the gravitational force at large time. The relative difference of the position of the wetting front between the horizontal and vertical infiltration increases with time.

4.2. Neuman´s boundary condition (NBC, flux condition)

On the boundary z_0 instead of the soil water content (or pressure head) as in DBC, the flux q_0 is defined. Either q_0 = const., or $q_0(t)$:

$$q_0 = -D(\theta)\frac{\partial \theta}{\partial z} + K(\theta) \qquad z=0 \qquad t>0 \qquad (I.5)$$

or

$$q_0 = -K(H)\frac{\partial H}{\partial z} + K(H) \qquad z=0 \qquad t>0$$

Field situations: Rain infiltration.
Irrigation: Sprinkler irrigation.
Infiltration tests: Rain simulators.

Since the infiltration rate q_o can be under certain circumstances lower than the rain intensity q_r, we have to use in the discussion these terms with the strict meaning:

q_r - rain intensity (= flux from the atmosphere),
q_o - infiltration rate (flux through the soil surface).

4.2.1. q_r = const.

Two classes of problems of rain infiltration exist, if q_r is the rain intensity:

A. $0 < q_r/K_s \leq 1$ and $q_o = q_r$

The boundary condition is (I.5). The profile of the soil water content is in Fig.6, the soil water content at the surface θ_o increases with time, approaching θ_c in $q_o = K(\theta_c)$.

B. $q_r/K_s > 1$

B.1. Time interval $0 < t \leq t_p$, $q_o = q_r$

The boundary condition is in the form of NBC:

$$q_o = -D(\theta)\frac{\partial\theta}{\partial z} + K(\theta) \qquad z = 0 \qquad 0 < t \leq t_p \qquad (\text{I}.6)$$

The soil water content at the surface θ_o increases with time, reaching $\theta_o = \theta_s$ at ponding time $t = t_p$ (Fig.7).

B.2 Time $t > t_p$

The boundary condition is in the form of DBC with:

either no runoff of excess water

$$H = H_o(t) \qquad z = 0 \qquad t > t_p \qquad (\text{I}.7)$$

or full runoff of excess water

$$\theta = \theta_s \qquad z = 0 \qquad t > t_p \qquad (\text{I}.8)$$

or $H = 0$

Time of ponding t_p separates from the rain the effective rain which can cause the immediate surface runoff. The greater is the rain intensity, the shorter is the period without ponding and t_p decreases (Fig.8, Rubin, 1969).

4.2.2. $q_r(t)$

Note: Infiltration with DBC (from the ponded surface with $H_o \to 0$) can be formulated as infiltration with the flux boundary condition and with $q_r/K_s \to \infty$ when the non-infiltrated excess water flows away (full runoff). The ponding time t_p for $q_r(t)$ is obtained from two equalities (Kutílek, 1982, Peschke and Kutílek, 1982, based upon the development of Mls, 1980 and Kutílek, 1980):

$$\int_0^{t_p} q_r(t)\,dt = \int_0^{t_x} q_D(t)\,dt \tag{I.9}$$

$$q_r(t_p) = q_D(t_x) \tag{I 10}$$

where q_D is the infiltration rate with DBC (= ponded surface with $H_o \rightarrow 0$). Graphical demonstration is in Fig.8. When q_r = const., and $q_r/K_s > 1$, we get

$$q_r\,t_p = \int_0^{t_x} q_D(t)\,dt \tag{I.11}$$

see Fig.8.

The infiltration rate q_o at $t > t_p$ is $q_D(\tau)$ with $\tau = t - (t_p - t_x)$, i.e. the q_D curve is shifted by $(t_p - t_x)$. Effective rain rate is

$$q_e = q_r(t) - q_D(\tau) \tag{I12}$$

The relations between the infiltration with DBC and NBC as discussed above in eq.(I.9) to (I.12) are valid if the soil surface is not affected by the rain in a different way from the ponded type of infiltration. The sealing of the surface due to the kinetic energy of rain drops is not considered. As it follows from the development, the ponded infiltration test with DBC should offer sufficient information for the

determination of the rainfall infiltration (with NBC) provided that the sealing by rain does not exist.

Graphical illustrations for rain intensities closer to reality and to the hydrological practice are in Fig.9 (Peschke, Kutílek, 1982).

5. CLASSES OF SOLUTIONS

5.1. Physically based solutions

The solution of the Richards´ equation (1.1) to the initial and boundary conditions (1.3) and (1.4) is searched and the form I(t) or $q_0(t)$ is preferred.

Philip´s (1957) procedure starts with the solution of the horizontal infiltration and the result is corrected with regard to the gravitational term, Fig.5 is instructive. The solution is based on the similarity solution of the one-dimensional absorption of the gravity-free Eq.(1.1), i.e. of

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left[D(\theta)\frac{\partial\theta}{\partial z}\right] \qquad (1.13)$$

for conditions (1.3) and (1.4). Solution of Eq.(1.13) is the first step of the solution of Eq.(1.1) and it has the form (Fig.10)

$$z_1(\theta,t) = \eta_1(\theta)\,t^{1/2} \qquad (1.14)$$

The cumulative volume of water thus infiltrated is

$$I_1 = \int_{\theta_i}^{\theta_0} z_1(\theta,t)\,d\theta \qquad (1.15)$$

When the term sorptivity S (Philip, 1957) is introduced

$$S_1 = \int_{\theta_i}^{\theta_0} \eta_1(\theta)\,d\theta \qquad (1.16)$$

we get

$$I_1 = S_1 t^{1/2} \qquad (1.17)$$

As Eq.(1.13) is equal to Eq.(1.1) when the gravitational term $\partial K/\partial z$ is introduced, the solution (1.14) has to be corrected by the term y and

$$z = z_1 + y \qquad (1.18)$$

where again $y(t,\theta)$. However, the solution allows to obtain the approximate of y only, denoted here y_1 and $y = y_1 + u_1$ where u_1 is the error of approximation and

$$z = z_1 + y_1 + u_1 + \cdots \qquad (1.19)$$

with transformations analogic to (1.14). The final solution of (1.1) has the form of a time serie solution

$$z(\theta,t) = \eta_1 t^{1/2} + \eta_2 t + \eta_3 t^{3/2} + \cdots \qquad (1.20)$$

In analogy to (1.15) and (1.17) Philip obtained

$$I = S_1 t^{1/2} + A_2 t + A_3 t^{3/2} + \cdots + A_n t^{n/2} + K_i t \qquad (1.21)$$

The last term expresses the flux at $\theta = \theta_i$, since

$$\int_0^\infty (\theta - \theta_i)\,dz = \int_{\theta_i}^{\theta_0} z\,d\theta = I(t) - K_i t \qquad (1.22)$$

Parlange´s (1971) approach was different. He integrated in the first step the equation of continuity, the result was combined with the Darcy-Buckingham equation, integrated again and this first approximate z_1 was used in the next second step etc.

The method was modified by Císler (1974) and by Philip and Knight (1974):

Philip (1973) first discussed the flux concentration relation (Morel-Seytoux, 1971)

$$F = \frac{\int_{\theta_i}^{\theta} \eta(\theta)\, d\theta}{\int_{\theta_i}^{\theta_0} \eta(\theta)\, d\theta} = \frac{q}{q_0} \qquad (1.23)$$

in various types of the unsaturated flow processes and for the envelope of possible solutions:

(a) the linear soil, D = const.,

(b) soil with $D(\theta)$ according to Dirac δ -function.

Further on, Philip and Knight (1974) have proposed an iterative scheme for the solution of the horizontal infiltration (absorption) with sorptivity expressed as

$$S_n = \left[2 \int_{\theta_i}^{\theta_0} \frac{(\theta - \theta_i)\, D(\theta)}{F_n(\theta)}\, d\theta\right]^{1/2} \qquad (1.24)$$

The index n denotes the n-th approximate, F_n is à priori not known except of linear soil and Dirac δ-function soil. Generally, $F\left(D, \theta, \theta_i, \theta_0\right)$. For the solution of the vertical infiltration, they started with an analogic development as for absorption and finally they obtained a relatively laborious iteration scheme. The principle disadvantage in applications is that $I(t)$ is not explicitly formulated.

Parlange et al. (1982) have utilized all tools up to now developed in the solution of the vertical infiltration. The procedure was extended to the ponded infiltration (Parlange et al., 1985) and modified (Haverkamp and Parlange, 1987). The description of the solution of the "double integration method" will be subdivided into 5 steps:

1st step: Richard´s eq. with a changed dependent variable is integrated. The explicit expression for z is obtained and the flux concentration ratio is introduced.

2nd step: In the cumulative infiltration $I = \int z\, d\theta$ the previous expression (1st integral of Richards´ eq.) is substituted. $D(\theta)$ is subdivided into two domains: 1. domain with $D(\theta)$ continuous. 2. domain with D defined by the Diarc δ-function. This expression is transcribed into saturated conductivity and the water entry value H_w is introduced here.

3rd step: In the previous expression a difficult solvable integral occurs. It is treated now by a shape factor. This parameter occurs together with $F(\theta)$ when the relative conductivity is related to the relative sorptivity. After substitution of a new variable in the relative sorptivity and derivation, the integral from the 2nd step is integrated.

4th step: The correction owing to the saturated zone at the surface is performed by Darcy´s law.

5th step: The equation is rewritten to dimensionless terms and I is obtained as a function inter alia of q_o. After derivation, rearrangement and integration, the final solution is obtained with the shape factor from the 3rd step equal unity:

$$[I - K_i t] = \frac{K_S (H_o - H_A)(\theta_o - \theta_i)}{q_o - K_S} + \frac{S^2 + 2 H_A K_S (\theta_o - \theta_i)}{2(K_S - K_i)} \ln\left[1 + \frac{K_S - K_i}{q_o - K_S}\right] \qquad (1.25)$$

$$t = \frac{K_S (H_o - H_A)(\theta_o - \theta_i)}{(q_o - K_S)(K_S - K_i)} - \frac{S^2 + 2 K_S H_A (\theta_o - \theta_i)}{2(q_o - K_i)(K_S - K_i)} + \frac{S^2 - 2K_S (H_o - 2H_A)(\theta_o - \theta_i)}{2(K_S - K_i)^2} \ln\left[1 + \frac{K_S - K_i}{q_o - K_S}\right] \qquad (1.26)$$

5.2. Approximate solutions

When the convergence of Eq.(1.21) is considered, a simple Philip´s algebraic equation (Philip, 1957) is obtained

$$I = S t^{1/2} + A t \qquad (1.27)$$

where A includes theoretically A_2, K_i and the truncation error. Philip (1969) has shown that A is related to K_s, generally $K_S/3 < A \lesssim 2/3\, K_S$. However, for large time $A \to K_s$.

In the method of Green and Ampt (1911), the soil water content profile is simplified to the step-like profile (Fig.11) with H_f acting as the accelerating force on the wetting front, with (**Neumann, 1976**)

$$H_f = \frac{1}{K_S} \int_0^{H_i} K(H)\, dH = \frac{1}{K_S} \int_{\theta_o}^{\theta_i} D(\theta)\, d\theta \qquad (1.28)$$

or

$$H_f = \frac{1}{2} \int_0^{H_i} \frac{\theta_o + \theta - 2\theta_i}{\theta_o - \theta_i} \, \frac{K(H)}{K_S} \, dH \qquad (1.29)$$

As the whole step-soil water profile is the saturated domain of the soil, infiltration is solved by the application of Darcy´s law:

$$q_o = K_A \frac{H_o - H_f + L_f}{L_f} \qquad (1.30)$$

Here, K_A is the approximate of K_S. $K_A \approx K_S$.

Philip (1957 and 1973) has shown that the step-like profile is exact when D-θ is the Dirac's δ-function, and then $K_A = K_S$.

As $q_0(t)$ and $L_f(t)$, too, $q_0(t)$ is expressed as dI/dt, $I = L_f(\theta_s - \theta_i)$, and after integration

$$t^* = I^* - \ln(1 + I^*) \qquad (I.31)$$

with

$$t^* = \frac{K_A\, t}{(\theta_s - \theta_i)(H_0 - H_f)} \qquad I^* = \frac{I}{(\theta_s - \theta_i)(H_0 - H_f)} \qquad (I.32)$$

since for horizontal infiltration $I = St^{1/2}$ with

$$S = \left[2 K_A (H_0 - H_f)(\theta_s - \theta_i)\right]^{1/2} \qquad (I.33)$$

eq.(I.31) can be transcribed to

$$K_A t = I - \frac{S^2}{t} \ln\left[1 + \frac{2 I K_A}{S^2}\right] \qquad (I.34)$$

or

$$I = K_A t + \lambda \ln\left[1 + \frac{I}{\lambda}\right] \qquad (I.35)$$

with $\lambda = (H_0 - H_f)(\theta_s - \theta_i)$.

From the Philip's serie solution (I.21) Kutílek and Krejča (1987) obtained the three parameters equation

$$I = C_1 t^{1/2} + C_2 t + C_3 t^{3/2} \qquad (I.36)$$

for dq/dt → 0 is

$$t_{lim} = \frac{C_1}{3C_3} \qquad (I.37)$$

and as approximation for $q(t_{lim}) \approx K_S$ is

$$K_S \approx (3C_1C_3)^{1/2} + C_2 \qquad (I.38)$$

Swartzendruber (1987) has adjusted the Philip's serie solution (I.21) to the limit t → ∞ through the exponential form and he got through this fully empirical step

$$I = \frac{S}{A_0}\left[1 - \exp\left(-A_0 t^{1/2} - B_0 t - C_0 t^{3/2} \ldots\right)\right] + K_S t \qquad (I.39)$$

When he neglects members of higher order, he gets

$$I = \frac{S}{A_0}\left[1 - \exp\left(-A_0 t^{1/2}\right)\right] + K_S t \qquad (I.40)$$

A simplification of (I.40) is the Stroosnijder's equation (1976). If we express $\exp(-A_0 t^{1/2})$ in eq. (I.40) as a serie and when we consider the first 4 members only, we get after rearrangement

$$I = S t^{1/2} + \bar{A} t + \bar{B} t^{3/2} \qquad (I.41)$$

$$\bar{A} = K_S - S A_0 / 2 \qquad (I.42)$$

$$\bar{B} = (S A_0)^2 / 6 \qquad (I.43)$$

and then using the symbols of (I.36) we get

$$K_S = \left(3/2\; C_1 C_3\right)^{1/2} + C_2 \qquad (I.45)$$

Brutsaert's (1977) equation is based upon the Philip's series solution (I.21), too. After simplifying the original treatment we get

$$I = S\sqrt{t}\;\frac{1}{1 + a\,\frac{K_s\sqrt{t}}{S}} + K_s t \qquad (1.46)$$

with $2/3 \leq a \leq 1$, most frequently $a=2/3$.
In Collis-George´s (1977) derivation, the empirical approaches prevail. He starts with the requirement that for very short time is $I = St^{1/2}$ while for $t \to \infty$ is $dI/dt = k_S$ according to our notation used here. With some simplifications is

$$I = I_0\,(\tanh\; T)^{1/2} + K_s t \qquad (1.47)$$

where $I_0 = S(t_c)^{1/2}$ and t_c is the time of the start of the quasi-steady infiltration, $T = t/t_c$.

5.3. Empirical equations

The empirical relations were proposed for the gradual decrease of the infiltration rate q_o with time t:

$q_o(t)$ as the hyperbole:

Kostiakov´s equation:

$$q_o = C_1 t^{-\alpha} \qquad (I.48)$$

$$I = \frac{C_1}{1-\alpha}\, t^{(1-\alpha)} \qquad (I.49)$$

where C_1, α are the empirical coefficients, C_1 should equal q_{o1}, the infiltration rate after the first time unit, usually at t = 1 min, $0 < \alpha < 1$. The equation is not appropriate to the description of infiltration at large time as $q_o \to 0$ at $t \to \infty$. To overcome this incovenience, Mezencev (1948) proposed the shift of q_o axis:

$$q_o = C_2 + C_3\, t^{-\beta} \qquad (I.50)$$

$$I = C_2 t + \frac{1}{1-\beta}\, C_3\, t^{(1-\beta)} \qquad (I.51)$$

where C_2, C_3, β are empirical coefficients. For $t \to \infty$ is $C_2 \to q_{oc}$, the constant infiltration rate, when the quasi-steady infiltration is reached. Theoretically $v_c = K_s.\,(C_2 + C_3)$ should equal q_{o1}, the infiltration rate after the first time unit and $0 < \beta < 1$.

Exponential decay of $q_o(t)$:

Horton's (1940) equation:

$$q_o = C_4 + C_5 \exp(-\gamma t) \qquad (I.52)$$

$$I = C_4 t + \frac{1}{\gamma} C_5 \left[1 - \exp(-\gamma t)\right] \qquad (I.53)$$

where C_4, C_5 and γ are empirical coefficients. At $t = 0$ has q_o a final value, in contradiction to the theory of flow in porous media. For $t \to \infty$ is $C_4 \to q_{oc}$ and $C_4 = K_s$ is frequently supposed.

Decay of q_o with I:

Holtan's (1961) equation:

$$q_o = C_6 (W - I)^{\varepsilon} + C_7 \qquad (1.54)$$

where C_6, C_7, ε are empirical coefficients, $C_7 = q_{oc}$, W is soil water storage above the impeding layer(?) ε is not integer, most frequently $\varepsilon > 1$.

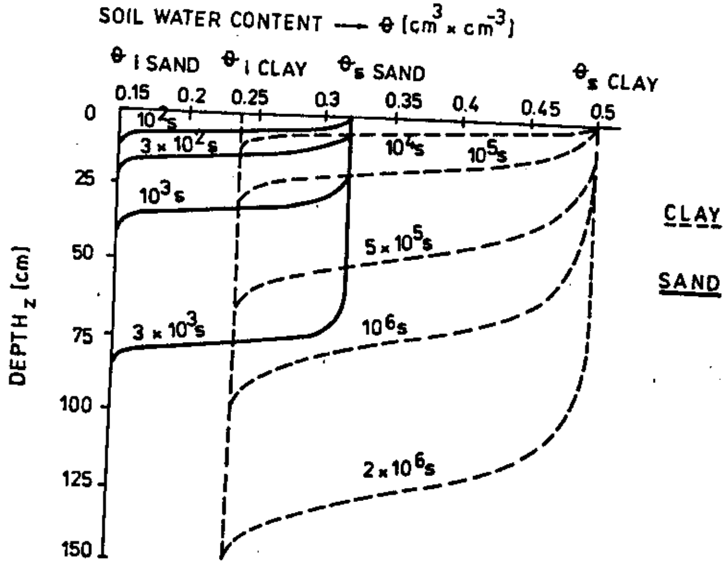

FIG. 1/I

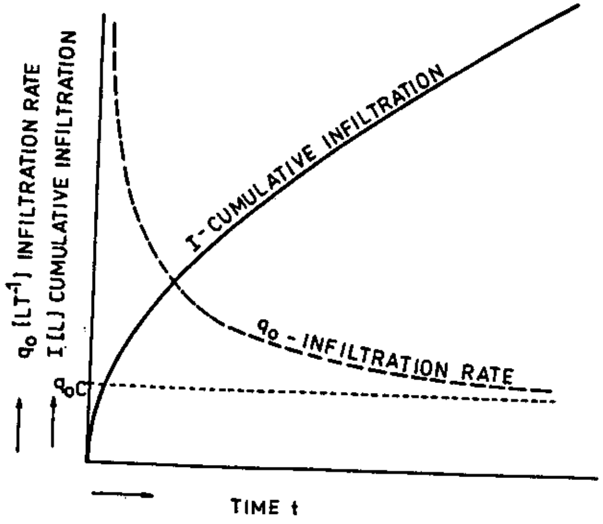

FIG. 2/I

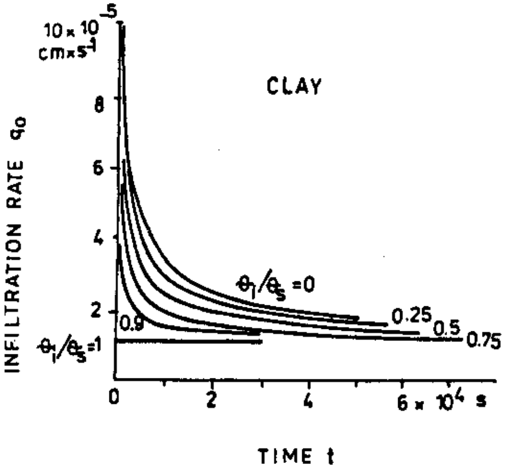

FIG 3/I

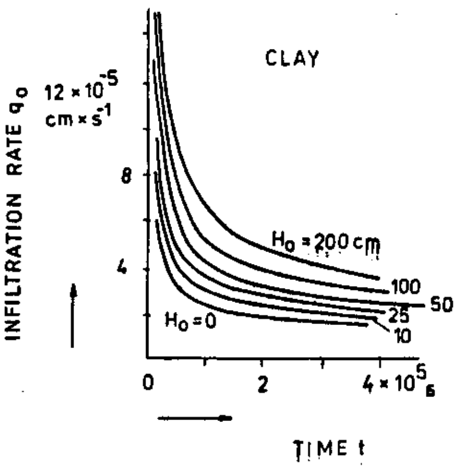

FIG. 4/I

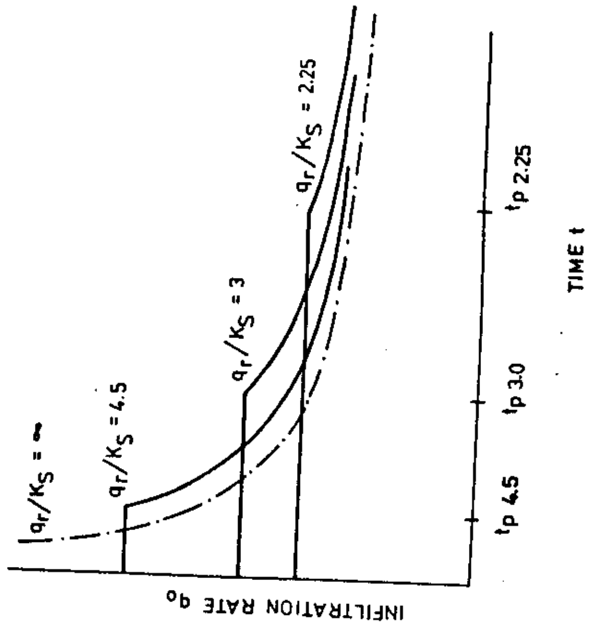

FIG. 7/I

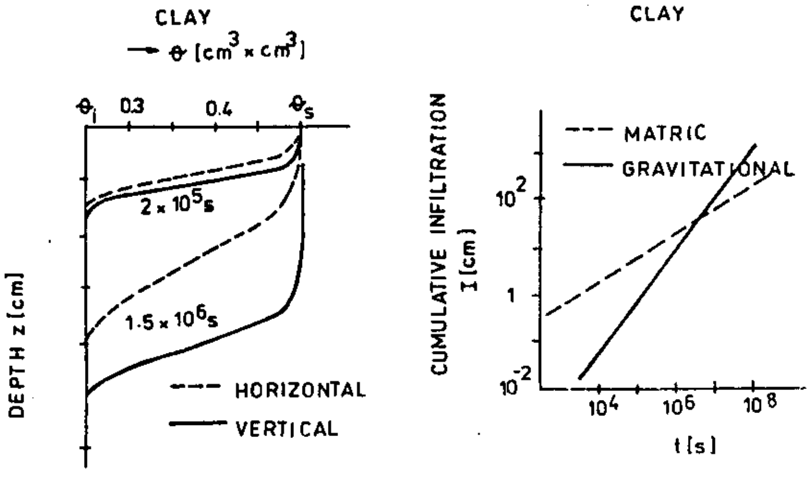

FIG. 5/I

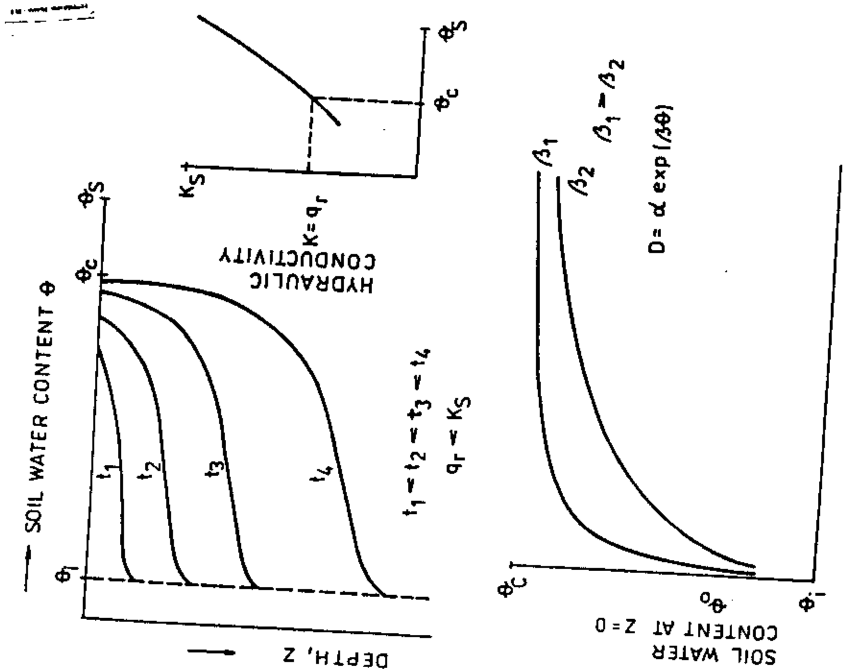

FIG. 6/I

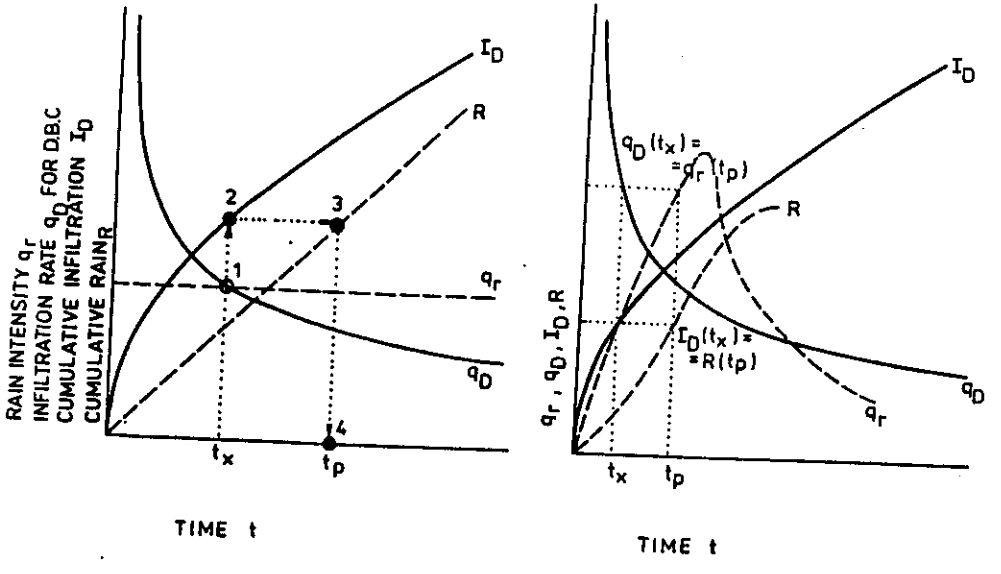

FIG. 8 /I

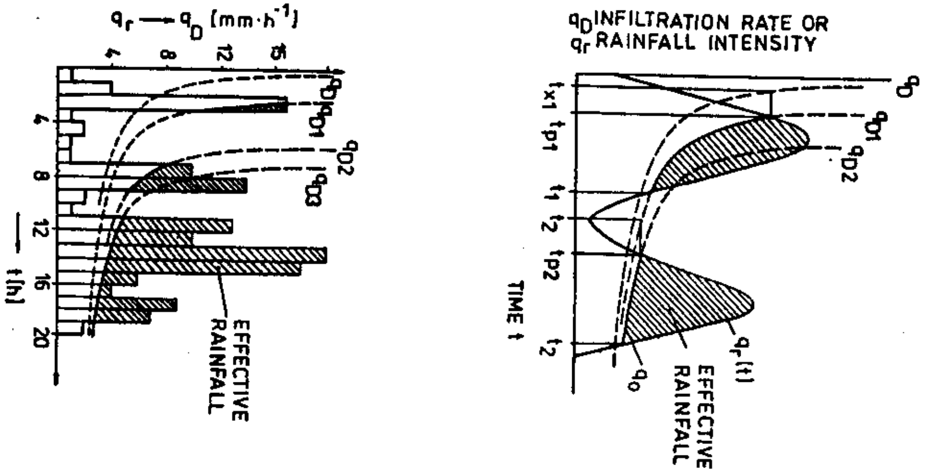

FIG. 9 /I

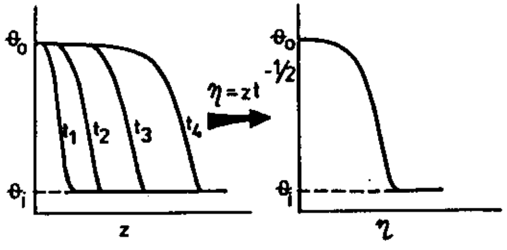

FIG. 10 /I

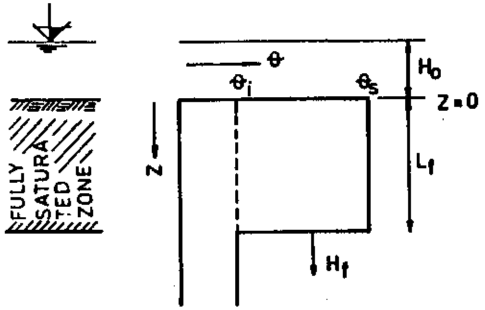

FIG. 11 /I