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The geometry, topology and existence of periodic minimal surfaces

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The Geometry, Topology and Existence of Periodic Minimal Surfaces (Lectures at Luminy)

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Contents

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1 Survey of the main theorems

1.1 Introduction

In recent years there has been great progress in the theory of properly embedded periodic minimal surfaces in $\mathbb{R}^3$. In this period there has been a simultaneous and balanced development of global theoretical results with the construction of rich collections of examples satisfying the hypotheses of the new theorems. Also, in this period, some of these special surfaces have been rediscovered by physical scientists, who have used them to model the geometry of liquid crystals and surface interfaces in microemulsions of block copolymers (see [2] and Section 10 of [44]).

This paper is based on my lectures at the geometry conference at Luminy in July 1989. It will cover the main theoretical results in this subject as well as the related technical tools that are used in proving the main theorems. A majority of the new theorems and examples can be found in the work of Callahan-Hoffman-Meeks [6, 7], Frohman [16], Frohman-Meeks [18, 19], Hoffman-Meeks [27, 31], Hass-Pitts-Rubenstein [23], Karcher [37, 39, 40], Karcher-Pitts [41], Meeks [44] and Meeks-Rosenberg [49, 50, 51]. Except for one new theorem, which will be explained in the next paragraph, the material in the present paper can be found in the above papers.

The only really new result that appears in this paper is Theorem 5.2 in Section 5.2. This theorem states that a properly embedded minimal surface in $\mathrm{T} \times \mathbb{R}$, T a flat two-torus, can have only a finite number of ends. The main application of this result is to give a slight improvement of the finite total curvature theorem of Meeks and Rosenberg for doubly-periodic minimal surfaces.

1.2 The finite total curvature theorem of Meeks and Rosenberg and applications

In this section I shall discuss a surprising relationship between the topology of a properly embedded periodic minimal surface in $\mathbb{R}^3$ and its global geometry. We shall call a minimal surface *periodic* if it is connected and invariant under a discrete group G of isometries that acts freely on $\mathbb{R}^3$. We will analyze these surfaces by studying their quotient surfaces in $\mathbb{R}^3/G$.

Recall that a surface has *finite topology* if it is homeomorphic to a closed surface with a finite number of points removed. The main theorem in [49] is:

Theorem 1.1 *A properly embedded minimal surface in a complete nonsimply connected flat three-manifold has finite total curvature if and only if it has finite topology.*

When the flat manifold is $\mathbb{R}^3$, the existence of the helicoid (which has finite topology and infinite total curvature) demonstrates that the condition that N be nonsimply connected is a necessary one. Theorem 1.1 has important topological and analytical consequences. One topological consequence is that a properly embedded orientable minimal surface of finite topology in an orientable flat nonsimply connected three-manifold always has an even number of ends or it is a plane (see Theorom 8 in [49]).

A theorem of Huber [34] states that a complete Riemannian surface with nonpositive Gaussian curvature whose total curvature is finite must be conformally diffeomorphic to a closed Riemann surface punctured in a finite number of points. Meeks and Rosenberg [49] proved that a complete minimal surface of finite total curvature in a flat three-manifold can be described in terms of meromorphic data on its conformal compactification. They went on to exploit these analytic conditions to prove the following uniqueness theorem.

Theorem 1.2 *The plane and the helicoid are the only properly embedded simply connected minimal surfaces in* $\mathbb{R}^3$ *with infinite symmetry group.*

Earlier in [51] Meeks and Rosenberg proved Theorem 1.1 in the case where the flat three-manifold N was isometric to the product $\mathsf{T}\times\mathsf{R}$ where T is some flat torus. In fact they proved that a properly embedded minimal surface M in $\mathsf{T}\times\mathsf{R}$ has finite total curvature $C(M) = 2\pi\chi(M)$. It follows from the classification of flat three-manifolds that a flat, noncompact, nonsimply-connected three-manifold is finitely covered by $\mathsf{T}\times\mathsf{R}$ or by R^3/S_θ where S_θ is the right hand screw motion obtained by rotation around the positive x_3-axis by θ followed by a nontrivial translation along the x_3-axis. Thus, to prove Theorem 1.1, it remained to consider only the case where the manifold N is isometric to R^3/S_θ for some θ, $0 \le \theta \le \pi$. However, the proof of Theorem 1.1 in [49] does not actually depend on the previous theorem in the special case of $\mathsf{T}\times\mathsf{R}$.

In Section 5 we shall prove that a properly embedded minimal surface $M \subset \mathsf{T}\times\mathsf{R}$ has a finite number of ends even when it has infinite topological type. This result together with Theorem 1.1 shows that such an M has finite total curvature if and only if it has finite genus.

We now describe the geometry of the annular ends of $M \subset \mathsf{T}\times\mathsf{R}$. Suppose $A \subset \mathsf{T}\times\mathsf{R}$ is a properly embedded minimal annulus with one compact boundary component and A finite total curvature. In this case A is asymptotic to a flat annulus in $\mathsf{T}\times\mathsf{R}$. This asymptotic behavior of A has important topological and geometrical consequences for a properly embedded minimal surface of finite genus in $\mathsf{T}^2\times\mathsf{R}$, which we now describe. We begin with the following definition.

Definition 1.1 $\mathsf{T}\times\mathsf{R}$ *has a* **commensurable** *lattice if* $\mathsf{T}\times\mathsf{R} = \mathsf{R}^3/\Lambda$ *and* Λ *contains two linearly independent vectors of equal length.*

Theorem 1.3 *Suppose* $M \subset \mathsf{T}\times\mathsf{R}$ *is a properly embedded minimal surface of finite topological type that is not flat. Then:*

1. *If M is orientable, then M separates $\mathsf{T}\times\mathsf{R}$. In this case the number of top ends of M, as well as the number of bottom ends of M, is even. In particular, M has at least four ends.*

2. *If M is nonorientable, then the number of top ends, as well as the bottom ends, is odd. In particular, whether M is orientable or nonorientable, the number of ends of M is even.*

3. *The top ends of M are parallel to the bottom ends of M if and only if the subgroup of $H_1(\mathsf{T}\times\mathsf{R})$ generated by the loops on the ends of M is cyclic. If the ends of M are parallel, then the number of top ends of M equals the number of bottom ends. In particular, by part 1, if M is orientable and has parallel ends, then the number of ends is a multiple of four.*

4. *If the ends of M are not parallel, then they are vertical and $\mathsf{T}\times\mathsf{R}$ has a commensurable lattice.*

The next theorem gives necessary conditions for a given doubly-periodic minimal surface to have nonparallel ends, which by part 4 of Theorem 1.3 forces the ambient space to have a commensurable lattice.

Theorem 1.4 *Suppose* $M \subset \mathsf{T}\times\mathsf{R}$ *is a properly embedded minimal surface of finite topological type that is not flat. Then the top ends of M are not parallel to the bottom ends of M if 1, 2 or 3 holds:*

1. *M is orientable and the number of ends is not a multiple of four.*

2. *M is a planar domain.*

3. $\chi(M)$ *is odd.*

Meeks and Rosenberg [51] also study the rigidity of doubly-periodic minimal surfaces.

Theorem 1.5 (Rigidity Theorem) *Suppose $f: M \to \mathbb{R}^3$ is a connected properly embedded doubly-periodic minimal surface. If $F: M \to \mathbb{R}^3$ is another isometric minimal immersion of M, then $F = I \circ f$ where I is an isometry of $\mathbb{R}^3$. In particular, intrinsic isometries of a properly embedded doubly-periodic minimal surface extend to isometries of $\mathbb{R}^3$.*

Theorem 1.5 fails to hold for singly-periodic minimal surfaces in $\mathbb{R}^3$ since it fails to hold for the helicoid. However, under the assumption that the quotient surface has finite topology, the last statement of Theorem 1.5 can be generalized. More precisely,

Theorem 1.6 ([44]) *Let M be a connected, properly embedded, minimal surface in $\mathbb{R}^3$, invariant under an infinite discrete group G of isometries of $\mathbb{R}^3$. If M/G has finite topology, then every isometry of M extends to an isometry of $\mathbb{R}^3$.*

Meeks and Rosenberg [49] also gave the following classification of the asymptotic geometry of the annular ends of the surfaces described in Theorem 1.1. As shown by work in [6, 7, 38, 41], every $\mathbb{R}^3/S_\theta$ has many examples with each possible end type.

Theorem 1.7 *An annular end of a properly embedded minimal surface of finite topology in $\mathbb{R}^3/S_\theta$ is asymptotic to a plane, a flat vertical annulus, or to an end of a helicoid (with horizontal limit tangent plane). If θ is nonzero and the end is asymptotic to a plane, then the plane is horizontal. If θ is irrational, then the end is not asymptotic to a flat vertical annulus.*

The total curvature of minimal surfaces of finite topology in $N = \mathbb{R}^3/S_\theta$ can be computed in terms of the winding numbers of its annular ends. Suppose A is the image of a proper embedding of the punctured disk D^* in N. Let γ be the geodesic representing the image of the x_3-axis in N. After removing a compact neighborhood of ∂A, we may assume that A is disjoint from the ε-tubular neighborhood T of γ with boundary torus ∂T. The torus is obtained as a quotient by S_θ of the flat cylinder C of distance ε from the x_3-axis. A basis for $\pi_1(\partial T)$ is obtained from the quotient α of the oriented circle $C \cap \mathbb{R}^2$ and the quotient β of the oriented right handed helical arc of least-length on C joining a point p with $S_\theta(p)$. The boundary curve of A is homotopic in $N - \gamma$ to a unique element of $\pi_1(\partial T)$. Suppose ∂A is homotopic to $n\alpha + m\beta$. The *winding number* of the end of A is then defined to be $\frac{1}{2\pi}|2\pi \cdot n + m \cdot \theta|$. If M is a complete embedded minimal surface of finite total curvature in $\mathbb{R}^3/S_\theta$, then define the *total winding number of M* to be the sum of the winding numbers of the ends of M. We let $W(M)$ denote the total winding number of M.

Theorem 1.8 ([49]) *If M is a properly embedded minimal surface of finite topological type in $\mathbb{R}^3/S_\theta$, then the total curvature of M is*

$$C(M) = 2\pi(\chi(M) - W(M)).$$

When the ends are asymptotic to flat vertical annuli, this formula yields $C(M) = 2\pi\chi(M)$. When there are k planar ends, $C(M) = 2\pi(\chi(M) - k)$.

1.3 Existence and global properties of singly periodic minimal surfaces with more than one end

In the special case of Theorem 1.8 where the lifted surface in $\mathbb{R}^3$ has an infinite number of ends, Callahan, Hoffman and Meeks [6] proved earlier that M has finite total curvature, it has k planar ends and $C(M) = 2\pi(\chi(M) - k)$.

The first examples of periodic minimal surfaces in $\mathbb{R}^3$ with an infinite number of ends were found by Riemann. Riemann [69] classified all minimal surfaces that can be expressed as a union of round circles in parallel planes. He proved that there exists a one-parameter family $\mathcal{M}_t$, $t \in (0, \infty)$, of singly-periodic minimal surfaces with the following property: Up to rigid motion and homothety, every minimal surface expressible as a union of circles in parallel planes is either a subset of some $\mathcal{M}_t$ or a subset of the catenoid. It is immediate that $\mathcal{M}_t$ are planar domains (homeomorphic to a subset of the plane) with an infinite number of annular ends and two limit ends. Callahan, Hoffman and Meeks have been able to generalize these fascinating surfaces [7].

Theorem 1.9 *For every positive integer k there exists a properly embedded minimal surface M_k with the following properties:*

1. *M_k has an infinite number of annular ends.*
2. *M_k is invariant under the group of translations $\mathbf{T}$ generated by $T: \vec{x} \mapsto \vec{x} + (0, 0, 2)$.*
3. *$\mathsf{M}_k/\mathbf{T}$ has genus $2k+1$ and two ends.*
4. *The symmetry group of $\mathsf{M}_k/\mathbf{T}$ has order $8(k+1)$.*
5. *Reflection in the plane $\{x_3 = n + \frac{1}{2}\}, n \in \mathbb{Z}$, is a symmetry of M_k.*
6. *$\mathsf{M}_k/\mathbf{T}$ has finite total curvature $-4\pi(2k+2)$.*
7. *All the ends of M_k are flat; they are asymptotic to the planes $x_3 = n$, $n \in \mathbb{Z}$.*
8. *$\mathsf{M}_k \cap \{x_3 = n\}$, $n \in \mathbb{Z}$, consists of $k+1$ equally spaced straight lines meeting at $(0, 0, n)$.*

9. *$\mathsf{M}_k \cap \{x_3 = c\}$, $c \notin \mathbb{Z}$ is a simple closed curve.*
10. *The subgroup $\mathbf{R}$ of the symmetry group of M_k consisting of rotations about the x_3-axis has order $k+1$ and is generated by rotation by $2\pi/(k+1)$.*
11. *M_k is symmetric under reflection through the $k+1$ vertical planes containing the x_3-axis and bisecting the lines of property 8.*
12. *The full symmetry group of M_k is generated by $\mathbf{T}, \mathbf{R}$, one of the reflections in 5., rotation about one of the lines in 8., and reflection through one of the planes in 11.*

Recently, Callahan, Hoffman and Meeks [6] have developed a theory to deal with existence and global properties of properly embedded singly-periodic minimal surfaces with an infinite number of ends. In particular they prove that every $N = \mathbb{R}^3/S_\theta$, S_θ a screw motion, admits for every odd positive integer k greater than 1, a properly embedded minimal surface $\mathsf{M}_k(\theta)$ of genus k with two ends, each end asymptotic to a horizontal plane in N. The examples $\mathsf{M}_k(\theta)$ are constructed by "twisting" the examples M_k in Theorem 1.9 by an angle θ around the x_3-axis.

Callahan, Hoffman and Meeks also study the global geometry of periodic minimal surfaces with an infinite number of ends. Before stating their main structure theorem, we recall the definition in [6] of a limit tangent plane. Suppose M is a properly embedded minimal surface in $\mathbb{R}^3$ and Σ is a properly embedded noncompact orientable minimal surface of finite total curvature, contained in the closure of one of the components of $\mathbb{R}^3 - M$, and such that $\partial\Sigma$ is compact and contained in M. Since Σ is embedded and has finite total curvature, it has a finite number of planar and catenoid type ends that are disjoint and parallel; the ends of Σ have the same limit tangent plane at infinity. This tangent plane passing through the origin is called a *limit tangent plane* of M. When M has more than one end, it is shown in Theorem 5 of [6] that M has a limit tangent plane and this plane is unique. Hence, when M has more than one end, we can speak of *the* limit tangent plane of M.

Theorem 1.10 (Structure Theorem [6]) *Suppose M is a properly embedded minimal surface in $\mathbb{R}^3$ with infinite symmetry group and more than one end. Then either M is a catenoid or else M possesses the following properties:*

1. *$\mathrm{Sym}(M)$ contains an infinite, cyclic subgroup $\mathbf{T}$ of finite index, generated by a screw motion T.*
2. *There exists a plane P, parallel to the limit tangent plane of M, whose intersection with M consists of a finite number of simple closed curves.*
3. *If the screw motion T has nontrivial rotational part, the limit tangent plane of M is orthogonal to the axis of T.*

Remark 1.1 *It should be pointed out that if T is a pure translation, the plane P in property 2 of the theorem is not necessarily orthogonal to the direction of T. For example, the axis of T in Riemann's example is not orthogonal to the associated plane.*

An immediate consequence of part 1 of the above structure theorem is:

Corollary 1.1 *A doubly-periodic minimal surface in $\mathbb{R}^3$ has one end.*

1.4 The topological uniqueness of complete one-ended minimal surfaces and Heegaard surface in $\mathbb{R}^3$

Frohman and Meeks [19] proved two fundamental theorems on the topological uniqueness of certain surfaces in $\mathbb{R}^3$. The first of these theorems, which will depend on the second theorem, shows that a properly embedded minimal surface in $\mathbb{R}^3$ with one end is unknotted. More precisely,

Theorem 1.11 *Two properly embedded one-ended minimal surfaces in $\mathbb{R}^3$ of the same genus are ambiently isotopic.*

Earlier Meeks [47] proved Theorem 1.11 in the case of finite genus. In this case the only known examples are the plane and the helicoid. However, the collection of properly embedded minimal surfaces of infinite genus and one end is extremely rich. In fact, except for the one-periodic examples of Riemann [69] (also see [7] for a description and computer graphics image of one of these surfaces) and the recent one-periodic examples of Callahan, Hoffman, and Meeks [6, 7], all known examples of infinite genus properly embedded minimal surfaces in $\mathbb{R}^3$ have one end. One reason for this is that most classical examples of these surfaces are doubly-periodic (i.e., they are invariant under translation in at least two independent directions) and in this case, Corollary 1.1 given above implies that the surface has one end and infinite genus when it is not a plane. Their result and Theorem 1.11 have the following corollary.

Corollary 1.2 *Any two properly embedded nonplanar minimal surfaces in $\mathbb{R}^3$ that are invariant under at least two linearly independent translations are ambiently isotopic.*

Essential in understanding the uniqueness theorems in this paper is the concept of a Heegaard surface in a noncompact three-manifold, which generalizes the usual notion of a Heegaard surface M in a closed 3-manifold

N^3. Recall that M is called a *Heegaard surface* if it separates N^3 into two genus-g handlebodies where g is the genus of M. (A handlebody of genus g is also frequently referred to in the literature as a solid g-holed torus, a g-holed doughnut or a pretzel.) Every closed three-manifold N^3 contains a Heegaard surface and so it can be constructed by glueing two handlebodies together along their boundary (see [24] for details).

Noncompact three-manifolds such as $\mathbb{R}^3$ fail to have compact Heegaard surfaces. However, there is a natural notion of Heegaard surface for these manifolds where the surface is allowed to be noncompact. We say that a properly embedded, one-ended surface M is a *Heegaard surface* in $\mathbb{R}^3$ if the closures of each of the two components of $\mathbb{R}^3 - M$ are one-ended handlebodies. This definition of course depends on the definition of a one-ended handlebody which we now give. A *one-ended handlebody of genus* g, $0 \leq g \leq \infty$, is a three-manifold with boundary that is diffeomorphic to the submanifold of $\mathbb{R}^3$ obtained by attaching, in a proper manner, g trivial 1-handles to the closed lower halfspace H in $\mathbb{R}^3$. When $g = \infty$, this attaching of handles on H can be performed on neighborhoods of the integer points on the x-axis contained in ∂H to obtain a one-periodic Heegaard surface in $\mathbb{R}^3$ (see Figure 1).

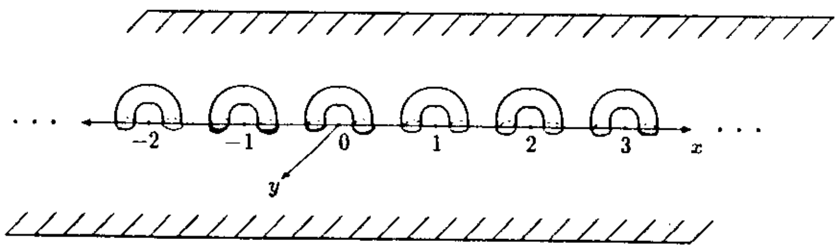

Figure 1:

The second main theorem of Frohman and Meeks is

Theorem 1.12 *Heegaard surfaces of the same genus in $\mathbb{R}^3$ are ambiently isotopic. Equivalently, given two diffeomorphic Heegaard surfaces in $\mathbb{R}^3$, there exists a diffeomorphism of $\mathbb{R}^3$ that takes one surface to the other surface.*

Frohman and Meeks prove that a properly embedded one-ended minimal surface in $\mathbb{R}^3$ is a Heegaard surface. This result, together with Theorem 1.12, proves Theorem 1.11.

We now wish to put Theorems 1.11 and 1.12 in historical perspective. First Waldhausen [82] proved the topological uniqueness of Heegaard surfaces in the unit three-sphere $S^3 \subset \mathbb{R}^4$. Later Lawson [43], using an argument of Frankel [14], proved that two closed minimal surfaces of genus g in S^3 are Heegaard surfaces and hence isotopic by Waldhausen's theorem. He was able to prove this same result whenever S^3 was endowed with a metric of positive Ricci curvature. Meeks [47] generalized Lawson's argument to the nonnegative Ricci curvature case. Finally, Meeks, Simon and Yau [52] proved that two diffeomorphic closed minimal surfaces in S^3 endowed with a metric of nonnegative scalar curvature are isotopic in S^3. Meeks [47] also proved some related topological uniqueness theorems for compact minimal surfaces with boundary in $\mathbb{R}^3$. More recently, Meeks and Yau [54] have proven a topological uniqueness result for properly embedded surfaces that is closely related to Theorem 1.11. Their main theorem states that two proper diffeomorphic minimal surfaces in $\mathbb{R}^3$ of finite topology are ambiently isotopic.

2 Related conjectures and unsolved problems

We first discuss some related questions for properly embedded minimal surfaces in $\mathbb{R}^3$ of finite topology.

Question 2.1 *Is the helicoid the only properly embedded minimal surface in $\mathbb{R}^3$ with finite topology but infinite total curvature?*

Question 2.2 *What are the possible topological types for properly embedded minimal surfaces in $\mathbb{R}^3$?*

The collection of properly embedded minimal surfaces of finite topology in $\mathbb{R}^3$ is very rich as demonstrated by the work in [5, 9, 28, 30, 32, 33, 68]. As Question 2.1 suggests, all of the new examples have finite total curvature. The finite total curvature property of a complete minimal surface $f: M \to \mathbb{R}^3$ has deep analytic and conformal consequences. In particular it follows from the work of Osserman [64, 66] that M is conformally diffeomorphic to a closed Riemann surface $\overline{M}$ punctured a finite number of points and the coordinates of $f(M)$ can be recovered analytically from a meromorphic 1-form on $\overline{M}$ and a meromorphic function $\overline{g} : \overline{M} \to \mathbb{C} \cup \{\infty\}$. Here, $\overline{g}$ is the extension to $\overline{M}$ of the stereographic projection of the Gauss map of $f(M)$. (See Section 3.6 for this representation.) When $f(M)$ is an embedded surface, this analytic representation of f implies each end of $f(M)$ is asymptotic to a catenoid end or to a plane [77].

To understand Question 2.1, it is necessary to understand the geometry of annular ends[1] of a properly embedded minimal surface in $\mathbb{R}^3$. When M has finite topology, all of its ends are annular. It was conjectured by Hoffman and Meeks that:

[1]An annular end of a surface M is a proper differentiable embedding of the punctured disk $D^* = \{z \in \mathbb{C} \mid 0 < |z| \leq 1\}$ into M.

Conjecture 2.1 *Suppose $M \subset \mathbb{R}^3$ is a properly embedded minimal surface with at least 2 ends. Then every annular end of M has finite total curvature.*

It would follow from Conjecture 2.1 that a properly embedded minimal surface with at least two ends and with finite topology has finite total curvature. Recently Hoffman and Meeks [27] proved that a properly embedded minimal surface in $\mathbb{R}^3$ can have at most two ends with infinite total curvature. Thus, to prove Conjecture 2.1, it remains to prove that the remaining two ends also have finite total curvature.

A more specific question related to Question 2.1 and Conjecture 2.1 is the following:

Question 2.3 *Suppose M is a properly embedded minimal surface of genus zero. Is M a plane, a catenoid, the helicoid or is it one of Riemann's one-periodic examples?*

For periodic minimal surfaces we conjecture

Conjecture 2.2 *Question 2.3 has an affirmative answer when the minimal surface has an infinite symmetry group.*

Now consider a properly embedded minimal planar domain M in $\mathbf{T} \times \mathbb{R}$. It follows from the discussion following the statement of Theorem 1.2 that M has a finite number of ends and hence has finite total curvature. Part 4 of Theorem 1.3 gives still further information on M; namely, the ends of M are vertical and the lattice of $\mathbf{T}$ is commensurable. For even n, $n \geq 4$, one can choose appropriate sublattices L_n of the lattice for a Scherk's doubly-periodic minimal surface $\mathcal{S} \subset \mathbb{R}^3$, so that $\mathcal{S}/L_n$ is a properly embedded genus 0 minimal surface with n ends in $\mathbb{R}^3/L_n = \mathbf{T}_n \times \mathbb{R}$. Meeks and Rosenberg [51] proved that a properly embedded minimal surface M of genus 0 and 4 ends in $T \times R$ lifts to a Scherk surface and F. Wei (personal communication) has shown the similar result when M has 6 ends. These results motivate the following

Conjecture 2.3 *A properly embedded minimal surface of genus 0 in $T \times R$ lifts to a Scherk surface in R^3.*

The rigidity results for periodic minimal surfaces given in Theorems 1.5 and 1.6 almost certainly can be improved. We conjecture

Conjecture 2.4 *If $f: M \to R^3$ is a properly embedded, nonsimply connected, minimal surface, then any other isometric minimal immersion of M in R^3 is congruent to f.*

Conjecture 2.5 *The symmetry group of a properly embedded minimal surface in R^3 is equal to its isometry group.*

We note that Conjecture 2.4 was proved in the special case M has more than one end [8].

For the upcoming discussion it is important to have a precise definition of an end of a noncompact surface. Intuitively, the ends of a noncompact surface can be thought as the number of different ways to travel to infinity on the surface. More precisely, an end of surface M is a set of equivalence classes of proper arcs on the surface that describe the different ways to travel to infinity. We now recall the definition of these equivalence classes.

Definition 2.1 *Consider two proper arcs $\alpha_1, \alpha_2: [0,\infty) \to M$. Then α_1 is* equivalent *to α_2, written $\alpha_1 \approx \alpha_2$, if there exists an exhaustion $M_1 \subset M_2 \subset \dots$ of M by smooth compact subdomains, such that for every i the noncompact components of $\alpha_1 - \mathrm{Int}(M_i)$ and $\alpha_2 - \mathrm{Int}(M_i)$ are contained in the same component of $M - \mathrm{Int}(M_i)$. The relation $\approx$ is an equivalence relation and*

we denote by $\overline{\alpha}$ the equivalence class of α and we call $\overline{\alpha}$ the end associated to the proper arc α.

With the above definition of end, it is easy to check that a closed surface punctured in n points has n ends, one corresponding to each removed point. It follows that R^2 has one end and the cylinder has two ends. However, in the general case, the structure of the ends of a noncompact surface can be much more complicated as occurs, for instance, in a surface obtained by removing a Cantor set from a closed surface.

In order to work with the ends of a surface, it is useful to make some further definitions.

Definition 2.2 *A smooth proper subdomain Σ of M with $\partial\Sigma$ compact is said to be an* end-representative *for an end $\overline{\alpha}$ of M if $\alpha \cap \Sigma$ is noncompact.*

Note that whether Σ is an end-representative of $\overline{\alpha}$ does not depend on the choice of representative in $\overline{\alpha}$.

On the basis of the topological uniqueness results for properly embedded minimal surfaces discussed in Section 1.4 (see [19] and [54]), one might be tempted to conjecture that two properly embedded diffeomorphic minimal surfaces in R^3 are isotopic. We strongly believe this conjecture to be false but a related conjecture to be true (see Conjecture 2.8 below). In order to understand the statement this related conjecture we first state an ordering theorem for the ends of a properly embedded minimal surface in R^3. The proof of the ordering theorem uses the existence and uniqueness of a limit tangent plane for these surfaces as proved in [6]. A more precise statement of the following theorem appears in Theorem 1.1 in [18].

Theorem 2.1 (Ordering Theorem) *Suppose M is a properly embedded minimal surface in R^3 with more than one end and whose limit tangent plane* *is the (x_1, x_2)-plane. Then the ends of M are naturally ordered by their "height" over the (x_1, x_2)-plane.*

Frohman and Meeks make three fundamental conjectures concerning the topology of properly embedded minimal surfaces with more than one end.

Conjecture 2.6 *Suppose M_1 and M_2 satisfy the hypotheses of M in Theorem 2.1 and $F: \mathbb{R}^3 \to \mathbb{R}^3$ is a diffeomorphism such that $F(M_1) = M_2$. Then F preserves or reverses the natural ordering of the ends of M_1 and M_2. In particular, if M satisfies the hypotheses of Theorem 2.1 and $F: \mathbb{R}^3 \to \mathbb{R}^3$ is a diffeomorphism such that $F(M) = M$, then F preserves or reverses the ordering of the ends of M.*

Conjecture 2.7 *The ordering of ends given in Theorem 2.1 is almost a well-ordering in the sense that it is equivalent to the ordering on a closed subset S of the interval $[0,1]$ with $S \cap (0,1)$ discrete.*

Conjecture 2.8 *Suppose M_1 and M_2 are two properly embedded minimal surfaces with more than one end and horizontal limit tangent plane. A necessary and sufficient condition for M_1 to be isotopic to M_2 is for there to exist a diffeomorphism $f: M_1 \to M_2$ that preserves or reverses the ordering of the ends of these surfaces induced by their minimal embeddings.*

It is important to note that Conjecture 2.7 implies that *a properly embedded minimal surface in $\mathbb{R}^3$ can have at most two limit ends and the total number of ends of the surface is countable.* In particular, the validity of Conjecture 2.7 would show that the surface obtained by taking $\mathbb{C} - \{0, 1\}$ and removing a closed discrete subset of points with limit points at $0, 1$, and ∞ can not properly minimally embed in $\mathbb{R}^3$.

A necessary and sufficient condition for a properly immersed minimal annulus in $\mathbb{R}^3$ with compact boundary to have finite total curvature is for it to have quadratic area growth. More generally,

Definition 2.3 *An end $\overline{\alpha}$ of a surface M in $\mathbb{R}^3$ is said to have* quadratic area growth *if $\overline{\alpha}$ has an end-representative E such that the area of E in balls of radius R is less than $cR^2 + K$ for some constants c and K.*

With this definition we can state the outstanding fundamental conjecture concerning the asymptotic geometry of properly embedded minimal surfaces in $\mathbb{R}^3$ with more than one end that generalizes Conjecture 2.1 and 2.7.

Conjecture 2.9 *Suppose M is a properly embedded minimal surface in $\mathbb{R}^3$ with more than one end and horizontal limit tangent plane and suppose $\overline{\alpha}$ is an end of M. A necessary and sufficient condition for $\overline{\alpha}$ to not have quadratic area growth is that it be a limit end of M and it be a highest or lowest end of M with respect to the the ordering given by the Ordering Theorem.*

3 Technical tools

3.1 Schoen's curvature estimates and global theorems concerning stable minimal surfaces

Recall that a minimal surface is stable if for every compact subdomain Δ and every nontrivial variation fixing $\partial\Delta$, the second derivative of area is positive. In [76] Schoen derived curvature estimates for stable minimal surfaces in Riemannian three-manifolds in terms of the distance to the boundary. These curvature estimates play an important role in the proof of many of the main theorems described in Section 1. In the case the ambient manifold is flat, as is our case, Schoen's curvature estimates can be described by

Theorem 3.1 *There exists a universal constant $c > 0$ such the following holds. Suppose M is a stable orientable minimal surface in a flat orientable three-manifold. Let $d: \mathrm{Int}(M) \to \mathbb{R}^+$ be the intrinsic distance function to the* *boundary of M and $K:\mathrm{Int}(M)\to\mathbb{R}$ be the Gaussian curvature. Then*

$$|K(p)|\le\frac{c}{d^2(p)}.$$

An immediate corollary to Theorem 3.1 is that a complete, stable, orientable, minimal surface in a flat three-manifold is totally geodesic. This result was proved earlier and independently by do Carmo and Peng [10] and by Fischer-Colbrie and Schoen [13] in a more general setting. Fischer-Colbrie [12] generalized her earlier work with Schoen to prove

Theorem 3.2 *Suppose M is a complete orientable minimal surface in a flat orientable three-manifold. Then the index of M with respect to the Jacobi operator (or stability operator) on M is finite if and only if M has finite total curvature.*

Since a complete minimal surface M of finite index is stable after removing a compact subset of M, the above theorem follows from the next theorem (see [12] and [54].

Theorem 3.3 *Suppose M is a stable minimal surface with smooth compact boundary in a flat orientable three-manifold. If M is complete as a metric space in the induced intrinsic distance function, then M has finite total curvature.*

3.2 Existence of least-area surfaces

In the classical Plateau problem one considers a smooth simple closed curve γ in $\mathbb{R}^3$ and asks: Does γ bound a map of a disk of least area? This problem was solved independently by Douglas and Rado. For the Douglas' solution one produces a map of the unit disk in C that minimizes energy and bounds γ; such an energy minimizing map is conformal and least-area. Meeks and

Yau [55, 56] gave the following natural condition for the Douglas solution be embedded: *The Douglas solution is one-to-one when γ is extremal,* i.e., the least-area disk is embedded when γ lies on the boundary of its convex hull. This result is a consequence of a more general result that holds in certain Riemannian three-manifolds with boundary and uses the generalization, due to Morrey [57], that if γ is a simple closed curve in a homogenously regular Riemannian n-manifold N and γ is homotopically trivial in N, then γ is the boundary of a least-energy map of the unit disk (and hence a least-area disk). This solution to Plateau's problem is called the *Morrey solution.* Based on Morrey's work and the proof of Dehn's lemma in three-manifold theory [24], Meeks and Yau generalized their embeddedness theorem for extremal γ in $\mathbb{R}^3$ to more general Riemannian three-manifolds. Before stating their result we describe the notion of a good barrier.

Definition 3.1 *Suppose N is a compact Riemannian three-manifold that embeds in the interior of another Riemannian three-manifold. Suppose that ∂N is geodesically convex or ∂N admits a triangulation by smooth two-simplices with interior angles less than or equal to π and such that the mean curvature of these two-simplices with respect to the outward pointing normal is nonnegative. If ∂N satisfies these conditions, it is said to be* a good barrier for solving Plateau-type problems in N.

Theorem 3.4 ([55, 56]) *Suppose N is a compact Riemannian three-manifold and ∂N is a good barrier for solving Plateau type problems in N. If γ is a simple closed curve in ∂N that is homotopically trivial in N, then:*

1. *γ is the boundary of a least-energy map $f\colon D\to N$.*

2. *Every least-energy map $f\colon D\to N$ with $f(\partial D)=\gamma$ is one-to-one and is a smooth immersion on* Int(D).

3. *f is as regular as γ and f is an immersion on D if γ is C^2.*

4. *If $f_1, f_2: D \to N$ are two least-energy maps, then $f_1(\mathrm{Int}(D)) \cap f_2(\mathrm{Int}(D)) \neq \emptyset$ implies that f_1 and f_2 differ by a conformal reparametrization of D.*

Suppose N is orientable and satisfies the hypotheses of Theorem 3.4 and Γ is a smooth collection of pairwise disjoint simple closed curves in ∂N and Γ is the boundary of an orientable two-chain in N, then geometric measure theory shows that Γ is boundary of a smooth embedded least-area compact surface where least-area means least-area with respect to all orientable two-chains with boundary Γ. Similarly, if Γ is a $\mathbb{Z}_2$-boundary, it is the boundary of a smooth embedded least-area compact surface. See [81] for these results. If Γ is the boundary of a map of a orientable surface into N of genus k, then Γ is the boundary of an embedded surface of genus at most k that is least-area in its homotopy class. This follows from a theorem of D. Gabai [20] that shows Γ is the boundary of an incompressible embedded surface Σ of smallest genus and a theorem of Freedman, Hass and Scott [15] that states there is a least-area embedded surface $\bar{\Sigma}$ with $\partial\bar{\Sigma} = \Gamma$ whose inclusion map induces the same map on fundamental groups as Σ. The existence part of the Freedman-Hass-Scott depends on the existence results Sachs-Uhlembeck [72] or Schoen-Yau [78] .

We now describe one final minimization method that is also useful. When Γ is the boundary of an embedded surface Σ in N, one can minimize area in the isotopy class of Σ to obtain a stable minimal surface of possibly less genus. This last minimization procedure can be found in [52] and is based on the earlier work of [1].

3.3 The barrier construction of Meeks-Yau and excellent exhaustions of surfaces

In this section we will be studying the geometry and topology of the closure of the components of the complement of properly embedded minimal surfaces in complete flat three-manifolds. These manifolds with boundary are almost-complete in the following sense.

Definition 3.2 *A Riemannian n-manifold with boundary is called* almost-complete *if it is complete as a metric space with respect to the natural distance function induced from the infimum the lengths of curves joining pairs of points in the manifold.*

Recall that a noncompact surface Σ in a Riemannian manifold N has *least area* if for any smooth compact subdomain $\Delta \subset \Sigma$, Δ is a surface of least area in N with boundary $\partial\Delta$. The following lemma and its proof appear in [18].

Lemma 3.1 *Suppose N is a connected, orientable, almost-complete Riemannian three-manifold with more than one boundary component. If ∂N has non-negative mean curvature with respect to the outward pointing normal, then N contains a properly embedded, orientable, stable minimal surface. When $H_2(N, \mathbb{Z}_2) = 0$, this surface can be chosen to be least area.*

Proof. The main idea in the proof of this lemma is taken from the proof of a similar result in [52].

Suppose ∂_1 and ∂_2 are two components of ∂M. Choose an arc δ in N that joins a point $p \in \partial_1$ with a point of ∂_2. Let $\tilde{\Sigma}_1 \subset \tilde{\Sigma}_2 \subset \ldots$ be a smooth compact exhaustion of ∂_1 with $p \in \tilde{\Sigma}_1$. Theorem 1 in [56] (together with the general regularity theory of area-minimizing currents in [22] and [81]) states that $\partial\tilde{\Sigma}_i$ is the boundary of a least-area surface Σ_i in N such that Σ_i is homologous to $\tilde{\Sigma}_i$ with $\mathbf{Z}_2$-coefficients. Note that Σ_i is orientable since $\Sigma_i \cup \tilde{\Sigma}_i$ is a boundary in N.

Since the surface Σ_i is area-minimizing in its relative homology class in the interior of N, for any ball $B \subset \text{Int}(N)$, a simple replacement argument using portions of ∂B shows that $\text{Area}(B \cap \Sigma_i) \leq \frac{1}{2}\text{Area}(\partial B)$. Similarly, if B is a smooth ball of geodesic radius ε centered at a point in ∂N, then $\text{Area}(B \cap \Sigma_i) \leq \text{Area}(\partial B)$. These estimates show that the family $\{\Sigma_i\}$ has uniform local area bounds. These area bounds are sufficient for applying the standard compactness and regularity theorems [81] of geometric measure theory that state that a subsequence Σ_{i_j} of these surfaces converges smoothly on compact subsets of N to a properly embedded stable surface Δ. If $H_2(N, \mathbf{Z}_2) = 0$, then Σ_i is area minimizing and hence Δ is area minimizing. (The property that a limit of area-minimizing surfaces is itself area-minimizing is well known and is proved in a similar context in the last paragraph of the proof of Theorem 3.1 in [54].) Note that each of the surfaces Σ_i has odd intersection number with the arc δ and so $\Delta \cap \delta \neq \emptyset$, which implies Δ is nonempty. This completes the proof of the lemma. □

Definition 3.3 *A smooth compact exhaustion $M_1 \subset M_2 \subset \cdots$ of M is called* good *if for all i, each component of $M - \text{Int}(M_i)$ is noncompact and has one boundary curve. It is called* excellent *if it is good and for all i, each component of $M - \text{Int}(M_i)$ has either one end or an infinite number of ends.*

The following lemma appears in [18]; it's proof is straightforward.

Lemma 3.2 *A noncompact surface M has an excellent exhaustion.*

The following is essentially contained in [54]. Its' proof is facilitated by the above lemma.

Theorem 3.5 *Suppose M is a properly embedded orientable minimal surface in a complete flat orientable three-manifold N with $H_2(N, \mathbf{Z}) = 0$ and M is not totally geodesic. Then M separates N into two components N_1, N_2. If M has more than one end, then,*

1. *There is a simple closed curve γ on M that separates M into two noncompact components, each of which is unstable.*
2. *The curve γ is not a $\mathbf{Z}_2$-boundary in N_1 or in N_2.*
3. *If γ is not homologous to zero mod 2 in N_i, then γ is boundary of a noncompact properly embedded least-area surface Σ in N_i with $\Sigma \cap M = \gamma$ and Σ has finite total curvature.*

Proof. Choose an excellent exhaustion $M_1 \subset M_2 \subset \cdots$ of M where M_1 is unstable. Since $M - M_1$ that has infinite total curvature, there exists a component E of $M - M_1$ has infinite total curvature. Since E has infinite total curvature it is unstable. Hence $\gamma = \partial E$ separates M into two unstable components M^+ and M^-. By Corollary 3.1 in the next section, a properly embedded, orientable, nontotally geodesic minimal surface in N separates N into two components N_1, N_2. Since $H_2(N, \mathbf{Z}_2) = 0$, elementary algebraic topology implies γ can not be homologous to zero mod 2 in both N_1 and N_2.

It remains to show the existence of the least-area surface Σ. Choose a compact exhaustion $E_1 \subset E_2 \subset \cdots$ of E and suppose γ is not a $\mathbf{Z}_2$-boundary in N_1. Let Σ_i be a least-area surface in N_1 with $\partial\Sigma_i = \partial E_i$. Since $H_2(N_1, \mathbf{Z}_2) = 0$, Σ_i separates N_1 and hence Σ_i is orientable. The argument in the proof of Lemma 3.1 shows that a subsequence of the Σ_i converge to a least-area orientable surface $\Sigma \subset N_1$ with $\partial\Sigma = \gamma$. Since Σ is stable and M^+ and M^- are unstable, the maximum principle implies $\Sigma \cap M = \gamma$. By Theorem 3.3, Σ has finite total curvature, which completes the proof of the theorem. □

3.4 The Strong Halfspace Theorem and the Convex Hull Theorem

Jorge and Xavier [36] answered the following question of Calabi: *Can a complete nonplanar minimal surface in $\mathbb{R}^3$ be contained in a halfspace?* They proved that there is such an example and, in fact, there is such an example in a slab. Later, Rosenberg and Toubiana [70] proved that there exists a complete minimally immersed annulus in $\mathbb{R}^3$, contained in an open slab, such that the mapping into the open slab is proper. In $\mathbb{R}^{n+1}, n \geq 3$, the n-dimensional $SO(n)$-invariant catenoid is a properly embedded minimal hypersurface in $\mathbb{R}^{n+1}$, contained in a slab. These results led to the general belief that there existed properly immersed minimal surfaces in $\mathbb{R}^3$ that were contained in a halfspace. However in 1984 Meeks and Hoffman proved that a nonplanar properly immersed minimal surface in $\mathbb{R}^3$ cannot be contained in a halfspace. Earlier work of Meeks, Simon and Yau [52] showed that this result had the following stronger consequence (see [31]):

Theorem 3.6 (Strong Halfspace Theorem) *Two properly immersed minimal surfaces must intersect unless they are parallel planes. In particular, a properly immersed minimal surface cannot lie in a halfspace.*

Proof. We first show how to reduce the theorem to the special case of the Halfspace Theorem: *A properly immersed connected minimal surface in $\mathbb{R}^3$ that is contained in a halfspace must be a plane.*

Suppose for the moment that M_1 and M_2 are two properly embedded disjoint minimal surfaces and neither M_1 or M_2 is a plane. Then $M_1 \cup M_2$ separates $\mathbb{R}^3$ into exactly three components where the closure N of one of these components has boundary $M_1 \cup M_2$. By Lemma 3.1, N contains a properly embedded orientable stable minimal surface which must be a plane by the results in [10] or [13]. This implies M_1 and M_2 are contained in a

halfspace. If M_1 and M_2 are immersed and not embedded, there is a unique component of $\mathbb{R}^3 - M_1 \cup M_2$ whose geodesic closure N contains portions of both M_1 and M_2 in its boundary and ∂N is a good barrier (see Definition 3.2 and note that we can relax the condition that N be compact since $\mathbb{R}^3$ is homogenously regular). Since ∂N is a good barrier, the proof of Lemma 3.1 works in this case to show that N contains a plane, which completes the reduction to the case of the Halfspace Theorem.

We now give a proof of the Halfspace Theorem that is a slight variation of the proof given in [31]. Without loss of generality we may assume that M is in the upper halfspace N determined by the (x_1, x_2)-plane $\mathbb{R}^2 = \partial N$, $dist(M, \partial N) = 0$ and $M \cap \partial N = \emptyset$.

Let D_t be the disk of radius t in $\mathbb{R}^2$ centered at the origin. Suppose $dist(D_1, \partial N) > \varepsilon$ and choose $\varepsilon < \frac{1}{4}$. Let γ denote the circle of height ε over ∂D_1. The curves $\gamma \cup \partial D_1$ bound a stable catenoid C_1 with $C_1 \cap \partial N = \partial D_1$. Furthermore it is clear that for t close to 1, $\gamma \cup \partial D_t$ is also the boundary of a stable catenoid C_t and the family C_t varies continuously with t. The maximum principle shows that the interiors of the one-parameter family of stable catenoids C_t are always disjoint from ∂N for $t \geq 1$ and for as long as C_t is defined in N. It is clear that C_t is defined for all $t \geq 1$ and C_{t_1} lies above C_{t_2} when $t_1 > t_2$. Since the C_t are catenoids, they converge to a noncompact totally geodesic surface C of height ε with $\partial C = \gamma$.

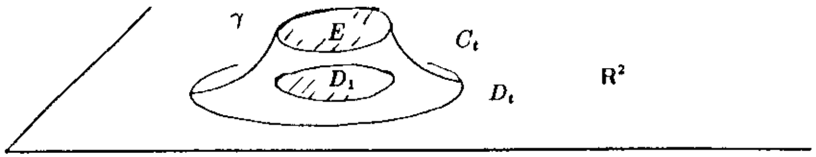

Figure 2:

Let E be the flat disk with $\partial E = \gamma$. Then $C_t \cup E \cup D_1$ bounds a compact region $R \subset N$. Clearly, the regions R_t converge to a slab of width ε that is isometrically embedded in N and one of its boundary planes is M. Hence, the distance from M to ∂N is positive, a contradiction of our earlier assumption that $M \cap \partial N = \emptyset$ and $dist(M, \partial N) = 0$. □

An immediate consequence of the Strong Halfspace Theorem is that a nontotally geodesic embedded periodic minimal surface in R^3 is connected. The following corollary is an immediate consequence of this observation.

Corollary 3.1 *If M is a properly embedded, nontotally geodesic minimal surface in a complete flat three-manifold N, then the induced map $\pi_1(M) \to \pi_1(N)$ is onto. In particular, if M and N are both orientable, M separates N.*

It follows from the Halfspace Theorem that the convex hull of a properly immersed minimal surface in R^3 is a plane or it is R^3. When the properly immersed minimal surface is allowed to have compact boundary, the convex hull of the surface is still rather special.

Theorem 3.7 (Convex Hull Theorem [31]) *Suppose M is a properly immersed noncompact minimal hypersurface in R^n and ∂M is compact. Then the convex hull $\mathcal{H}(M)$ is one of the following:*

1. *$\mathcal{H}(M)$ is a plane;*
2. *$\mathcal{H}(M)$ is a slab;*
3. *$\mathcal{H}(M)$ is a halfspace;*
4. *$\mathcal{H}(M)$ is R^n.*

When $n = 3$, ∂M intersects each boundary component of $\mathcal{H}(M)$.

3.5 Maximum principles at infinity

The usual maximum principle for minimal hypersurfaces implies rather easily that the distance between two proper disjoint minimal hypersurfaces $M_1, M_2 \subset R^n$ can not obtain its minimum value at a pair $(p_1, p_2) \in \text{Int}(M_1) \times \text{Int}(M_2)$ unless they are parallel hyperplanes. On the other hand, it may be that $dist(M_1, M_2)$ is never obtained by a pair $(p_1, p_2) \in M_1 \times M_2$ as occurs for example on the three-dimensional $SO(3)$ invariant "catenoid" C^3 in R^4 whose convex hull is a slab and whose distance from the boundary of the slab is zero. This behavior of course is not possible for properly immersed minimal surfaces in R^3 as seen by the Strong Halfspace Theorem (Theorem 3.6). Thus, one can view the Strong Halfspace Theorem as an example of a maximum principle at infinity. Rosenberg and Meeks [50] have given the following useful generalization of the Strong Halfspace Theorem. (See [8] and [42] for some less general versions of this theorem.)

Theorem 3.8 (Strong Maximum Principle at Infinity) *Suppose M_1 and M_2 are two disjoint properly immersed minimal surfaces with compact* *boundary in a complete flat three-manifold N. If $\partial M_1 = \partial M_2 = \emptyset$, then M_1 and M_2 are flat. Otherwise,*

$$dist(M_1, M_2) = \min\{dist(M_1, \partial M_2), dist(M_2, \partial M_1)\}.$$

We briefly sketch the proof of the above maximum principle in the case $N = \mathbb{R}^3$. Suppose it fails for two properly immersed minimal surfaces M_1 and M_2 with compact boundary. If $\partial M_1 = \partial M_2 = \emptyset$, then the Strong Halfspace Theorem implies M_1 and M_2 are parallel planes, and so we may assume that $\partial M_1 \neq \emptyset$. Since the minimum distance between M_1 and M_2 occurs at infinity there exist pairs $(p_i, q_i) \in M_1 \times M_2$ such that $dist(M_1, M_2) = \lim_{i\to\infty} |p_i - q_i|$. Note that a subsequence of the vectors $\{v_i = p_i - q_i\}$ converge to a vector v and the translated surface $\widetilde{M}_2 = M_2 + v$ is disjoint from M_1 and $dist(M_1, \widetilde{M}_2) = 0$. Thus, to derive a contradiction we need only prove the special case where $dist(M_1, M_2) = 0$.

Assume now that $dist(M_1, M_2) = 0$. Then using $M_1 \cup M_2$ as a barrier, we obtain (by a modification of the proof of Lemma 3.1) disjoint stable orientable properly embedded minimal surfaces Σ_1 and Σ_2 in $\mathbb{R}^3 - (M_1 \cup M_2)$ and with compact boundary that separate M_1 from M_2 in the sense that any short arc that is far from the origin and joins a point of M_1 to a point of M_2 must intersect Σ_1 and Σ_2. Since Σ_1 and Σ_2 have finite total curvature (Theorem 3.3), the ends of Σ_1 and Σ_2 are asymptotic to planes or catenoids that are graphs. This reduces the problem to the special case of graphs where one easily obtains a contradiction (see [50] for details).

3.6 The Weierstrass representation for periodic minimal surfaces of finite total curvature

In a series of papers Osserman [64, 65] proved many of the basic theorems concerning complete orientable minimal surfaces $f: M \to \mathbb{R}^3$ with finite total

curvature. Among other things, he showed that M is conformally diffeomorphic to a closed Riemann surface $\overline{M}$ punctured in a finite number of points, the stereographic projection $g: M \to \mathbb{C} \cup \{\infty\}$ of the Gauss map is a meromorphic function that extends to a meromorphic function $\overline{g}: \overline{M} \to \mathbb{C} \cup \{\infty\}$, and, after a translation of a point $p_0 \in M$ to the origin, there exists a holomorphic one-form $\overline{\eta}$ on M (that extends meromorphically to $\overline{M}$) so that the coordinate functions of f can be expressed by

$$f(p) = \mathrm{Re} \int_{p_0}^{p} \left[(1 - g^2, (1 + g^2)i, 2g)\right] \eta .$$

The above representation of M in the above formula is called the *Weierstrass representation of M*.

Let G be a discrete group of translations of $\mathbb{R}^3$ and $\widetilde{M}$ is a complete minimal surface in $\mathbb{R}^3$ that is invariant under translation by G in an orientation preserving manner. Suppose the quotient surface $f: M = \widetilde{M}/G \to \mathbb{R}^3/G$ has finite total curvature. Then M is conformally diffeomorphic to a closed Riemann surface (Huber's Theorem [34]), the steoreographic projection of the Gauss map is well defined and, as in the case of $\mathbb{R}^3$, one has an analytic representation of f as given in the above formula.

When $\widetilde{M}$ is a properly embedded minimal surface in $\mathbb{R}^3$ invariant under a screw motion symmetry S_θ, $0 < \theta < 2\pi$, the Gauss map $M = \widetilde{M}/S_\theta$ is not well defined in $\mathbb{R}^3/S_\theta$. Since the Gauss map for M is not defined, it is more difficult to analyze the analytic and geometric behavior of M, especially when this surface has finite total curvature. Still, when M has finite total curvature, M is conformally diffeomorphic to a closed Riemann surface $\overline{M}$ punctured in a finite number of points. In particular the ends of M are annuli of finite total curvature. We now describe how Osserman's results and the Weierstrass representation generalize for such M. This discussion is taken from Section 3 in [49].

Let A be a finite total curvature minimal annulus embedded in $N = \mathbb{R}^3/S_\theta$, where $0 < \theta < 2\pi$. We will derive meromorphic data on the disk that parametrizes A, and describes its aymptotic behavior at infinity.

We take S_θ to be a translation along the x_3-axis followed by rotation by θ about the x_3-axis. Since A has finite total curvature, A is conformally the punctured disk D^*. We no longer have a single valued Gauss map g on A; g is a multi-valued meromorphic map on D^* whose values differ by multiplication by λ^m, $\lambda = e^{2\pi i\theta}$. To see this, let E be a connected lifting of A, slit along a radial segment (say $\theta = 0$), to $\mathbb{R}^3$. The Gaussian image of the normal vector to E at $p \in E$ and the image of the normal vector to E at $S_\theta(p)$, differ by rotation about the x_3-axis by θ. Hence, the stereographic projections of these vectors on the sphere, differ by rotation by θ in $\mathbb{C}$, i.e., by multiplication by $\lambda = e^{2\pi i\theta}$.

Lifting g to the Riemann universal covering surface of D^*, we have a well defined meromorphic map $\tilde{g}$, on the half plane $H = \{x \le 0\}$, satisfying $\tilde{g}(z + 2\pi m i) = \lambda^m \tilde{g}(z)$, for $z \in H$. Then $g = \tilde{g}(\exp^{-1})$.

We wish to show that A has a limiting tangent plane at ∞, i.e., g extends continuously to 0 (even though g is multi-valued). This will follow from the fact that the area of the spherical image of g (i.e. a single valued branch of g on the slit punctured disk D') is finite (see Theorem 3.9 below).

Theorem 3.9 *Let g be a multi-valued meromorphic map on D^*, $g = \tilde{g}(\exp^{-1})$, with $\tilde{g}(z + 2\pi i) = \lambda\tilde{g}(z)$, for $z \in H$, and some λ, $|\lambda| = 1$. If $Area(g(D'))$ is finite, then g extends continuously to 0.*

Now we shall use Theorem 3.9 to obtain a Weierstrass representation on the disk D, for finite total curvature annuli A in $N = \mathbb{R}^3/S_\theta$. By Theorem 3.9, the multi-valued g extends continuously to 0 and since the limiting value is fixed by multiplication by λ and $\lambda \ne 1$, the limiting value is 0 or ∞; so

we can assume $g(0) = 0$. Write $\lambda = e^{2\pi i a}$ with $0 < a < 1$. Since $\tilde{g}(z + 2\pi i) = \lambda\tilde{g}(z)$, the map $z^{-a}g(z)$ is indeed single valued on D^*. Furthermore, $z^{1-a}g(z)$ is bounded in a neighborhood of 0, hence $g(z) = z^{a-1}h(z)$ where h is holomorphic in a neighborhood of 0. Hence, dg/g is a well defined meromorphic one-form on D^*, and 0 is a removable singularity. The multi-valued g on D, is obtained from this form by $g(z) = \exp(\int dg/g)$.

Notice that the third coordinate function X_3 is well defined on M. Define the holomorphic one-form $\eta = dX_3 + i * dX_3$. A straightforward and similar argument proves that η extends meromorphically at 0.

We take as Weierstrass data on A the pair $(dg/g, \eta)$; these forms are meromorphic at the puncture and A is obtained from this data by the formula $g = \exp(\int dg/g)$,

$$X(z) = \mathrm{Re} \int \left(g + \frac{1}{g},\ ig - \frac{i}{g},\ 2\right) \eta.$$

In particular, we have:

Theorem 3.10 *Let M be a complete finite total curvature minimal surface in $\mathbb{R}^3/S_\theta$. Then there exists a conformal compactification $\overline{M}$ of M, and meromorphic forms $(dg/g, \eta)$ on $\overline{M}$, such that M is parametrized by*

$$X(z) = \mathrm{Re} \int \left(g + \frac{1}{g},\ ig - \frac{i}{g},\ 2\right) \eta$$

$$\textit{where} \quad g = \exp\left(\int \frac{dg}{g}\right).$$

Remark 3.1 *H. Karcher has given many new examples of such M with this data [38].*

3.7 The sum of finite total curvature minimal surfaces

Let M_1, M_2 be finite total curvature complete nonplanar minimal surfaces in $\mathbb{R}^3$, with Gauss maps g_1, g_2. Let $p_1, \ldots, p_n$, $q_1, \ldots, q_m$ be the punctures of M_1, M_2, and $\overline{M}_1, \overline{M}_2$ the compactified Riemann surfaces. If one fixes a unit vector $z \in S^2$, one can add (in $\mathbb{R}^3$) all points on M_1 and M_2 having z as normal. As z varies over a domain in S^2, this yields a minimal surface in $\mathbb{R}^3$ (or a point). More precisely, one has the following [71].

Theorem 3.11 *For some subset W of $g_1(p_1, \ldots, p_n) \cup g_2(q_1, \ldots, q_m)$, the set*

$$M_1 + M_2 = \{\Sigma_{x \in g_1^{-1}(z)} x + \Sigma_{y \in g_2^{-1}(z)} y \mid z \in (S^2 - W)\}$$

is a proper minimal surface in $\mathbb{R}^3$, or a point. In general $M_1 + M_2$ has branch points.

Since $M_1 + M_2$ can be paramatrized by the sphere punctured at a finite number of points (the normal to $M_1 + M_2$ at $\Sigma_{x \in g_1^{-1}(z)} x + \Sigma_{y \in g_2^{-1}(z)} y$ is z), the total curvature of $M_1 + M_2$ is -4π (or zero).

We denote by $\widehat{g}: S^2 - W \to M_1 + M_2$ the natural conformal parametrization of $M_1 + M_2$ (or constant map, if $M_1 + M_2$ is a point). One can just as well define this sum for unoriented normal lines. The result is a proper minimal surface parametrized by the projective plane $\mathbb{P}^2$ punctured in a finite number of points [71]. Hence its total curvature is -2π or 0. Again denote by $\widehat{g}: \mathbb{P}^2 - W \to M_1 + M_2$, the natural parametrization.

Since $\mathbb{T} \times \mathbb{R}$ is an abelian group under addition and the Gauss map is invariant under translation. The sum $M_1 + M_2$ is well defined in $\mathbb{T} \times \mathbb{R}$ for nonplanar complete minimal surfaces $M_1, M_2 \subset \mathbb{T} \times \mathbb{R}$ of finite total curvature. The orientable sum has total curvature -4π (or 0) and the nonorientable sum

has total curvature -2π (or 0), and we denote again the natural parametrization by $\widehat{g}$. Similarly, one could define for M_1, M_2 in $S^1 \times \mathbb{R}^2$ or $\mathbb{T}^3$ the sum $M_1 + M_2$ and $\widehat{g}$.

Definition 3.4 *We define* $M \oplus M = \frac{1}{2}(M + M)$

Note that $M \oplus M$ makes sense in $\mathbb{T} \times \mathbb{R}, S^1 \times \mathbb{R}^2$ or $\mathbb{T}^3$, and equals the set of points that can be expressed as the sum of points on M with the same normal vector. We also denote the natural parametrization by $\widehat{g}$. In [51] Meeks and Rosenberg prove

Theorem 3.12 *Let $F: M \to \mathbb{T} \times \mathbb{R}$ be a connected properly immersed minimal surface with finite (nonzero) total curvature. Then:*

1. *If M has parallel flat ends, then $\widehat{g}$ is constant (flat means each end converges to a flat cylinder).*

2. *If M is embedded and has nonparallel ends, then $\widehat{g}$ is a Scherk's surface (orientable or nonorientable).*

Corollary 3.2 *Suppose M is a properly embedded minimal surface of finite topology in $\mathbb{T} \times \mathbb{R}$.*

1. *If $\mathbb{T} \times \mathbb{R}$ has a incommensurable lattice, then $\widehat{g}$ is constant.*

2. *If M is an embedded minimal surface of genus one with four parallel ends, then after a translation of M (so that a zero of Gaussian curvature occurs at the origin) M is invariant under the isometry $p \mapsto -p$. After this translation, the order 2 points in the group $(\mathbb{R}^2/\Lambda) \times \mathbb{R}$ are the zeros of the Gaussian curvature of M. (We consider the identity element to have order 2.) In this case M separates $\mathbb{T} \times \mathbb{R}$ into two components that are isometric.*

Theorem 3.12 as well as the definition of the sum surface $M \oplus M$ for minimal surfaces of finite total curvature, was motivated by earlier work of Meeks [45] on minimal surfaces in a flat three torus T^3. Meeks proved that the sum map $\widehat{g}$ for a closed minimal surface $M \subset T^3$ was constant and used this result to prove that when M had genus 3, then it disconnected T^3 into isometric pieces (see Section 4.1). The sum map is also useful for studying surfaces of finite total curvature in R^3/G where G is an infinite cyclic group of translations. In particular, the sum map plays a fundamental role in the proof of Theorem 1.7, which characterizes the asymptotic behavior of these surfaces.

4 Triply-periodic minimal surfaces

4.1 The geometry of hyperelliptic minimal surfaces and Abel's Theorem

Many of the more intricate and beautiful examples of minimal surfaces in R^3 have the additional property of being preserved by a group of three linearly independent translations. During the middle of the nineteenth century, a thorough investigation of triply-periodic minimal surfaces was carried out by Schwarz [79]. By extending Plateau's construction to polygonal curves and then extending them by repeated reflection across the line boundaries, he found an effective method for generating surfaces invariant under a lattice L of translations. The resulting quotient surfaces in R^3/L gave the first examples of compact minimal surfaces in flat three-tori.

We will call a closed Riemann surface M *periodic* if it conformally minimally immerses in a flat three-torus T^3. By lifting to the universal cover of T^3, these periodic surfaces become the proper triply-periodic minimal surfaces in R^3.

The compactness of a minimal surface M in T^3 gives rise to restrictions on the conformal type of M. Frequently, these conformal restrictions give nontrivial geometric information about the lifted minimal surface in R^3. For these reasons, we consider the following fundamental questions:

1. *Which compact Riemann surfaces are periodic?*
2. *How does the conformal structure of a periodic surface influence its geometry?*

In this section, we study the geometry and conformal structure of these periodic surfaces. The geometric tools of this investigation are the Gauss map and the Gauss-Bonnet Theorem. Note that the Gauss map of an orientable minimal surface in a flat three-torus is well defined. Some of the results in this section were found independently by Nagano and Smyth [59, 60, 62].

Theorem 4.1 (Gauss-Bonnet Theorem) *If $f\colon M_g \to T^3$ is a minimal surface of genus g, then the Gauss map $G\colon M_g \to S^2$ represents M_g as a $(g-1)$ conformal branched cover of S^2.*

Proof. Recall that the Gauss map $G\colon M \to S^2$ for a minimal surface in R^3 is holomorphic. Similarly, when $f\colon M_g \to T^3 = R^3/L$ is a minimal surface, G is also holomorphic, and hence, exhibits M as a conformal branched cover of S^2. The usual Gauss-Bonnet Theorem states that the degree of G is $g-1$, where g is the genus of M_g. □

Straightforward applications of the Gauss-Bonnet Theorem give rise to the following restrictions on the topological and conformal type of minimal surfaces in T^3.

Corollary 4.1 *A surface of genus two is never periodic.*

Proof. If a surface of genus 2 was periodic, then, by the Gauss-Bonnet Theorem, the Gauss map would represent the surface as a one-sheeted branched cover of S^2. But any one-sheeted cover of S^2 is again S^2. □

Corollary 4.2 *A periodic surface of genus three is hyperelliptic.*

Proof. By definition, a Riemann surface is hyperelliptic if it can be represented as a two-sheeted covering of S^2. In the case of genus three, the Gauss map provides this representation. □

Corollary 4.3 *If $f: M_g \to T^3$ is a minimal surface of genus g, then M_g has $4(g-1)$ zeros of Gauss curvature, counted with multiplicities.*

Proof. Since the Gauss map $G: M_g \to S^2$ has degree $g-1$, the Riemann-Hurwitz formula implies there are $4(g-1)$ branch points counted with multiplicity. Since we can identify the zeros of Gauss curvature with the branch points, the corollary now follows. □

The rest of this section is devoted to the study of minimal immersions of hyperelliptic surfaces in flat tori.

Proposition 4.1 *If $f: M \to T^n$ is a hyperelliptic minimal surface, then:*

1. *The hyperelliptic automorphism is an isometry that is induced by an inversion symmetry in T^n through any hyperelliptic point of $f(M)$;*

2. *After a translation, the hyperelliptic points are contained in the set of order two points in T^n. (Note 0 trivially has order two in the abelian group T^n.)*

Proof. Let $\Theta: M \to M$ denote the hyperelliptic automorphism of M. Since $M/\Theta = S^2$ and S^2 has no harmonic one-forms, Θ^* is multiplication by (-1) on the harmonic one-forms. After a translation, suppose that f is represented by $f(p) = \int_{p_0}^{p}(h_1, \ldots, h_n)$, where h_i is a harmonic one-form and p_0 is a hyperelliptic point. Then

$$f(\Theta(p)) = \int_{p_0}^{\Theta(p)}(h_1, \ldots, h_n) = \int_{\Theta(p_0)}^{\Theta(p)}(h_1, \ldots, h_n) = \int_{p_0}^{p}\Theta^*(h_1, \ldots, h_n) = -f(p),$$

Hence, $(-1): T^3 \to T^3$ leaves M invariant and fixes the hyperelliptic points.

If x is a hyperelliptic point, then the above equation shows that $f(x) = f(\Theta(x)) = -f(x)$. Hence, every hyperelliptic point has order two or is $0 \in T^n$. □

One can easily verify that the following theorem holds on the four Schwarz surfaces of genus three.

Theorem 4.2 *If $f: M_3 \to T^3 = \mathbb{R}^3/L$ is a minimal surface of genus three, then:*

1. *M_3 is hyperelliptic;*

2. *The hyperelliptic automorphism is an isometry and is induced by inversion symmetry in T^3 through any hyperelliptic point;*

3. *If f is an embedding, then after a translation, the set of zeros of Gauss curvature can be identified with $\frac{1}{2}L$ = order two points of T^3. (Note $0 \in \frac{1}{2}L$ trivially has order two.)*

Proof. The Gauss map $G: M_3 \to S^2$ represents M_3 as a two-sheeted cover of S^2 with simple branch points. Hence, by the Riemann-Hurwitz formula, we get eight branch points or zeros of Gauss curvature. Since there are precisely eight order two points in T^3, the theorem follows from the previous proposition. □

The following theorem gives a natural interpretation of the classical Abel's Theorem for periodic minimal surfaces.

Theorem 4.3 (Abel's Theorem for Periodic Surfaces) *Let $f: M_g \to T^3$ be a minimal surface and let $G: M_g \to S^2$ be its Gauss map. Then for all $s \in S^2$, $q = \sum_{p \in G^{-1}(s)} p \in T^3$ (summed with multiplicity) is independent of $s \in S^2$.*

Proof. Since G is holomorphic, the continuous map $\tilde{G}: S^2 \to T^3$ defined by $\tilde{G}(s) = \sum_{p \in G^{-1}(s)} p$, where the sum is taken with multiplicity, is locally a sum of harmonic maps and hence is itself a harmonic map. Since S^2 is simply connected, $\tilde{G}$ lifts to the universal cover R^3 of T^3 with harmonic coordinate functions. Since a harmonic function on a closed Riemann surface is constant, the lifted map is constant. Hence, $\tilde{G}$ is constant, which proves the theorem. □

Remark 4.1 *The above theorem holds for any meromorphic function $F: M_g \to S^2$, not just the Gauss map.*

Corollary 4.4 *After a fixed translation, the three points of a periodic surface of genus 4 with the same unit normal are coplanar, i.e. they lie on the quotient of a plane passing through the origin.*

Proof. Suppose M_4 is a genus 4 minimal surface in a flat three-torus T^3. After a fixed translation of M_4, we may assume by Theorem 4.3 that $p_1 + p_2 + p_3 = 0 \in T^3$ for $\{p_1, p_2, p_3\} \subset G^{-1}(s)$ for every $s \in S^2$ not in the branch locus of G. This implies p_1, p_2, and p_3 are coplanar. □

4.2 The main existence theorems

One of the beautiful classical theorems on Riemann surfaces states that every closed Riemann surface of positive genus holomorphically embeds in a complex torus called its Jacobian. In particular, every closed Riemann surface of positive genus conformally embeds as a complex minimal submanifold in some flat complex torus.

Theorem 4.4 (Abel-Jacobi Embedding Theorem) *Let M be a closed Riemann surface of positive genus g and let $\{\omega_1, \ldots, \omega_g\}$ be a basis for $H^{1,0}(M)$. Then $f(z) = \int_{z_0}^{z}(\omega_1 \ldots, \omega_g): M \to C^n/L = J(M)$ (= Jacobian of M) is a holomorphic embedding, where L is the lattice of period vectors $\{\int_\gamma(\omega_1, \ldots, \omega_g) \mid \gamma \in H_1(M, Z)\}$.*

It is important to consider the conformal structure of a Riemann surface when searching for minimal surfaces in tori. In fact, on two-dimensional Riemannian manifolds, $(M, <>)$, there is usually much to be gained by using appropriate coordinate charts. When M is orientable, it is possible to pick coordinates so that the metric $ds^2 = F(dx^2 + dy^2)$, and under change of coordinates, angles are preserved. Such coordinates are called *isothermal coordinates* and give M a conformal or complex structure.

A nonorientable "Riemann surface" M with a conformal-anticonformal structure no longer has holomorphic forms but it still has harmonic one-forms. By integration of a basis of the space of harmonic one-forms, one obtains a harmonic map $f: M \to A(M)$ where $f(p) = \int^p(h_1, \ldots, h_n)$ and $A(M) = R^n/P$ where $P = \{\int_\gamma(h_1, \ldots, h_n) \mid \gamma \in H_1(M, Z)\}$. As shown in [48], f is not always one-to-one, however, it is a smooth one-to-one immersion when f is a branched minimal immersion. This regularity theorem will be used to prove the regularity of the surfaces described in the next theorem.

Theorem 4.5 *There is a real five-dimensional family V of periodic hyperelliptic Riemann surfaces of genus three. These are the surfaces which can be represented as two-sheeted covers of S^2 branched over four pairs of antipodal points. Furthermore,*

1. *There exist two distinct isometric minimal immersions for each $M_3 \in V$;*

2. *These immersions are embeddings;*

3. *V induces a five-dimensional family $\tilde{V}$ of embedded non-orientable minimal surfaces of Euler characteristic $\chi = -2$.*

Proof. We first give explicit analytic formulae for the periodic minimal surfaces in the family V. Suppose $M \in V$ with Gauss map $G: M \to S^2 = \mathbf{C} \cup \{\infty\}$. We may assume, after a possible rigid motion of the lifted surface in $\mathbf{R}^3$, that G is a branched cover of $\mathbf{C} \cup \{\infty\}$ with branch points $P = \{a_1, \ldots, a_4, a_5 = -1/\bar{a}_1, \ldots, a_8 = -1/\bar{a}_4\}$ in the complex plane and where the product $a_1 a_2 a_3 a_4$ is a positive real number.

In this case the plane curve of M is $y^2 = (z - a_1)\ldots(z - a_8)$. In this representation G is the meromorphic function $z: M \to \mathbf{C} \cup \{\infty\}$. If $\eta = (1/y)dz$ and $\omega = [(1 - z^2), (1 + z^2)i, 2z)\eta]$, then $f(z) = \int^z \omega: M \to J(M)$ induces the Jacobi map of M. The projections $f_1 = \mathrm{Re}(f)$ and $f_2 = \mathrm{Im}(f)$ are the two minimal embeddings described in Theorem 4.1.

It is straightforward to check that the antipodal map on S^2 has two lifts σ_1 and σ_2 to M, each acting freely on M and of order two. Furthermore, it is easily checked that one can index σ_1 and σ_2 so that $f_1 \circ \sigma$ (resp. $f_2 \circ \sigma_2$) differs from f_1 (resp. f_2) by a translation v_1 (resp. v_2) of order two in the quotient torus of f_1 (resp. f_2). After quotienting out by this additional translation one obtains a map $\tilde{f}_1: M/\sigma_1 \to \mathrm{T}_1^3$ (resp. $\tilde{f}_2: M/\sigma_2 \to \mathrm{T}_2^3$). Since $\tilde{f}_1$ (resp.

$\tilde{f}_2$) is the Albanese map of the nonorientable minimal surface M/σ_1 (resp. M/σ_2), $\tilde{f}_1$ (resp. $\tilde{f}_2$) is a smooth minimal surface embedded by the earlier stated regularity theorem for the Albanese map of a nonorientable closed Riemannian surface [48]. Since $\tilde{f}_1$ and $\tilde{f}_2$ are one-to-one, so are f_1 and f_2, which completes the proof of the theorem. □

Theorem 10.1 in [44] states that every flat three-torus contains an infinite number of examples in the family V of genus 3 periodic surfaces described in Theorem 4.1. In fact, every $\mathbf{T}^3$ contains an infinite sequence $\overline{\Sigma}_1, \ldots, \overline{\Sigma}_k, \ldots$ of nonorientable embedded minimal surfaces with Euler characteristic $\chi = -2$ and such that $\lim_{i\to\infty} Area(\overline{\Sigma}_i) = \infty$. The existence of these new examples is based on an abstract mini-max type proof that is independent of Theorem 4.5.

Let $T^3 = \mathbf{R}^3/\mathbf{Z}^3$ where $\mathbf{Z}^3$ is the integer lattice in $\mathbf{R}^3$ and let $F \subset T^3$ be the quotient torus of the x_1x_2-plane in T^3. Let $\sigma: T^3 \to T^3$ be the diagonal translation of order 2. Consider the surface Σ in T^3 obtained from F and $F + (0, 0, \frac{1}{2})$ by taking their connected sum along vertical line segments ℓ and $\sigma(\ell)$. Do this so that $\sigma(\Sigma) = \Sigma$ and let $\overline{\Sigma} = \overline{\Sigma}/\sigma \subset \overline{T}^3 = T^3/\sigma$. See Figure 3 below for a picture of Σ. Let $\overline{F}$ denote the image of F in $\overline{T}^3$. It is straightforward to check that $\overline{\Sigma}$ is isotopic in $\overline{T}^3$ to $\mathcal{P}/\sigma$ where $\mathcal{P}$ is the Schwarz primitive surface. Let $R: T^3 \to T^3$ be the rotation around the diagonal vector $(1, 1, 1)$ by 120° and note that R commutes with σ. Let $\overline{R}: \overline{T}^3 \to \overline{T}^3$ denote the associated quotient linear isometry. Since $\mathcal{P}$ is invariant under R, $\mathcal{P}/\sigma$ is invariant under $\overline{R}$. Hence, $\overline{R}(\overline{\Sigma})$ is isotopic to $\overline{\Sigma}$.

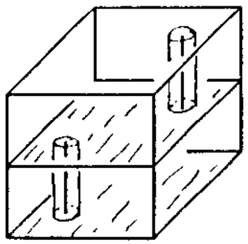

Figure 3:

If $\tilde{F}$ is a subtorus of $\overline{T}^3$ that represents the same Z_2-homology class as $\overline{F}$, then there exists a linear automorphism $\overline{L}: \overline{T}^3 \to \overline{T}^3$ with $\overline{L}(\overline{F}) = \tilde{F}$ and such that $\overline{L}$ lifts to a $L: T^3 \to T^3$. Note that the linear automorphisms of T^3 are generated by R and Δ, Δ defined by $e_1 \to e_1 + e_2$, $e_2 \to e_2$, $e_3 \to e_3$. Note also that $\overline{\Sigma}$ is isotopic to $\overline{\Delta}(\overline{\Sigma})$ where $\overline{\Delta}: \overline{T}^3 \to \overline{T}^3$ is the associated quotient map. Recall that the surface $\overline{\Sigma}$ is obtained from $\overline{F}$ by taking and adding a handle along a vertical line segment in $\overline{T}^3$. Since $\overline{\Delta}(\overline{F}) = \overline{F}$ and $\overline{\Delta}$ preserves the vertical, $\overline{\Delta}(\overline{\Sigma})$ is isotopic to $\overline{F}$ by adding a vertical handle. Hence $\overline{\Delta}(\overline{\Sigma})$ is isotopic to $\overline{\Sigma}$. Since $\overline{L}$ is a composition of products of $\overline{\Delta}$ and R, both of which preserve the isotopy class of $\overline{\Sigma}$, the isotopy class of $\tilde{F}$ can be obtained by a single surgery on $\overline{\Sigma}$. This proves the following lemma.

Lemma 4.1 *Suppose $\tilde{F} \subset \overline{T}^3$ is a subtorus that represents the Z_2-homology class of $\overline{\Sigma}$. Then $\tilde{F}$ is isotopic to a surface obtained by doing surgery on $\overline{\Sigma}$.*

This lemma is used in the proof of the following theorem in [44] that has been proved independently by Hass, Pitts, and Rubenstein [67].

Theorem 4.6 *Let T^3 be an arbitrary flat three-torus. Then there exists an infinite sequence of embedded minimal surfaces $\Sigma_1, \ldots, \Sigma_k, \ldots$ in the family*

V given in Theorem 4.5. Furthermore, the Σ_k can be choosen to have area greater than k.

We briefly outline the main idea of the proof of the theorem. After lifting to two-sheeted covers of flat three-tori, it is sufficient to prove that our original T^3 contains an infinite sequence $\overline{\Sigma}_1, \ldots, \overline{\Sigma}_k, \ldots$ of nonorientable minimal surfaces the family $\tilde{V}$ described in Theorem 4.5 and such that $Area(\overline{\Sigma}_k) > k$.

After composing with a linear isomorphism of T^3 with T^3, consider $\overline{\Sigma}$ to be contained in T^3. By Lemma 4.1, the surface $\overline{\Sigma}$ is "isotopic by surgery" to any fixed flat two-torus in the Z_2-homology class of $\overline{\Sigma}$. Suppose T_1 and T_2 are two flat tori in T^3 which represent the Z_2-homology class of $\overline{\Sigma}$ but represent different Z-homology classes. Furthermore, choose T_1 and T_2 so that $Area(\mathrm{T}_1) > Area(\mathrm{T}_2) > n$. Notice that flat two-tori in T^3 are strong local minima to the area functional on the space of Z_2-currents representing a Z_2-homology class and the local minima $\mathrm{T}_1, \mathrm{T}_2$ can be joined by a path, Σ_t, $1 > t > 2$, such that the Σ_t limit as varifolds to $\mathrm{T}_1, \mathrm{T}_2$, respectively, as $t \to 1$ or 2 and for any such path $Area(\Sigma_t) > Area(\mathrm{T}_2)$. The general of mini-max principle for minimal surfaces, first developed by Morse-Tompkins [58] and Shiffman [73], states that in the space of paths Σ_t joining the local area minima $\mathrm{T}_1, \mathrm{T}_2$, there should exist an a path whose maximum area surface $\tilde{\Sigma}$ has area which is minimal over all such paths. The surface $\tilde{\Sigma}$ is then called a *mini-max* and it is an unstable minimal surface. Since $\tilde{\Sigma}$ is a mini-max, its area is at least as big as $Area(\mathrm{T}_2) > n$.

While the above guiding mini-max principle is easy to state, in principle it is usually difficult to apply because the spaces involved are infinite dimensional. By working with paths of harmonic maps, one can reduce the question of finding the minimal $\tilde{\Sigma}$ to finding the required mini-max on a finite dimensional space which is the Teichmoeller space of $\overline{\Sigma}$. This proof of the existence of $\tilde{\Sigma}$ was found around 1980. It seemed clear to the author at that time that the results of Meeks, Simon and Yau [52] and Simon [80] should generalize to prove the existence of the minimax $\tilde{\Sigma}$, by doing the minimax procedure in space of all embedded surfaces. This second approach has been made rigorous by Hass, Pitts and Rubenstein. Their proof of the existence of $\tilde{\Sigma}$ is much more general than the author's and is applicable to the general theory of closed minimal surfaces in closed Riemannian three-manifolds [67].

5 Doubly-periodic minimal surfaces

5.1 Existence of examples

Before discussing examples of properly embedded minimal surfaces M in a given $\mathrm{T} \times \mathrm{R}$ of finite topological type, recall that Theorem 1.3 and Theorem 1.4 give certain topological and geometric restrictions on what is permitted. For example, when $\mathrm{T} \times \mathrm{R}$ does not have a commensurable lattice, M is not a planar domain. On the other hand, the natural quotient of Scherk's dobuly periodic minimal surfaces is a planar domain with four ends in a $\mathrm{T} \times \mathrm{R}$ where T is a square torus. This surface can be described as the solution set to the equation $\cos(x)e^z - \cos(y) = 0$. Scherk's surface fits naturally into a one-parameter family of doubly-periodic minimal surfaces where each example is invariant under translation by two linearly independent unit vectors with varying angle between the vectors.

Actually the full lattice L of translational symmetries of Scherk's surface S contains orientation reversing translations and $S/L \subset \mathrm{R}^3/L$ is a properly embedded minimal surface diffeomorphic to the projective plane punctured in two points. A simple analysis carried out in [51] shows that for every closed orientable surface punctured in four points or for every closed nonorientable surface punctured in two points, there exists a sublattice of L such that the quotient of S by this sublattice is diffeomorphic to the surface being

considered. This result should be compared to the nonexistence statements in Theorem 1.3 and 1.4.

Karcher [38] has constructed a large number of geometrically distinct examples of finite total curvature minimal surfaces in various $\mathrm{T} \times \mathrm{R}$. Based on one of Karcher's examples and Meeks' existence theorem for triply-periodic minimal surfaces (Theorem 4.5), Meeks and Rosenberg found the following family of embedded doubly-periodic minimal surfaces.

Theorem 5.1 *Let $\overline{M}$ be the elliptic curve defined by*

$$\omega^2 = (z - x_1)(z - x_2)(z - x_3)(z - x_4),$$

where $x_1, x_3 \in \mathrm{C} - \{0\}$ with $\arg(x_1) = -\arg(x_3)$ and $x_2 = \frac{-1}{\bar{x}_1}$ and $x_4 = \frac{-1}{\bar{x}_3}$. Consider z and ω to be meromorphic functions on $\overline{M}$. Define $M = \overline{M} - (Z \cup P)$ where Z and P denote the zeroes and poles of $z: \overline{M} \to \mathrm{C} \cup \{\infty\}$. Choose a base point p_1 such that $z(p_1) = x_1$. Let $\eta = \frac{dz}{z\omega}$ and $\Phi = ((1 - z^2)\eta, (1 + z^2)\sqrt{-1}\eta, 2z\eta)$ and $\Lambda = \{\int_\gamma \Phi \mid \gamma \in H_1(M, Z)\}$. Then

1. *The real and imaginary projections $\mathrm{Re}(\Lambda), \mathrm{Im}(\Lambda) \subset \mathrm{R}^3$ are each generated by two linearly independent vectors.*

2. *The maps $f_{\mathrm{Re}}: M \to \mathrm{R}^3/\mathrm{Re}(\Lambda)$ and $f_{\mathrm{Im}}: M \to \mathrm{R}^3/\mathrm{Im}(\Lambda)$, defined by $f_{\mathrm{Re}}(p) = \mathrm{Re}\int_{p_1}^{p} \Phi$ and $f_{\mathrm{Im}}(p) = \mathrm{Im}\int_{p_1}^{p} \Phi$, are each proper one-to-one minimal immersions of a surface of genus one with four horizontal ends.*

3. *The surface $f_{\mathrm{Re}}(M) \subset \mathrm{R}^3/\mathrm{Re}(\Lambda)$ is invariant under a translation of order 2 which is orientation reversing and whose quotient surface is a Klein bottle with two ends. A similar statement holds for $f_{\mathrm{Im}}(M)$. Furtheremore, all embedded Klein bottles with two parallel ends in $\mathrm{T} \times \mathrm{R}$ arise from these families.*

4. *$f_{\mathrm{Re}}(M) \subset \mathbb{R}^3/\mathrm{Re}(\Lambda)$ is invariant under the isometry $p \longmapsto -p$. A similar statement holds for $f_{\mathrm{Im}}(M)$.*

5. *Every $\mathbb{T} \times \mathbb{R}$ contains an infinite collection of nonhomotopic examples in the above families of minimal tori and Klein bottles.*

6. *The extended Gauss maps for $f_{\mathrm{Re}}(M)$ and $f_{\mathrm{Im}}(M)$ are the same and are equal to the meromorphic function $z: \overline{M} \to \mathbb{C} \cup \{\infty\}$.*

5.2 Generalizations of Theorem 1.1 for doubly-periodic minimal surfaces

In this section we shall generalize Theorem 1.1 for doubly-periodic minimal surfaces in several ways. One generalization is to show that a properly embedded minimal surface in M in $\mathbb{T} \times \mathbb{R}$ of finite genus has finite total curvature. This will be proved by showing that any properly embedded minimal surface $\Sigma \subset \mathbb{T} \times \mathbb{R}$ can have only a finite number of ends and by showing each end $\overline{\alpha}$ of such a Σ, which has an end-representative E such that the induced map $\pi_1(E) \to \pi_1(\mathbb{T} \times \mathbb{R})$ is not onto, has linear area growth. Throughout this section, the surfaces we will consider are noncompact, have a finite number of connected components, and have compact boundary that may be empty.

We begin with a definition.

Definition 5.1 *A surface M in $\mathbb{T} \times \mathbb{R}$ has* linear area growth *in $\mathbb{T} \times \mathbb{R}$ if there exist constants K_1 and K_2 such that for t large, $K_1 t \leq Area(M_t = M \cap \mathbb{T} \times [-t, t]) \leq K_2 t$. We will say that M has* area growth bounded from below [resp. above] by Kt *if for t, large $Kt \leq Area(M_t)$* [resp. $Area(M_t) \leq Kt$].

The next lemma is well known.

Lemma 5.1 *Suppose $A \subset \mathbb{T} \times \mathbb{R}$ is a proper annulus with smooth compact boundary, nonpositive curvature and A has area growth bounded from above by Kt. Then A has finite total curvature.*

Proof. Since A has one end and nonpositive Gaussian curvature, we may assume that $A \subset \mathbb{T} \times [0, \infty)$. Consider $\Gamma(t) = A \cap (\mathbb{T} \times \{t\})$. Again using the nonpositive Gaussian curvature of property A, it is straightforward to show that for t greater than the height of $\partial A, \mathbb{T} \times \{t\}$ is transverse to A and $\Gamma(t)$ is a simple closed curve. Since A has area growth bounded from above by Kt, there is a sequence of $t_1, < t_2, < \dots$ with $\lim t_i \to \infty$ such that $\Gamma(t_i)$ has length at most $\widetilde{K} = K + \varepsilon$ for some fixed positive ε. Hence when t_i is large, we can choose a point $p_i \in A_{t_i}$ and replace $\Gamma(t_i)$ by an embedded least length geodesic arc α_i passing through p_i and with a possible exterior angle at p_i. Since the subdomain of A with boundary $\alpha_i \cup \partial A$ is an annulus A_i, the Gauss-Bonnet formula shows

$$\int_{A_i} K\, dA = -\int_{\partial A_i} \kappa_g \geq -\int_{\partial A} \kappa_g - 2\pi.$$

Since each compact subdomain of A is eventually contained in A_i for i large, A has finite total curvature. □

Remark 5.1 *If the area of $A \subset \mathbb{T} \times \mathbb{R}$ is bounded from below by Kt, the intrinsic area of $A(t)$ grows at most linearly with respect to the distance from ∂A. The proof of the above lemma can be easily generalized to show that if A is an almost-complete Riemannian annulus with compact boundary and nonpositive curvature and at most linear area growth, then A has finite total curvature. (In fact quadratic area growth implies finite total curvature.)*

Proposition 5.1 *Suppose $\Sigma \subset \mathbb{T} \times [0, \infty)$ is a properly embedded minimal surface with compact boundary and more than one end. Then Σ has area* *growth bounded from below by $C(\mathrm{T}) \cdot t$ where $C(\mathrm{T})$ is a positive constant that only depends on T. Furthermore there exists a noncompact, flat minimal annulus with compact boundary in $\mathrm{T} \times [0,\infty)$ that is disjoint from Σ.*

Proof. Without loss of generality we may assume that $\partial\Sigma$ is nonempty and contained in $\mathrm{T} \times \{0\}$. Since Σ has more than one end, after removing a compact subdomain of Σ, we may assume that Σ is not connected and contains two noncompact components Σ_1 and Σ_2. Let N denote the closure of one of the components of $\mathrm{T} \times [0,\infty) - M$ that contains at least two distinct components of Σ on its boundary. We will assume that two of these components are Σ_1 and Σ_2. [Note if Σ_1 (resp. Σ_2) does not separate $\mathrm{T}\times[0,\infty)$, then Σ_1 (resp. Σ_2) appears on the boundary of N two times, once from each side of Σ_1 (resp. Σ_2). If Σ_i appears in ∂N two times, fix and denote one of these occurrences as $\Sigma_i \subset \partial N$.]

Since ∂N is a good barrier for solving Plateau type problems in N, if $\partial\Sigma_1$ bounds in N, $\partial\Sigma_1$ is the boundary of a compact embedded minimal surface Δ in N, which by the maximum principle must be contained in $\partial N \cap (\mathrm{T}\times\{0\})$. Clearly, in this case, $\Delta = \partial N \cap (\mathrm{T}\times\{0\})$, which is impossible since $\partial(\partial N \cap \mathrm{T} \times \{0\})$ contains $\partial\Sigma_2$.

By the proof of Theorem 3.5, $\partial\Sigma_1$ is the boundary of a smooth properly embedded orientable stable noncompact minimal surface F of least area in N such that F separates N. The least-area surface F has finite total curvature (Theorem 3.3) and so the annular ends of F, which are asymptotic to flat annuli in $\mathrm{T}\times[0,\infty)$, must have area growth bounded from below by $(\tilde{C}(\mathrm{T}) - \varepsilon)t$ where ε is any small positive number and $\tilde{C}(\mathrm{T})$ is the minimum length of a closed geodesic on T. In particular F has area growth bounded from below by $\frac{1}{2}\tilde{C}(\mathrm{T})t$. Let $C(\mathrm{T}) = \frac{1}{2}\tilde{C}(\mathrm{T})$. Since F separates N, $(\Sigma_1 \cup F) \cap (\mathrm{T} \times \{t\})$ bounds a subdomain of $\mathrm{T} \times \{t\}$ of area less than $Area(\mathrm{T})$. Hence, if the area growth of F is greater than $(\tilde{C}(\mathrm{T}) - \varepsilon)t$ for any positive ε, then the area

growth of Σ_1 is also greater than $C(\mathrm{T})t$. This implies the area growth of Σ is also greater than $C(\mathrm{T})t$.

It remains to prove that there exists a flat annulus in the complement of Σ. Let $\tilde{\Sigma}_1 \subset N$ denote the union of Σ_1 and a small ε-neighborhood of $\partial\Sigma_1$ in ∂N. Using $\tilde{\Sigma}_1$ in place of Σ_1 in the construction of F, we obtain a properly embedded minimal surface $\tilde{F}$ in N with $\partial\tilde{F} = \partial\tilde{\Sigma}_1$ and the ends of $\tilde{F}$ are minimal annuli with finite total curvature. Note, by the maximum principle, $\tilde{F}$ is disjoint from Σ since every component of $\partial N - \partial\tilde{\Sigma}_1$ that intersects Σ is not smooth. Hence, there is a stable minimal annulus $\tilde{A}$, which is an end of $\tilde{F}$, that is contained in $\mathrm{T} - \Sigma$ and $\partial\tilde{A}$ is compact. By the maximum principle at infinity $dist(\tilde{A},\Sigma) > 0$. Since $\tilde{A}$ is stable, it has finite total curvature. A simple calculation using the Weierstrass representation of $\tilde{A}$ shows $\tilde{A}$ is asymptotic to a flat minimal annulus A which is also a positive distance from Σ. In particular $A \cap \Sigma = \emptyset$, which completes the proof of the proposition. □

Actually the proof of Proposition 5.1 shows that each end of a Σ satisfying the hypotheses of Σ in Proposition 5.1 must have area growth bounded from below by $C(\mathrm{T})t$. This observation proves:

Corollary 5.1 *Suppose $\Sigma \subset \mathrm{T} \times [0,\infty)$ satisfies the hypotheses of Proposition 5.1. If Σ has at least n ends, then its area growth must be at least $n \cdot C(\mathrm{T})$. In particular if Σ has an infinite number of ends, the area growth of Σ is not bounded from below by Kt for any positive constant K.*

The second statement in Proposition 5.1, concerning the existence of a flat annulus, implies that every end $\overline{\alpha}$ of Σ has an end-representative $E(\overline{\alpha})$ such that the induced map $\pi_1(E(\overline{\alpha})) \to \pi_1(\mathrm{T}\times\mathbb{R})$ is not onto. On the other hand, if a properly embedded minimal surface $\Sigma \subset \mathrm{T} \times [0,\infty)$ has one end $\overline{\alpha}$ and Σ contains an end-representative $E(\overline{\alpha})$ such that $\pi_1(E(\overline{\alpha})) \to \pi_1(\mathrm{T} \times [0,\infty))$

not onto, then there is a two-sheeted cover of $\mathsf{T} \times [0,\infty)$ such that the inverse image of Σ to this covering space satisfies the hypotheses of Proposition 5.1. These observations prove

Corollary 5.2 *Suppose $\Sigma \subset \mathsf{T} \times [0,\infty)$ is a properly embedded minimal surface with compact boundary and at least one end. Then Σ satisfies the conclusions of Proposition 5.1 if and only if Σ has an end-representative E such that the induced map $\pi_1(E) \to \pi_1(\mathsf{T})$ is not onto.*

Proposition 5.2 *Suppose $\Sigma \subset \mathsf{T} \times [0,\infty)$ is a connected properly embedded noncompact minimal surface with compact boundary. If Σ has more than one end or Σ has one end with end-representative E such that $\pi_1(E) \to \pi_1(\mathsf{T} \times [0,\infty))$ is not onto, then Σ has linear area growth.*

Before proving Proposition 5.2 we state and prove some of its important corollaries.

Corollary 5.3 *If $M \subset \mathsf{T} \times \mathsf{R}$ is a properly embedded minimal surface, then M has a finite number of ends.*

Proof. If M has an infinite number of ends, then we may assume that M is transverse to $\mathsf{T} \times \{0\}$ and $\Sigma = M \times [0,\infty)$ has an infinite number of ends. By Corollary 5.1, Σ does not have linear area growth but Proposition 5.2 implies Σ has linear area growth, a contradiction. □

The next corollary is an immediate consequence of the statement of Proposition 5.2.

Corollary 5.4 *If $M \subset \mathsf{T} \times \mathsf{R}$ is a properly embedded minimal surface and E is an end representative of M with $\pi_1(E) \to \pi_1(\mathsf{T} \times \mathsf{R})$ not onto, then E has linear area growth.*

Lemma 5.1 and Corollary 5.3 and 5.4 imply

Theorem 5.2 *Suppose M is a properly embedded minimal surface in $\mathsf{T} \times \mathsf{R}$. Then M has a finite number of ends and each annular end of M has finite total curvature. In particular if M has finite genus, then it has finite total curvature.*

It remains to prove Proposition 5.2.

Proof of Proposition 5.2. Suppose Σ satisfies the hypotheses of Proposition 5.2. By Proposition 5.1 there exists a flat annulus A in $\mathsf{T} \times [0,\infty)$ that is disjoint from Σ. Since $\partial((\Sigma \cup A) \cap \mathsf{T} \times [t,\infty)) \subset \mathsf{T} \times \{t\}$ for t large, we may assume that both A and Σ have their boundary on $\mathsf{T} \times \{0\}$. Let $\widetilde{N}$ be the geodesic closure of $(\mathsf{T} \times [0,\infty)) - A$ and note that $\widetilde{N}$ has two sides A_1, A_2 that correspond to parallel copies of A. Note that $\partial \widetilde{N} = A_1 \cup A_2 \cup A_3$ where A_3 is a compact flat horizontal annulus. Consider the set of compact parallel flat annuli in $\widetilde{N}$ with boundary in $A_1 \cup A_2$ and that are orthogonal to $A_1 \cup A_2$. Let Δ be one of these flat annuli which is transverse to Σ. Let N be submanifold of $\widetilde{N}$ obtained by removing the component of $\widetilde{N} - \Delta$ containing ∂A_1. This new N has the advantage over $\widetilde{N}$ in that, after a rotation of coordinates Δ is horizontal and the sides of N are vertical. See Figure 4 for a picture.

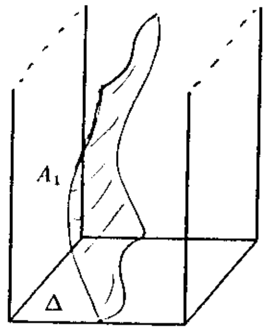

Figure 4:

Suppose now that Σ has area that grows faster than Kt for any constant K. We shall derive a contradiction by considering two possible cases.

Case 1 There exists a positive number ε such that the subdomain $\Sigma(\epsilon)$ of Σ whose normal vector makes an angle of at least ε radians with the vertical line has area greater than Kt, for any positive K and for some values of t that are arbitrarily large and that depend on K.

Let $F(t)$ be the horizontal compact annulus of height t in N. Since the normal vector to $\Sigma(\varepsilon)$ stays a bounded distance ε away from the horizontal, there exists a t_0 such that $F(t_0)$ is transverse to $\Sigma(\varepsilon)$ and the following equation holds:

$$\sin(\varepsilon)\cdot \text{Length } (F(t_0)\cap\Sigma(\varepsilon)) \geq \text{Length } (\partial\Sigma).$$

Let $\widehat{\Sigma} = \Sigma \cap (\mathbb{T}\times[0,t_0])$. Recall that the height function X of $\widehat{\Sigma}$ is harmonic. Thus, if we let η denote the conormal to $\widehat{\Sigma}$, the divergence theorem implies

$$\sin(\varepsilon)\cdot\text{Length } (F(t_0)\cap\Sigma(\varepsilon)) \leq \int_{F(t_0)\cap\Sigma} \nabla X_3\cdot\eta = \int_{\partial\Sigma}\nabla X_3\cdot\eta \leq \text{Length } (\partial\Sigma),$$

which contradicts our previous equation. This contradiction implies Case 1 can not occur.

Case 2 $W = \Sigma - \Sigma(\frac{1}{4})$ has area growth that is faster than Kt for any constant K.

This time let $F(t)$ be one of the two families of flat compact minimal annuli in N with boundary curves on the sides of N and whose normal vector makes an angle of $\frac{1}{2}$ radian with the vertical. As in Case 1 there exists a t_0 such that $F(t_0)$ is transverse to $\Sigma(\frac{1}{4})$ and the length of this intersection curve is at least equal to the length of $\partial\Sigma$ divided by $\sin(\frac{3}{4})$. Let $\widehat{\Sigma}$ denote the compact subdomain of Σ with boundary $\partial\Sigma\cup(F(t_0)\cap\Sigma)$. As in Case 1 we obtain a contradiction via the divergence theorem applied to the harmonic function on $\widehat{\Sigma}$ that is induced by the linear function on N that has level sets containing the foliation $\{F(t)\}$.

Since Case 1 or Case 2 must occur, the proposition is proved. □

6 Singly-periodic minimal surfaces

6.1 Proof of the Structure Theorem for one-periodic minimal surfaces with more than one end.

Recall from Section 1.3 the definition of limit tangent plane for a properly embedded minimal surface M in $\mathbb{R}^3$: *A limit tangent plane for M is the limit tangent plane passing through the origin of any properly embedded noncompact orientable minimal surface Σ in $\mathbb{R}^3$ with compact boundary $\partial\Sigma\subset M$*

and finite total curvature and such that Σ is contained in the closure N of one of the components of $\mathbb{R}^3 - M$. Theorem 5 in [6] states that when M has at least two ends, M has a unique limit tangent plane. Assume now that M has at least two ends and we shall sketch the proof that M has a unique limit tangent plane.

By Theorem 3.5 M has a limit tangent plane. It remains to prove the uniqueness of the limit tangent plane. Suppose Σ_1 and Σ_2 are two minimal surfaces satisfying the conditions of Σ in the previous paragraph and which have nonparallel limit tangent planes. Clearly we may assume that Σ_1 and Σ_2 are connected. Since the ends of Σ_1 are parallel, as are the ends of Σ_2, if an end-representative of Σ_1 is disjoint from an end-representative of Σ_2, then Σ_1, would have the same limit tangent plane. Hence we may assume:

1. Every end-representative of Σ_1 intersects every end-representative of Σ_2;
2. Σ_1 and Σ_2 are contained in the closure N of the same component of $\mathbb{R}^3 - M$;
3. $\Sigma_1 \cap N = \partial\Sigma_1$ and $\Sigma_2 \cap \partial N = \partial\Sigma_2$.

It follows directly from the proof of Theorem 3.5 that there exists a simple closed curve γ on M that separates M into two components M_2, M_2 where $\partial\Sigma_1 \cup \partial\Sigma_2$ are contained in M_2, and such that γ is the boundary of a properly embedded minimal surface Σ_3 of finite total curvature where Σ_3 is contained in the closure of one of the components of $\mathbb{R}^3 - (M \cup \Sigma_1 \cup \Sigma_2)$. Since the ends of Σ_3 are disjoint from the ends of Σ_1 and of Σ_2, the limit tangent planes of Σ_1 and of Σ_2 are parallel to the limit tangent planes of Σ_3 and hence parallel to each other. This contradiction completes the outline of the existence and uniqueness of the limit tangent plane for M.

The uniqueness of the limit tangent plane to M will be used in the proof of the Structure Theorem (Theorem 1.10). We will also need:

Lemma 6.1 *Suppose M is a properly embedded minimal surface in $\mathbb{R}^3$ with more than one end and infinite total curvature. Then there exists an end E of a plane or a catenoid with a circle boundary such that $E \subset \mathbb{R}^3 - M$ and the limit tangent planes of E and of M are the same. Furthermore, the boundary circle of E is not homologous to zero in the component of $\mathbb{R}^3 - M$ in which it is contained and E is a positive distance from M.*

Proof. If M has finite total curvature, then E can be chosen to be the end of any plane that is parallel to the limit tangent plane and such that this plane is not asymptotic to one of the planar ends of M. Suppose now that M has infinite total curvature.

Theorem 3.5 implies there exists a simple closed curve $\gamma \subset M$ that separates M and such that γ is not homologous to zero mod 2 in the closure N of one of the components of $\mathbb{R}^3 - M$. Furthermore, γ is the boundary of a properly embedded orientable minimal surface Σ of finite total curvature in N and $\Sigma \cap \partial N = \gamma$. It follows that at least one of the annular ends A of Σ has ∂A not homologous to zero in N. By the maximum principle at infinity $dist(A, M) > 0$. The annulus A is asymptotic to the end $\tilde{E}$ of a plane or catenoid that has an end E that is also a positive distance from M and $E \subset N$. Clearly ∂E is not homologous to zero in N, which completes the proof of the lemma. □

We now sketch the proof of the Structure Theorem in Section 1.3. See [6] for further details in this proof.

Proof of Theorem 1.10. Suppose M is a properly embedded minimal surface in $\mathbb{R}^3$ with infinite symmetry group and more than one end and suppose M is not a catenoid. We shall prove that there exists a plane P, parallel to the limit tangent plane at infinity for M, whose intersection with M consists of a finite number of simple closed curves. Theorem 1.10 follows easily from the existence of the plane P.

Suppose that the (x_1, x_2)-plane is the limit tangent plane to M. Since M has more than one end and it is not invariant under a rotation of infinite order, it must be invariant under a screw motion symmetry σ of infinite order whose linear part fixes the limit tangent plane to M, which is the (x_1, x_2)-plane. Hence σ is either a translation or, after a horizontal translation of M, a screw motion S_θ.

Let E be the minimal annulus in Lemma 6.1. If E is the end of a plane, then a small translation of this plane, yields a new plane P that is transverse to M and whose end is still a positive existence from M. In the case P is the required plane. It remains only to show E can not be the end of a catenoid.

Assume, after a rigid motion that ∂E is a circle centered at the origin in the (x_1, x_2)-plane and E is a nonnegative graph over this plane. Let D be the flat disk with $\partial D = \partial E$. Since ∂E is not homologous to zero in $\mathbb{R}^3 - M$, the subdomain $M^+ = \{x \in M \mid x \text{ lies above } E \cup D\}$ has nonempty intersection with M.

We first show that σ is not a translation by a vector in the (x_1, x_2)-plane. If it were, consider the σ-orbit Γ_3 of E. Let H denote the upper half space in $\mathbb{R}^3$. Since M^+ is disjoint from Γ, M^+ is contained in one of the components C of $H - \Gamma$ which is contained between two vertical half planes. In particular the convex hull of M^+ is not $\mathbb{R}^3$ or a slab, a contradiction of Theorem 3.7. This proves σ must have a translation component that is vertical.

Suppose now that the symmetry σ has a positive vertical translational part. Thus, for a large power f of σ, we may assume that $f(M_+) \subset M_+$. Since ∂M_+ is compact, there is a large k such that $\partial(M_+ - f^k(M)) \neq \partial(f^k(M))$.

Let T be the (x_1, x_2)-plane. Let $M^+(k)$ denote the portion of M^+ below

$f^k(T)$. By the divergence theorem the flux of the vector field ∇X_3 on $M^+(k)$ across the top boundary curves of $\partial M^+(k)$ equals the negative of the flux of ∇X_3 across the lower boundary curves ∂M^+ of $M^k(k)$. But since ∇X_3 is invariant by f^k, the flux across $f^k(\partial M^+)$ equals the flux across the portion $f^k(\partial M^+) \subsetneq M^+(k) \cap f^k(T)$, a contradiction since there is also flux across the boundary curves $(M^+(k) \cap f^k(T)) - f^k(\partial M^+)$. This contradiction completes the proof of the existence of the plane P and our outline of the proof of Theorem 1.10. □

6.2 Outline of the proof of Theorem 1.1.

In this section we will give a brief outline the proof of Theorem 1.1. By Theorem 5.2 we know that Theorem 1.1 is true for doubly-periodic minimal surfaces and so we need only prove the theorem for singly-periodic surfaces. The first step is to study the geometry of a properly embedded minimal annulus A (with one boundary curve) in a complete flat three-manifold N with fundamental group $\mathbb{Z}$. Here, $N = \mathbb{R}^3/S_\theta$ where S_θ is a screw motion symmetry of $\mathbb{R}^3$ generated by rotation by θ around the x_3-axis followed by a nontrivial vertical translation. Next, by applying Theorem 3.5, one proves that $A \subset N$ can be "trapped" between two minimal annuli of finite total curvature. The third step is to show that the finite total curvature trapping annuli can be chosen to be flat vertical annuli or to be planar or helicoid-type ends. This step of the proof involves the development of an analytic representation for a minimal surface M of finite total curvature in N in terms of two meromorphic forms on the conformal completion of M (see Theorem 3.10). The final and most difficult step is to use the trapping of A by standard examples of finite total curvature minimal annuli to prove A must have finite total curvature. This proof breaks up into three cases depending on the geometry of the trapping annuli.

Recall that the three types of trapping annuli are planar, vertical flat annuli or helicoid-type. The proofs of all three cases depend partly on a new technical tool which involves the construction of a foliation of compact minimal annuli. By examining the intersection of A with the leaves of the foliation we are able to prove that A contains an end that is stable and, therefore, has finite total curvature (Theorem 3.3). This foliation construction first appeared in work of Hoffman and Meeks [27] and we will use it to understand the case where A is trapped between two minimal annuli F_1, F_2 with planar-type ends.

After lifting A to $\mathbb{R}^3$, we see that A can be trapped between two parallel planes P_{-1} and P_1, which we may assume are the horizontal planes of height 1 and -1, respectively. After a homothety of A and the removal of a compact portion of A, we may assume that ∂A is contained in the cylinder radius 1 centered along the x_3-axis. Choose a compact minimal catenoid C with axis the x_3-axis, invariant under reflection in the (x_1, x_2)-plane, has its boundary circles on $P_{-1} \cup P_1, C$ is a radial graph transverse to A, and ∂A is contained inside the bounded component of $\mathbb{R}^3 - (P_{-1} \cup P_1 \cup C)$.

Let S denote the slab between $P_{-1} \cup P_1$. For $t \geq 1$ let $C(t) = (t \cdot C) \cap S$ be the portion of the homothetic expansion of C that lies inside S. Let $\Gamma = \bigcup_{t \geq 1} C(t)$ and let $F: \Gamma \to [1, \infty)$ be the function whose level set at t is $C(t)$. Let $A_\Gamma = A \cap \Gamma$. Since the level sets of F are minimal and A is minimal, $f = F|A_\Gamma$ has no local maximum or minima in $\text{Int}(A_\Gamma)$, f obtains its minimal value along ∂A_Γ and the critical points of f have strictly negative index (since minimal surfaces that are tangent at a point intersect in a hyperbolic manner).

Since $f: A_\Gamma \to [1, \infty)$ is proper and A_Γ is a planar domain with, say, $n+1$ boundary components, the Morse inequalities imply that f has exactly n critical points connected with multiplicity. Since A has one end, for any

t greater than the largest critical value of f, $f^{-1}(t)$ is a simple closed curve on A. This means that for t large, $C(t) \cap A$ is a simple closed curve on the compact catenoid $C(t)$ and let t_0 be such a value of t. Hence, after replacing A by $(\frac{1}{t_0})A$ and removing the portion of A inside $C = C(1)$, we may assume that $A \subset \Gamma$ and $\partial A \subset C$.

We want to show A contains a subannulus $\tilde{A}$ such that the normal vector to $\tilde{A}$ is never horizontal, which will prove $\tilde{A}$ is stable and hence has finite total curvature. If such a $\tilde{A}$ fails to exist, then there exists a sequence $\{p(i)\}$ of points on A with vertical tangent planes $\{T(i)\}$ and for all i and we can choose p_i so that $|p_i| > i$.

Case 1 For some i, $T(i) \cap C = \emptyset$ and p_i is in the (x_1, x_2)-plane.

In this case there exists circle α_1 in P_1 that is tangent to the line $T(i) \cap P_1$, its projection onto the (x_1, x_2)-plane contains p_i, and such that the planar disk with boundary α_1 contains the circle $\alpha_0 = C \cap P_1$ in its interior. Suppose $p \in P_1$ is the center of $\alpha(1)$ and $\alpha(1)$ has radius R. For $t \geq 1$ let $\alpha(t)$ denote the circle of radius $t + R - 1$ in P_1 with center p. Fill in the annulus in P_1 with boundary $\alpha(0) \cup \alpha(1)$ by circles $\alpha(t), 0 \leq t \leq 1$, so that $\mathcal{C}(1) = \{\alpha(t) \mid t \geq 0\}$ is a smooth foliation of the annulus $\cup \mathcal{C}$. Let $\mathcal{C}(-1)$ denote the vertical translate to P_1 and let $\mathcal{F} = \{F(t)\}$ be the related foliation of catenoids where $F(t)$ is the stable catenoid with boundary $\alpha(t) \cup (\alpha(t) - (0, 0, 2))$. Note by construction of $\mathcal{F}$, $F(0) = C$ and the catenoid $F(t_0)$ containing $p(i)$ has the same tangent plane as A at $p(i)$.

Let $f: A \to [0, \infty)$ denote the induced proper function corresponding to the minimal foliation $\mathcal{F}$. As in the earlier considered case f, has n critical points where $n+1$ is the number of boundary curves of A. Since $n + 1 = 1$, f has no critical points. However f has a critical point at $p(i)$. This contradiction proves that Case 1 can not occur.

In the general case, one modifies the above construction of $\mathcal{F}$ to show that, when $|p(i)|$ is sufficiently large, C is the leaf of a foliation $\tilde{\mathcal{F}}(i)$ of the exterior of C in S by compact minimal annuli and such that some leaf of $\tilde{\mathcal{F}}(i)$ is not transverse to A at $p(i)$. The construction of $\tilde{\mathcal{F}}(i)$ in the case $A \subset S$ is carried out in detail in [27] and [49]. This completes our outline, by way of an example, of how foliations of minimal annuli can be used to prove a trapped minimal annulus has finite total curvature.

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