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Elliptic operators on manifolds (*)

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(*) Notes used by P. Berard (Université de Grenoble I, France) for his course "Linear Analysis for the Geometer"

ELLIPTIC OPERATORS ON MANIFOLDS

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Abstract

ELLIPTIC OPERATORS ON MANIFOLDS.

I. An abstract Dirichlet problem: Hilbert space background; Representation of functionals; Compact transformations; The Dirichlet problem. II. Elliptic operators on the torus: Group duality and Fourier analysis; The Hilbert spaces; Dirichlet's problem on T^n; Regularity of solutions; Zero boundary values in $\mathbb{R}^n$; Strongly elliptic systems. III. Differential operators on vector bundles. Sheaves of modules; Vector bundles; Smooth manifolds and vector bundles; Certain operators on vector bundles; Riemannian structures; Differential operators. IV. The existence theorem and applications: The existence theorem; Hodge's theorem.

INTRODUCTION

The primary object of this paper is to present an elementary and self-contained proof of the following fundamental existence theorem. *Let X be a compact smooth manifold (without boundary), and let ξ, η be smooth vector bundles (finite fibre dimension) over X. Let A be a smooth elliptic operator from the sections of ξ to the sections of η. If ψ is a smooth section of η, then there is a smooth section ϕ of ξ such that $A\phi = \psi$ if and only if ψ is orthogonal to the kernel of the adjoint of* A. Furthermore, $\dim \mathrm{Ker}(A) < \infty$, so that deviation from uniqueness is somewhat restricted.

There are several well established approaches to this problem; for example:

(1) **The methods of potential theory**, centering around the properties of singular integral equations and pseudodifferential operators; see Seeley [17] for an exposition in appropriate generality.

(2) **Schwarz's alternating method** (Hildebrandt [7]).

(3) **The heat equation method** of Milgram-Rosenbloom [12], put in a Hilbert space framework by Gaffney [4].

(4) **The theory of coercive quadratic forms** in Hilbert space, based on Gårding's inequality. It is this fourth method that we shall develop. It should be clear that our exposition owes a great deal to Bers-John-Schechter [2] (their chapters on Hilbert space methods are the basis for our Section II), Koszul [9], Singer [18]. A treatment of Hodge's theorem in this direction was given by Morrey-Eells [13].

The material was presented at Cornell University in the Fall of 1964, and at the University of Amsterdam in the Spring of 1966. It is included in the present volume primarily because that fundamental theorem has been so frequently used during our Summer Course.

I. AN ABSTRACT DIRICHLET PROBLEM

1. HILBERT SPACE BACKGROUND

(A) Let us review briefly certain aspects of the theory of Hilbert spaces, over the real number field $\mathbb{R}$; the modifications necessary to produce the analogous theory for the complex field $\mathbb{C}$ will be taken for granted.

Definition. A *pre-Hilbert space* is a vector space E together with an inner product $\langle\,,\rangle$. We will let $\langle\,,\rangle_E$ denote $\langle\,,\rangle$ if we wish to emphasize the space to which the inner product belongs. Thus $\langle\,,\rangle$ is a symmetric bilinear form $E \times E \to \mathbb{R}$ such that $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0$ when and only when $x = 0$. For each $x \in E$ we write $|x| = +\sqrt{\langle x, x\rangle}$; then we have the

Schwarz inequality: For any $x, y \in E$, $|\langle x, y\rangle| \leq |x|\,|y|$. It follows at once that the function $x \to |x|$ is a norm on E:

(1) $|x| = 0$ if and only if $x = 0$;
(2) $|ax| = |a|\,|x|$ for all $(a, x) \in \mathbb{R} \times E$, where $|a|$ denotes the absolute value of a;
(3) For any $x, y \in E$ we have $|x + y| \leq |x| + |y|$.

In particular, setting $\rho(x, y) = |x - y|$ defines a *metric on* E; and that in turn determines a Hausdorff topology on E relative to which the algebraic operations are continuous.

We have also the

Parallelogram law: $|x+y|^2 + |x - y|^2 = 2(|x|^2 + |y|^2)$

Pythagoras' law: If x and y are orthogonal, then $|x + y|^2 = |x|^2 + |y|^2$.

Definition. A *Hilbert space* is a pre-Hilbert space which is complete in the metric ρ. Every pre-Hilbert space E has a unique completion to a Hilbert space E_1, whose points can be viewed as the totality of Cauchy sequences of E; and E is a dense subspace of E_1.

If $(E, |\;|_1)$ and $(E, |\;|_2)$ are Hilbert spaces, we say that they are *equivalent* if there is a number $c > 0$ such that

$$c^{-1}|x|_1 \leq |x|_2 \leq c|x|_1 \qquad \text{for } x \subset E$$

This is an equivalence relation; the equivalence classes are called *Hilbertian spaces*. Thus a Hilbertian space is a topological vector space whose topology can be given by an inner product whose induced metric is complete.

Example. Let E be the totality of smooth functions $x : I \to \mathbb{R}$, where $I = [0, 1]$. Then

$$\langle x, y\rangle_1 = \int_I \langle x(t) y(t)\, dt + \int_I x'(t)\, y'(t)\, dt$$

$$\langle x, y\rangle_2 = x(0)\, y(0) + \int_I x'(t)\, y'(t)\, dt$$

are both inner products on E; let E_1 and E_2 be the completions of E in the indicated metric. Then E_1 and E_2 are topologically equivalent Hilbert spaces, and hence are two representations of the same Hilbertian space, really the fundamental entity. The elements are those absolutely continuous functions on I having square integrable first derivatives.

Example. Any two inner products on an n-dimensional vector space determine equivalent Hilbert

(B) **Theorem.** *A Hilbertian space is locally compact if and only if it is finite dimensional.*

Proof of the sufficiency. Let $(e_i)_{1 \leq i \leq n}$ be a base for E. Define the isomorphism $\phi : \mathbb{R}^n \to E$ by

$$\phi(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i e_i$$

Then ϕ is a homeomorphism; for this is true for $n = 1$ by the axioms for a norm, and we can apply induction. Thus any n-dimensional vector space E is topologically isomorphic to $\mathbb{R}^n$, whence E is locally compact.

Proof of the necessity. We can suppose $B = \{x \in E : |x| \leq 1\}$ is compact without loss of generality. Suppose $\dim E = \infty$; then there is an infinite orthonormal set $(e_i)_{i \geq 1}$ in B, which we can suppose is convergent. But that contradicts Pythagoras' law: $2 = |e_i|^2 + |e_j|^2 = |e_i - e_j|^2$.

(C) **Theorem.** *Let* V *be a closed subspace of the Hilbert space* E. *Then*
(1) for any $x \in E$, *there is a unique* $y \in V$ *such that* $|x - y| = \rho(x, V)\ (= \inf[|x - z| : z \in V]$. So y is the point in V nearest to x.
(2) y is the only point in V *for which* $x - y \perp V$.

Proof (1). Let $\alpha = \rho(x, V)$, and let $(y_i)_{i \geq 1} \subset V$ be a sequence such that $|x - y_i| \to \alpha$. Then (y_i) is Cauchy. By the parallelogram law

$$|y_i - y_j|^2 = |y_i - x + x - y_j|^2 = 2(|x - y_i|^2 + |x - y_j|^2) - 4|x - (y_i + y_j)/2|^2$$

But $(y_i + y_j)/2 \in V$, whence $4|x - (y_i + y_j)/2|^2 \geq 4\alpha^2$. Given $\epsilon > 0$, choose i_0 such that $i, j \geq i_0$ implies $|x - y_i|^2 \leq \alpha^2 + \epsilon$, $|x - y_j|^2 \leq \alpha^2 + \epsilon$. Then

$$|y_i - y_j|^2 \leq 2(\alpha^2 + \epsilon + \alpha^2 + \epsilon) - 4\alpha^2 = 4\epsilon$$

Since $(y_i) \subset V$ is Cauchy and V is closed, we find that y_i approaches some $y \in V$, whence $\rho(x, y_i) \to \rho(x, y)$ and $\rho(x, y) = \alpha$. Suppose y' were another such point. Then

$$|y - y'|^2 = |y - x + x - y'|^2 = 2(|y - x|^2 + |y' - x|^2) - 4|x - (y + y')/2|^2$$

$$\leq 2(\alpha^2 + \alpha^2) - 4\alpha^2 = 0$$

whence $y = y'$.

Proof (2). For any $z \in V$ such that $z \neq 0$ and any number $\lambda \neq 0$, we have $|x - (y + \lambda z)|^2 > \alpha^2$. Thus $2\lambda\langle z, y - x\rangle < \lambda^2|z|^2$ for all real $\lambda \neq 0$. But this cannot be (for all $|\lambda|$ small) unless $\langle z, y - x\rangle = 0$.

Suppose now $y' \in V$ is such that $x - y' \perp V$. Then

$$\alpha^2 = |x - y|^2 = |x - y'|^2 + |y - y'|^2$$

But $\alpha \leq |x - y'|$, whence $|y - y'|^2 = 0$.

(D) Proposition. *If* E *and* F *are Hilbert spaces and* $\phi: E \to F$ *a linear map, then* ϕ *is continuous if and only if there is a real number* b *such that*

$$|\phi(x)|_F \leq b|x|_E \quad \text{for all } x \in E$$

Lemma. *If* $\phi: E \to F$ *is a linear map for which there exists a number* $b > 0$ *such that*

$$b^{-1}|\phi(x)|_F \leq |x|_E \leq b|\phi(x)|_F \quad \textit{for all } x \in E$$

then ϕ *is continuous, injective, and* $\phi(E)$ *is closed in* F.

Proof. The first two assertions are immediate. To prove that $\phi(E)$ is closed, let $(y_i)_{i \geq 1}$ be a Cauchy sequence in $\phi(E)$, and let $x_i \in E$ be the points for which $\phi(x_i) = y_i$. Then $|x_i - x_j|_E \leq b|y_i - y_j|_F \to 0$ as $i, j \to \infty$. Let $x \in E$ be the limit of x_i in E, and set $y = \phi(x)$. Then $|y - y_i|_F = |\phi(x - x_i)|_F \leq b|x - x_i|_E \to 0$, whence $y \in \phi(E)$ is the limit of the y_i.

(E) Let $\pi: E \to V$ be the nearest-point map $x \to y$ given by (1) in Theorem 1C. Then clearly π is surjective and is linear, and is continuous. (Proof of linearity: $x + y - \pi(x + y)$ is $\perp V$, and so is $x + y - (\pi(x) + \pi(y))$. But there is *only one* vector z such that $(x + y) - z \perp V$, whence $\pi(x + y) = \pi(x) + \pi(y)$. Similarly for $\pi(ax) = a\pi(x)$ for $(a, x) \in \mathbb{R} \times E$.)

Proposition. Kernel $\pi = \{x \in E : \pi(x) = 0\}$ *is the orthogonal complement of* V, *written* $V^\perp$. *We have* $E = V \oplus V^\perp$; i.e. *every* $x \in E$ *can be written uniquely* $x = \pi(x) + (x - \pi(x))$ *with* $\pi(x) \in V$, $x - \pi(x) \in V^\perp$ *and these components are orthogonal.*

Corollary. *If* $V \neq E$, *there exists a* $u \neq 0$ *in* E *such that* $u \perp V$.

Proof. Take any $x \in E$, $x \notin V$. Then we can write $x = \pi(x) + u$, $u \perp V$. $u \neq 0$, for otherwise $x = \pi(x)$, whence $x \in V$.

2. REPRESENTATION OF FUNCTIONALS

(A) Representation theorem. *Let* $f: E \to \mathbb{R}$ *be a continuous linear form. Then there exists a unique* $w \in E$ *such that* $f(x) = \langle x, w\rangle$ *for all* $x \in E$; *if* $|f| = \sup\{|f(x)| : |x| = 1 \text{ in } E\}$, *then* $|f| = |w|$.

Proof. Let V = kernel f; then V is a closed linear subspace, for if $(x_i)_{i \geq 1} \subset V$ and $|x_i - x| \to 0$ then $|f(x)| = |f(x - x_i)| \leq \text{const}\,|x - x_i| \to 0$; whence $x \in V$. If $V = E$, take $w = 0$. Otherwise there exists a u in E such that $|u| = 1$ and $u \perp V$. Then $f(u) \neq 0$, and for all $x \in E$

$$f\left(x - \frac{f(x)}{f(u)}u\right) = f(x) - f(x) = 0$$

Thus $x - \dfrac{f(x)}{f(u)}u \in V$, whence $\left\langle x - \dfrac{f(x)}{f(u)}u, u\right\rangle = 0$, i.e. $\langle x, u\rangle = \dfrac{f(x)}{f(u)}|u|^2$. It follows that $f(x) = \langle x, f(u)u/|u|^2\rangle$. Taking $w = f(u)u/|u|^2$ satisfies the conditions of the theorem. If w' is another representation of f, then $f(x) = \langle x, w\rangle = \langle x, w'\rangle$, whence $(w' - w) \perp E$, and by Corollary 1E we have $w' - w = 0$.

To prove $|f| = |w|$, we first observe that $|w|^2 = \dfrac{f(u)}{|u|^2} \cdot \dfrac{f(u)}{|u|^2}\langle u, u\rangle$, whence $|w| = \dfrac{|f(u)|}{|u|} \leq |f|$. Conversely, for every $\epsilon > 0$ there exists an $x_\epsilon \in E$ such that $|x_\epsilon| = 1$ and $|f| - \epsilon \leq f(x_\epsilon) = \langle x_\epsilon, w\rangle \leq |x_\epsilon||w| = |w|$. Since this is true for all $\epsilon > 0$, we have $|f| \leq |w|$.

(B) Consider now a continuous linear map $\phi: E \to F$. For each $y \in F$ the map $f: x \to \langle\phi(x), y\rangle_F$ is a continuous linear form on E. By Theorem 2A there is a unique element $x_y \in E$ such that

$$f(x) = \langle x, x_y\rangle_E \quad \text{for all } x \in E$$

Thus we have a map $F \to E$ defined by $y \to x_y$, which we will call ϕ^*, *the adjoint of* ϕ. Clearly ϕ^* is a continuous linear map, satisfying $\langle\phi(x), y\rangle_F = \langle x, \phi^* y\rangle_E$ for all $x \in E$, $y \in F$. In particular, $\phi^{**} = \phi$.

(C) Representation theorem. *Let* $\beta: E \times E \to \mathbb{R}$ *be a bilinear form such that:*
(1) $|\beta(x, y)| \leq \text{const}\,|x||y|$
(2) $|x|^2 \leq b\beta(x, x)$
for all x, y ∈ E *for some strictly positive* $b \in \mathbb{R}$. *Given any continuous linear form* $f: E \to \mathbb{R}$, *there is a unique* $v \in E$ *such that*

$$f(x) = \beta(x, v) \quad \textit{for all } x \in E$$

Suppose first that β is symmetric. Then $|x|^2 \leq b\beta(x, x) \leq \text{const}|x|^2$ shows that β is a topologically equivalent inner product on E, and the theorem follows from Theorem 2A. Thus the emphasis of the present theorem is absence of symmetry of β.

Proof. Take any $y \in E$; then the map $x \to \beta(x, y)$ is a continuous linear form on E, whence by Theorem 2A there is a unique element, which we will call $Sy \in E$, such that

$$\beta(x, y) = \langle x, Sy\rangle$$

It follows that S is a linear endomorphism of E which is continuous: $|Sx|^2 = \langle Sx, Sx\rangle = \beta(Sx, x) \leq \text{const}\,|Sx|\,|x|$, whence $|Sx| \leq \text{const}\,|x|$ for all $x \in E$.

Furthermore, $|x| \leq \text{const}|Sx|$ for all $x \in E$, so that S maps E bijectively onto a closed linear subspace (by Lemma 1D). In fact, S is surjective, for otherwise there is a $w \perp S(E)$ and $w \neq 0$. This would imply that $\langle w, Sw\rangle = 0$ so that $\beta(w, w) = 0$; i.e. $w = 0$, a contradiction.

Now take the form f, and let w be its representative: $f(x) = \langle x, w\rangle$. Then there is a unique $v \in E$ such that $Sv = w$, so that $f(x) = \langle x, Sv\rangle = \beta(x, v)$ for all $x \in E$.

3. COMPACT TRANSFORMATIONS

(A) A linear transformation $\phi: E \to F$ is *compact* (or *completely continuous)* if ϕ maps bounded subsets of E into relatively compact subsets of F. (A subset is relatively compact if its closure is compact.) Such a ϕ is bounded (i.e. is continuous), for otherwise there is a sequence $(x_i)_{i \geq 1} \subset E$ with $|x_i|_E = 1$, $|\phi x_i|_F \to \infty$; but $(\phi(x_i))_{i \geq 1}$ is relatively compact, so that a subsequence would converge to an element of F.

Proposition. *Let* $\phi: E \to F$ *be a compact linear transformation, and set* $\psi = I - \phi$. *Then* $\psi: E \to E$ *is a continuous linear map whose kernel* $K(\psi)$ *has finite dimension.*

Proof. Clearly ψ is continuous and linear. Let $(x_i)_{i \geq 1} \subset K(\psi)$ be a bounded sequence. Then $x_i = \phi(x_i)$ for all $i \geq 1$, whence $(x_i)_{i \geq 1}$ is a relatively compact subset of E. Thus every bounded sequence in $K(\psi)$ has a convergent sequence, which implies that Kernel ψ is locally compact.

Example. Let E be the completion of the smooth functions $x: I \to \mathbb{R}$ with the inner product

$$\langle x, y\rangle_E = \int x(t)y(t)\,dt + \int x'(t)\,y'(t)\,dt$$

Let F be the completion of the same space in

$$\langle x, y\rangle_F = \int_I x(t)\, y(t)\, dt$$

Then the inclusion map $\phi : E \to F$ is compact, for if (x_i) is a bounded sequence in E, then it can be proved that (x_i) is equicontinuous, whence by Ascoli's theorem (x_i) has a convergent subsequence.

Let $K^\perp(\psi) = \{x \in E : \langle x, y\rangle = 0 \text{ for all } y \in \operatorname{Ker}\psi\}$. $K^\perp(\psi)$ is called the *orthogonal complement of* $K(\psi)$; it is clearly a closed linear subspace of E.

Lemma. *There is a number* $c > 0$ *such that* $c^{-1}|x| \leq |\psi(x)| \leq c|x|$ *for all* $x \in K^\perp(\psi)$.

Proof. The second inequality is clear. If the first were false, there would exist a sequence $(x_i)_{i\geq 1} \subset K^\perp(\psi)$ such that $|x_i| = 1$ and $|\psi(x_i)| \to 0$. But $x_i = \phi(x_i) + \psi(x_i)$ and $(\phi(x_i))_{i\geq 1}$ has a convergent subsequence, whence $(x_i)_{i\geq 1}$ has a convergent subsequence (still called $(x_i)_{i\geq 1}$) converging to a point $x \in K^\perp(\psi)$. The continuity of ϕ implies that $\phi(x) = x$, so that $x \in K(\psi)$; i.e. $x = 0$. On the other hand, continuity of the norm shows that $|x| = 1$, contradiction.

(B) **Lemma.** *If* $\phi : E \to E$ *is compact, then so is its adjoint* ϕ^*.

Proof. Suppose (x_i) is a bounded sequence in E. Since ϕ^* is continuous, $(\phi^*(x_i))$ is also bounded, so that $(\phi\phi^*(x_i))$ has a convergent subsequence, still called $(\phi\phi^*(x_i))$. Then

$$|\phi^*(x_i) - \phi^*(x_j)|^2 = \langle \phi^*(x_i - x_j), \phi^*(x_i - x_j)\rangle = \langle x_i - x_j, \phi\phi^*(x_i - x_j)\rangle \leq \text{const}\, |\phi\phi^*(x_i - x_j)| \to 0$$

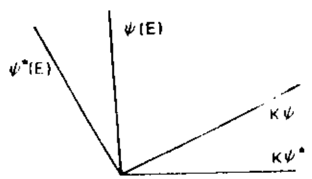
i.e. $(\phi^*(x_i))$ is convergent.

Proposition. *Given* $y \in E$, *there is a* $v \in E$ *such that* $\psi^*(v) = v - \phi^*(v) = y$ *if and only if* $y \in K^\perp(\psi)$.

Proof. The necessity is clear. To prove the sufficiency, we first remark that by Lemma 3A the functions $x \to |x|$ and $x \to |\psi(x)|$ are equivalent norms on the Hilbertian space $K^\perp(\psi)$. Since the form $f(x) = \langle x, y\rangle$ is continuous on $K^\perp(\psi)$, it follows from Theorem 2A that there is some $u \in K^\perp(\psi)$ for which

$$\langle x, y\rangle = f(x) = \langle \psi(x), \psi(u)\rangle \quad \text{for all } x \in K^\perp(\psi)$$

In fact, this holds for all $x \in E$. For $K(\psi)$ has finite dimension and is therefore closed; by Proposition 1E we can write every $x \in E$ uniquely in the form $x = x' + x''$ with $x' \in K^\perp(\psi)$, $x'' \in K(\psi)$. Then $\langle \psi(x), \psi(u)\rangle = \langle \psi(x'), \psi(u)\rangle = \langle x', y\rangle = \langle x, y\rangle$, the last equality because y is orthogonal to every $x'' \in K(\psi)$. If we now set $v = \psi(u)$, we find that $\langle x, \psi^*(v)\rangle = \langle \psi(x), v\rangle = \langle x, y\rangle$ for all $x \in E$, whence $\psi^*(v) = y$.



Corollary. *Given* $y \in E$ *there is a* $v \in E$ *such that* $\psi(v) = v - \phi(v) = y$ *if and only if* $y \in K^\perp(\psi^*)$; i.e. $K^\perp(\psi) = \psi^*(E)$ and $K^\perp(\psi^*) = \psi(E)$. In particular, ψ and ψ^* have closed ranges and finite-dimensional kernels and cokernels.

(C) It is an easy matter to check that the composition and linear combinations of compact linear endomorphisms of E are again compact. Thus the n^{th} iterate:

$$\psi^n = (I - \phi)^n = I + \sum_{k=1}^{n} \binom{n}{k} (-1)^k \phi^k = I - T$$

where T is a compact linear endomorphism of E. In particular, setting $K^p = K(\psi^p)$, we find that (K^p) is a non-decreasing sequence of finite-dimensional subspaces of E.

Lemma. *There is a positive integer* n *such that*

$$K^p \subsetneqq K^{p+1} \quad \text{for } p < n$$

$$K^p = K^n \quad \text{for } p > n$$

Proof. First of all, that $K^p = K^{p+1}$ implies $K^p = K^{p+2} = \dots = K^{p+k}$ follows at once. If the lemma were false, we could find a sequence $(x_p)_{p\geq 1} \subset E$ such that $|x_p| = 1$, $x_p \in K^{p+1}$ and $\langle x_p, K^p\rangle = 0$. Then $(\phi x_p)_{p\geq 1}$ has no convergent subsequence, because $|\phi x_p - x_{p-q}|^2 \geq 1$. Namely, $\phi(x_p - x_{p-q}) = x_p - (\psi x_p + \phi x_{p-q})$, and $\psi^p(\psi x_p + \phi x_{p-q}) = \psi^{p+1} x_p + \phi\psi^p x_{p-q} = 0$, whence x_p and $\psi x_p + \phi x_{p-q}$ are orthogonal. By the law of Pythagoras, $|\phi x_p - \phi x_{p-q}|^2 = |x_p|^2 + |\psi x_p + \phi x_{p-q}|^2 \geq 1$. But this contradicts the compactness of ϕ.

Theorem. *If* $\phi : E \to E$ *is compact, then* $\psi = I - \phi$ *is injective if and only if* ψ *is surjective.*

Proof of the sufficiency. Suppose there is a non-zero element $x_0 \in K(\psi)$. Let $(x_i)_{i\geq 1}$ be chosen inductively so that $\psi x_i = x_{i-1}$. Then $\psi^p x_p = x_0 \neq 0$, $\psi^{p+1}(x_p) = \psi x_0 = 0$; i.e. $K^p \subsetneqq K^{p+1}$ for all p, contradicting the lemma. Thus $K(\psi) = 0$.

Proof of the necessity. The hypothesis $K(\psi) = 0$ implies that ψ^* is surjective by Proposition 3B. We then apply the preceding argument to ψ^* to conclude that $K(\psi^*) = 0$. Corollary 3B shows that ψ is surjective.

(D) **Theorem.** $K(\psi)$ *and* $K(\psi^*)$ *have the same finite dimension.*

Proof. Let these spaces have dimensions n and n^*. Without loss of generality we can suppose $n^* \geq n$; and we shall show that strict inequality leads to a contradiction.

Let (u_i) and (u_i^*) be orthonormal bases for $K(\psi)$ and $K(\psi^*)$ respectively. The operator

$$x \to \phi(x) - \sum_{i=1}^{n} \langle u_i, x\rangle u_i^*$$

is compact, being the sum of two compact operators. We define

$$\theta(x) = \psi(x) + \sum_{i=1}^{n} \langle u_i, x\rangle u_i^*$$

Now if $\theta(x) = 0$, then $x = 0$. For

$$0 = \langle u_j^*, \theta(x)\rangle = \langle u_j^*, \psi(x)\rangle + \sum_{i=1}^{n} \langle u_i, x\rangle\langle u_i^*, u_j^*\rangle = \langle \psi^* u_j^*, x\rangle + \langle u_j, x\rangle = \langle u_j, x\rangle$$

for all $1 \leq j \leq n$, whence $\psi(x) = 0$, and therefore $x = 0$. Thus θ is an injective map of the type considered in Theorem 3C, from which we conclude that there is an element v of E such that $\theta(v) = u_{n+1}^*$. But

$$1 = |u_{n+1}^*|^2 = \langle u_{n+1}^*, \theta v\rangle = \langle u_{n+1}^*, \psi v\rangle + \sum_{i=1}^{n} \langle u_i, v\rangle\langle u_{n+1}^*, u_i^*\rangle = 0$$

contradicting the assumption $n^* > n$.

(E) For any $\lambda \in \mathbb{R}$, the endomorphism $\lambda\phi$ is compact if ϕ is.

Proposition. *Let $\psi_\lambda = 1 - \lambda\phi$. Then $\dim K(\psi_\lambda) > 0$ for at most countably many λ having no finite accumulation point.*

Proof. Suppose there were a sequence $(\lambda_i)_{i \geq 1}$ of distinct bounded non-zero numbers such that each $\dim K(\psi_{\lambda_i}) > 0$. Choose $0 \neq x_i \in K(\psi_{\lambda_i})$ for each i; then for each n, the elements $x_1, \ldots, x_n$ are linearly independent. For, suppose $x_1, \ldots, x_{n-1}$ are linearly independent and

$$\sum_{j=1}^{n} c_j x_j = 0$$

Then

$$0 = \psi_{\lambda_n}\left(\sum_{j=1}^{n} c_j x_j\right) = \sum_{j=1}^{n} c_j x_j - \sum_{j=1}^{n} \lambda_n c_j \phi(x_j)$$

and because $\phi(x_j) = x_j/\lambda_j$, we find

$$\sum_{j=1}^{n-1} \left(1 - \frac{\lambda_n}{\lambda_j}\right) c_j x_j = 0$$

so that $c_1 = \ldots = c_{n-1} = 0$; it follows that $c_n = 0$ too; i.e. $x_1, \ldots, x_n$ are linearly independent.

Let V_n be the subspace spanned by $x_1, \ldots, x_n$; then there are elements $v_n \in V_n$ such that $|v_n| = 1$ and $v_n \perp V_{n-1}$. Further, if $w \in V_n$, then $w - \lambda_n\phi(w) \in V_{n-1}$.

$$w = \sum^{n} c_j x_j$$

whence

$$w - \lambda_n\phi(w) = \sum_{j=1}^{n}{}' c_j x_j - \lambda_n \sum_{k=1}^{n}{}' \frac{c_j}{\lambda_j} x_j = \sum_{j=1}^{n-1}{}' \left(1 - \frac{\lambda_n}{\lambda_j}\right) c_j x_j$$

But $(\phi(\lambda_n v_n))_{n \geq 1}$ has no convergent subsequence since $\phi(\lambda_n v_n - \lambda_m v_m) = v_n - (v_n - \lambda_n\phi(v_n) + \lambda_m\phi(v_m))$, and for $n > m$ we can apply Pythagoras' law:

$$|\phi(\lambda_n v_n - \lambda_m v_m)|^2 = |v_n|^2 + |v_n - \lambda_n\phi(v_n) + \lambda_m\phi(v_m)|^2 \geq 1$$

But because ϕ is compact, this contradicts the boundedness of $(\lambda_i)_{i \geq 1}$. Hence the numbers $\lambda : \dim K(\psi_\lambda) > 0$ can be counted.

Let us collect several of the preceding results as a theorem, often referred to as the

Fredholm Alternative Theorem. *Let ϕ be a compact endomorphism of the Hilbertian space E, and form the endomorphism $\psi_\lambda = 1 - \lambda\phi$. Then there is:*

(1) at most a countable sequence of real numbers λ for which $\dim K(\psi_\lambda) > 0$. For such values there is a solution of $\psi_\lambda(x) = x - \lambda\phi(x) = y$ if and only if $y \perp K(\psi_\lambda^*)$.

(2) If λ is a value for which $\dim K(\psi_\lambda) = 0$, then for all $y \in E$, there is a unique $x \in E$ for which

$$\psi_\lambda(x) = x - \lambda\phi(x) = y$$

4. THE DIRICHLET PROBLEM

(A) Let V and E be Hilbert spaces, and suppose we have a compact injection of V into E; in particular, considering V as a vector subspace of E (from the algebraic viewpoint only), there is a constant such that $|x|_E \leq \text{const}\,|x|_V$ for all $x \in V$. Suppose furthermore that V is dense in E.

Let $\alpha : V \times V \to \mathbb{R}$ be a bilinear form for which:

(1) there exists $a \in \mathbb{R}$ such that $|\alpha(x, y)| \leq a|x|_V|y|_V$ for all $x, y \in V$.

(2) there are numbers $c > 0$, $\lambda_0 > 0$, such that $\alpha(x, x) + \lambda_0|x|_E^2 \geq c|x|_V^2$ for all $x \in V$. This will be called the *coercivity condition* on α. In physical terms $\alpha(x, x)$ has an interpretation as a sort of energy.

For any positive $\lambda \in \mathbb{R}$, set $\alpha_\lambda(x, y) = \alpha(x, y) + \lambda\langle x, y\rangle_E$. Then $|\alpha_\lambda(x, y)| \leq a|x|_V|y|_V + \lambda|x|_E|y|_E \leq \text{const}|x|_V|y|_V$, where the constant depends on λ. Also, for $\lambda \geq \lambda_0$ we have $\alpha_\lambda(x, x) = \alpha(x, x) + \lambda_0|x|_E^2 + (\lambda - \lambda_0)|x|_E^2 \geq c|x|_V^2$ for all $x \in V$.

(B) Suppose now we are given $u \in E$; then the linear form $f : V \to \mathbb{R}$ given by $f(x) = \langle x, u\rangle_E$ is continuous on V, for $|\langle x, u\rangle_E| \leq |x|_E|u|_E \leq \text{const}|x|_V$ for all $x \in V$. Thus by Theorem 2C for any $\lambda \geq \lambda_0$ there is a unique $v \in V$ such that $f(y) = \alpha_\lambda(v, y)$ for all $y \in V$; i.e. for $u \in E$, there is a unique $v \in V$ such that

$$\alpha(v, y) + \lambda\langle v, y\rangle_E = \langle u, y\rangle_E \quad \text{for all } y \in V$$

Thus for $\lambda \geq \lambda_0$ we have a linear map $G_\lambda : E \to V$ given by $u \to v$ (G_λ plays the role of Green's function in potential theory). We have whence $|G_\lambda(u)|_V^2$ const$|u|_E |G_\lambda(u)|_E$ so that $|G_\lambda(u)|_V \leq$ const$|u|_E$; i.e. G_λ is a continuous linear map. Because the injection $V \to E$ is compact, we obtain the

Proposition. *Take* $\lambda \geq \lambda_0$. *The composition* $G_\lambda : E \to E$ *is a compact endomorphism such that for any* $u \in E$, $G_\lambda(u)$ *is the unique element such that*

$$\alpha(G_\lambda(u), y) + \lambda\langle G_\lambda(u), y\rangle_E = \langle u, y\rangle_E$$

for all $y \in V$.

This result leads us to consideration of the operator $I - \lambda G_\lambda$, to which we can apply the methods of Section 3.

(C) **Definition.** The *first null space* of α is $K_1(\alpha) = \{x \in V : \alpha(x, y) = 0$ for all $y \in V\}$. Similarly for the *second null space* $K_2(\alpha)$. If we define $\alpha^*(x, y) = \alpha(y, x)$, then clearly $K_1(\alpha^*) = K_2(\alpha)$.

Proposition. $x \in K_1(\alpha)$ *if and only if for* $\lambda \geq \lambda_0$, $x - \lambda G_\lambda(x) = 0$. *In particular,* $\dim K_1(\alpha) < \infty$.

Proof. For any $x \in V$, $\alpha(G_\lambda(x), y) + \lambda\langle G_\lambda(x), y\rangle_E = \langle x, y\rangle_E$, so that $\alpha(G_\lambda(x), y) = \langle x - \lambda G_\lambda(x), y\rangle_E$. If $x = \lambda G_\lambda(x)$, then $0 = \alpha(G_\lambda(x), y) = \lambda^{-1}\alpha(x, y)$ for all $y \in V$.

Conversely, we compute

$$\alpha_\lambda(x - \lambda G_\lambda(x), y) = \alpha_\lambda(x, y) - \lambda\alpha_\lambda(G_\lambda(x), y)$$

$$= \alpha_\lambda(x, y) - \lambda\langle x, y\rangle_E$$

$$= \alpha(x, y) = 0 \quad \text{for all } y \in V$$

The coercivity condition now shows that $x - \lambda G_\lambda(x) = 0$.

Lemma. *If* G_λ^* *is Green's function associated with* α^*, *then* G_λ *and* G_λ^* *are adjoints relative to the inner product of* E *for* $\lambda \geq \lambda_0$.

Proof.

$$\langle x, G_\lambda^*(y)\rangle_E = \alpha_\lambda(G_\lambda(x), G_\lambda^*(y))$$

$$= \alpha(G_\lambda(x), G_\lambda^*(y)) + \lambda\langle G_\lambda(x), G_\lambda^*(y)\rangle_E$$

$$= \alpha^*(G_\lambda^*(y), G_\lambda(x)) = \langle y, G_\lambda(x)\rangle_E$$

It follows that G_λ^* is a compact endomorphism of E, and the corresponding results above apply to G_λ^*. For instance, $\dim K_2(\alpha) = \dim K_1(\alpha^*) < \infty$.

(D) **Proposition.** Given $u \in E$, *there is a solution* $v \in V$ *of* $\alpha(v, y) = \langle u, y\rangle_E$ *for all* $y \in V$ *if and only if* $u \perp_E K_2(\alpha) = K_1(\alpha^*)$.

Proof. First of all, $\alpha(v, y) = \langle u, y\rangle_E$ for all $y \in V$ if and only if $(I - \lambda G_\lambda)v = G_\lambda u (\lambda \geq \lambda_0)$; for

$$\alpha(v - \lambda G_\lambda(v), y) + \lambda\langle v - \lambda G_\lambda(v), y\rangle_E = \langle u, y\rangle_E$$

$$\alpha(v, y) - \lambda[\alpha(G_\lambda(v), y) + \lambda\langle G_\lambda v, y\rangle_E] + \lambda\langle v, y\rangle_E = \langle u, y\rangle_E$$

whence $\alpha(v, y) = \langle u, y\rangle_E$. The necessity is obtained by reversing the steps, using the uniqueness of $G_\lambda(u)$.

By Corollary 3B there is a $v \in E$ with $v - \lambda G_\lambda(v) = G_\lambda(u)$ (or $G_\lambda(u) \in$ image $I - \lambda G_\lambda$) if and only if $G_\lambda(u) \perp_E K(I - \lambda G_\lambda^*)$. But then $v \in V$. By Proposition 4C, $K(I - \lambda G_\lambda^*) = K_1(\alpha^*)$, where $\lambda \geq \lambda_0$. But $G_\lambda(u) \perp K(I - \lambda G_\lambda^*)$ when and only when $u \perp K(I - \lambda G_\lambda^*)$, for $\langle u, y\rangle_E = \lambda\langle u, G_\lambda^* y\rangle_E = \lambda\langle G_\lambda u, y\rangle_E$ for all $y \in K(I - \lambda G_\lambda^*)$.

For future reference let us formulate our results as follows:

Theorem. *Let* V *and* E *be Hilbert spaces and* $V \to E$ *a compact dense injection. Suppose that* $\alpha : V \times V \to \mathbb{R}$ *is a* V*-continuous coercive bilinear form. Then*

(1) *For all* $u \in E$ *and* $\lambda \geq \lambda_0$, *there is a unique solution* v *of* $\alpha(v, y) + \lambda\langle v, y\rangle_E = \alpha_\lambda(v, y) = \langle u, y\rangle_E$ *for all* $y \in V$.

(2) $\alpha(v, y) = \langle u, y\rangle_E$ *has a solution if and only if* $u \perp_E K_2(\alpha)$.

(3) $\dim K_1(\alpha) = \dim K_2(\alpha) < \infty$.

(E) Consider $N = \{x \in V :$ the form $y \to \alpha(x, y)$ is E continuous on $V\}$; because V is dense in E it follows that we can extend this form to be defined on E. By Theorem 2A, for each $x \in N$ there is an element $A(x) \in E$ such that $\alpha(x, y) = \langle A(x), y\rangle_E$ for all $y \in V$. Then A is linear on N, called *the domain of* A and written henceforth Dom(A), but not necessarily continuous.

Lemma. Dom(A) *is dense in* E.

Proof. Take $u \in E$ such that $\langle x, u\rangle_E = 0$ for all $x \in$ Dom(A). For $\lambda \geq \lambda_0$, there is a unique $v \in$ Dom(A^*) for which $A_\lambda^*(v) = u$, whence $\langle A_\lambda(x), v\rangle_E = \langle x, A_\lambda^*(v)\rangle_E \langle x, u\rangle = 0$ for all $x \in$ Dom(A). But A_λ is surjective, whence v is E-orthogonal to all E. It follows that $v = 0$, whence $u = 0$.

Remark. Let V and E be as above, and let $A : V \to E$ be a continuous linear map with closed range and $\dim K(A) < \infty$. Then the bilinear form $\alpha(x, y) = \langle Ax, Ay\rangle_E$ on V is a special case of the preceding situation, and is the object of the study [6].

In Part II we shall be interested in densely defined operators $A :$ Dom(A) $\to$ E (not necessarily satisfying the strong conditions of the preceding Remark). These seem to be most easily studied – as we have done in this section – through the associated bilinear form $\alpha(x, y) = \langle Ax, y\rangle_E$, with solutions given by Theorem 4D; if we think of A as a differential operator, then these solutions are thus given, so to speak, in integrated form.

II. ELLIPTIC OPERATORS ON THE TORUS

1. GROUP DUALITY AND FOURIER ANALYSIS

(A) In the next section we shall study elliptic differential operators on the torus. There are certain special features of this case which provide simplification in the analytic theory. Basically this happens because the Fourier transform is a tool used essentially in the study of differential operators, and on the torus it is especially simple. (Here we have an example of a principle prevalent in modern mathematics: in certain problems a dualization is possible, and the dual problem is sometimes easier to handle. For instance, cohomology is richer in structure than homology; the dual space of a Banach space has special features not always present in the given space.) In our case dualization is provided by the Fourier transform. We illustrate the idea briefly now before specializing to the torus.

Let G be a locally compact abelian group; its dual group $\hat{G}$ is the totality of continuous homomorphisms of G into the unit circle S^1. With natural (weak) topology, $\hat{G}$ also has the structure of a locally compact abelian group. The Pontrjagin duality theorem asserts that the natural map $G \times \hat{G} \to S^1$ is a dual pairing, written $x, y \to (x, y)$ and called the character function; i.e. the canonical map $G \to \hat{\hat{G}}$ is a topological isomorphism.

With every such G we have an (essentially unique) invariant Radon measure, called *Haar measure*, relative to which we can form the complex Hilbert space H(G) of square integrable complex functions on G. Given $f \in H(G)$ we can construct its *Fourier transform* $\hat{f} = F(f) \in H(\hat{G})$ by

$$\hat{f}(y) = \int_G f(x)\,\overline{(x, y)}\,dx$$

When the Haar measures on G and $\hat{G}$ are suitably normalized, the Plancherel theorem states that $F: H(G) \to H(\hat{G})$ is a bijective isometry.

Example. $G = \mathbb{R}^n$. Then $\hat{G} = \mathbb{R}^n$ too, and the character function is $(x, y) = \exp(i\langle x, y\rangle)$ for all $x, y \in \mathbb{R}^n$, where $\langle x, y\rangle = x_1 y_1 + \ldots + x_n y_n$. The Haar measure is (except for a multiplicative factor) Lebesgue measure. The Fourier transformation of $u \in H(\mathbb{R}^n)$ is

$$\hat{u}(y) = \int_{\mathbb{R}^n} u(x) e^{-i\langle x, y\rangle} dx$$

The Plancherel theorem asserts that

$$\int_{\mathbb{R}^n} |u(x)|^2\,dx = (2\pi)^n \int_{\mathbb{R}^n} |\hat{u}(y)|^2\,dy$$

(B) *Example.* The following example is basic motivation for all that follows. Let $\mathbb{Z}^n$ be the lattice subgroup of $\mathbb{R}^n$ consisting of the n-tuples of integers. The quotient group $G = T^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ is called the n-dimensional torus; its points can be thought of as n-tuples of real numbers modulo 2π. Then clearly $\hat{G} = \mathbb{Z}^n$ with character function $T^n \times \mathbb{Z}^n \to S^1$ defined by $(x, k) \to \exp(i\langle k, x\rangle)$. We form the Hilbert space $H(T^n)$ of square integrable (relative to the natural invariant measure on T^n) complex functions on T^n with inner product

$$\langle u, v\rangle_{H(T^n)} = \int_{T^n} u(x)\bar{v}(x)\,dx$$

The functions $x \to \exp(i\langle k, x\rangle)/(2\pi)^{n/2}$ for $k \in \mathbb{Z}^n$ form an orthonormal base for $H(T^n)$; i.e. if $e_k = \exp(i\langle k, x\rangle)/(2\pi)^{n/2}$, then $\langle e_p, e_q\rangle = \delta_{pq}$ for all $p, q \in \mathbb{Z}^n$, and their finite linear combinations (called *trigonometric polynomials*) are dense in $H(T^n)$, by the Riesz-Fischer theorem.

Now $H(\mathbb{Z}^n)$, the totality of maps $\mathbb{Z}^n \to \mathbb{C}$ of the form $k \to a_k$ with

$$\sum_{k \in \mathbb{Z}^n}' |a_k|^2 < \infty$$

is also a Hilbert space, relative to the inner product

$$\langle a, b\rangle_{H(\mathbb{Z}^n)} = (2\pi)^n \sum_{k \in \mathbb{Z}^n} a_k b_k$$

The *Fourier transform* $F: H(T^n) \to H(\mathbb{Z}^n)$ is given by $F(u) = \hat{u}$, where

$$\hat{u}(k) = \int_{T^n} u(x) e^{-i\langle k, x\rangle} dx$$

Thus we transfer the study of $H(T^n)$ to $H(\mathbb{Z}^n)$, thereby obtaining a definite simplification. Furthermore, if $H^s(T^n)$ denotes the completion of the smooth functions in the inner product (with s a positive integer)

$$\langle u, v\rangle_{H^s(T^n)} = \sum_{|\alpha| \leq s}' \langle D^\alpha u, D^\alpha v\rangle_{H(T^n)}$$

then the Fourier transform defines a topological isomorphism of $H^s(T^n)$ onto a Hilbert space $H^s(\mathbb{Z}^n)$; in fact, $H^s(\mathbb{Z}^n)$ is the completion of the trigonometric polynomials in the inner product

$$\langle a, b\rangle_{H^s(\mathbb{Z}^n)} = (2\pi)^n \sum_{k \in \mathbb{Z}^n}' (1 + |k|^2)^s a_k b_k$$

where $|k|^2 = k_1^2 + \ldots + k_n^2$. We shall not prove this, but mention it now only as motivation for the constructions to follow.

A powerful property of the Fourier transform is that it changes differentiation into multiplication. This fact lies behind the equivalence of the norms on $H^s(T^n)$ expressed in Proposition 2A below.

2. THE HILBERT SPACES

(A) We wish to consider real-valued *trigonometric polynomials on* T^n, i.e. functions u expressible in the form $u(x) = \sum u_k \exp(i\langle k, x\rangle)$, where the sum is extended over all $k \in \mathbb{Z}^n$, where the $u_k \in \mathbb{C}$ satisfy $u_{-k} = \bar{u}_k$, and where only finitely many $u_k \neq 0$. For any real number s let H^s denote the completion of these trigonometric polynomials in the inner product

$$\langle u, v\rangle_s = (2\pi)^n \sum' (1 + |k|^2)^s u_k v_{-k}$$

Note that

$$|u|_s \leqslant |u|_t \quad \text{if} \quad s \leqslant t$$

whence we have a continuous injection $H^t \to H^s$ if $s \leqslant t$.

Define $H^{\infty} = \cap\{H^t : t \in \mathbb{R}\}$ and $H^{-\infty} = \cup\{H^t : t \in \mathbb{R}\}$:

$$H^{\infty} \subset \ldots \subset H^s \subset \ldots \subset H^0 \subset \ldots \subset H^{-s} \subset \ldots \subset H^{-\infty}$$

Lemma (Schwar(t)z inequality). *For any trigonometric polynomials* u, v *we have*

$$|(u, v)_s| \leqslant |u|_{s+t} |v|_{s-t}$$

In particular, $|u|_s^2 \leqslant |u|_{s+t}|u|_{s-t}$.

This follows from

$$\left|\sum (1 + |k|^2)^{\frac{s+t}{2}} u_k (1 + |k|^2)^{\frac{s-t}{2}} v_{-k}\right|^2 \leqslant \left|\sum (1 + |k|^2)^{s+t} u_k u_{-k}\right| \left|\sum (1 + |k|^2)^{s-t} v_k v_{-k}\right|$$

Lemma. *For any* $0 < \epsilon < 1$ *and* $t_2 < s < t_1$ *we have*

$$|u|_s^2 \leqslant \epsilon |u|_{t_1}^2 + \epsilon^{-(s-t_2)/(t_1-s)} |u|_{t_2}^2$$

for all trigonometric polynomials. Also,

$$|u|_s \leqslant \epsilon |u|_{t_1} + \epsilon^{-(s-t_2)/(t_1-s)} |u|_{t_2}$$

Proof. We first note the inequality $a^s \leqslant \epsilon a^{t_1} + \epsilon^{-(s-t_2)/(t_1-s)} a^{t_2}$ for $a \geqslant 1$. Namely,

$$\epsilon a^{t_1-s} + \epsilon^{-(s-t_2)/(t_1-s)} a^{t_2-s} = \epsilon a^{t_1-s} + (\epsilon a^{t_1-s-(s-t_2)/(t_1-s)})$$

But this is always $\geqslant 1$, as we see by considering the cases $\epsilon a^{t_1-s} \geqslant 1$ and $\epsilon a^{t_1-s} \leqslant 1$; in either case one of the terms in question is $\geqslant 1$.

Applying this to $a = 1 + |k|^2$ for $k \in \mathbb{Z}^n$, multiplying by $u_k u_{-k}$ and summing gives the desired inequality for all trigonometric polynomials $x \to u(x) = \sum u_k \exp(i\langle k, x\rangle)$. The second inequality follows, replacing ϵ by ϵ^2.

Lemma. *For all* ϵ *such that* $0 < \epsilon < 1$ *we have*

$$|u|_s |u|_{s-1} \leqslant \epsilon |u|_s^2 + (\epsilon^{-2s+1}/2) |u|_0^2$$

for all trigonometric polynomials u, *if* $s \geqslant 1$. (This lemma is for use in Section 3.)

Proof. Assume first that $s > 1$.

$$|u|_s^2 |u|_{s-1}^2 \leqslant |u|_s^2 [\epsilon^2 |u|_s^2 + \epsilon^{-2(s-1)} |u|_0^2]$$

... ding lemma with ϵ replaced by ϵ^2 and taking $0 < s - 1 < s$.

By completion of the square

$$|u|_s^2 |u|_{s-1}^2 \leqslant \epsilon^2 [|u|_s^4 + \epsilon^{-2s} |u|_s^2 |u|_0^2 + (\epsilon^{-2s}/2)^2 |u|_0^4] \leqslant \epsilon^2 [|u|_s^2 + (\epsilon^{-2s}/2)|u|_0^2]^2$$

Now observe that the same estimates hold for $s = 1$, even without the preceding lemma.

Lemma. *The differential operator* $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ *(where* $|\alpha| = \alpha_1 + \ldots + \alpha_n$*) is a continuous map* $D^\alpha : H^{s+|\alpha|} \to H^s$, *with* $|D^\alpha u|_s \leqslant |u|_{s+|\alpha|}$ *for all* $u \in H^{s+|\alpha|}$. *There is a constant depending only on* s *such that*

$$|u|_s \leqslant \text{const} \sum_{|\alpha| \leqslant s} |D^\alpha u|_0$$

Proof. First of all, $D^\alpha u$ is only defined for trigonometric polynomials u. $D^\alpha u(x) = \sum (ik)^\alpha u_k \times \exp(i\langle k, x\rangle)$, where $(ik)^\alpha = (ik_1)^{\alpha_1} \ldots (ik_n)^{\alpha_n}$. Then

$$|D^\alpha u|_s^2 = (2\pi)^n \sum (1 + |k|^2)^s (ik)^\alpha u_k u_{-k} (-ik)^\alpha = (2\pi)^n \sum (1 + |k|^2)^s k^{2\alpha} u_k u_{-k}$$

But $k^{2\alpha} \leqslant (|k|^2)^{|\alpha|} \leqslant (1 + |k|^2)^{|\alpha|}$, so that

$$|D^\alpha u|_s^2 \leqslant (2\pi)^n \sum (1 + |k|^2)^{s+|\alpha|} u_k u_{-k} = |u|_{s+|\alpha|}^2$$

It now follows that D^α has a unique extension to a continuous linear map $H^{s+|\alpha|} \to H^s$.

On the other hand,

$$(1 + |k|^2)^s \leqslant \text{const} \sum_{|\alpha| \leqslant s} k^{2\alpha}$$

hence

$$|u|_s^2 = (2\pi)^n \sum (1 + |k|^2)^s u_k u_{-k} \leqslant \text{const} \sum_{|\alpha| \leqslant s} k^{2\alpha} u_k u_{-k} \leqslant \text{const} \sum_{|\alpha| \leqslant s} |D^\alpha u|_0^2$$

Putting these together, we obtain the

Proposition. *Suppose* s *is a positive integer. There are numbers* $a > 0$ *and* $c > 0$ *such that*

$$a^{-1} |u|_s^2 \leqslant \sum_{|\alpha| \leqslant s} |D^\alpha u|_0^2 \leqslant a |u|_s^2$$

$$c^{-1} \sum_{|\alpha| \leqslant s} |D^\alpha u|_0^2 \leqslant \left[|u|_0^2 + \sum_{|\alpha| = s} |D^\alpha u|_0^2 \right] \leqslant c \sum_{|\alpha| \leqslant s} |D^\alpha u|_0^2$$

for all $u \in H^s$.

Proof. The first equivalence of inner products follows at once from the preceding lemma. To prove the first inequality of the second equivalence we note that for every $\epsilon > 0$ there is a number b such that (taking $t_2 = 0$, $t_1 = t$, in a lemma above)

$$|D^\alpha u|_0 \leq |u|_{|\alpha|} \leq b|u|_0 + \epsilon |u|_t$$

for all α for which $|\alpha| < t$.

Thus we regard H^s as a Hilbertian space, and are free to choose any one of the above three inner products to describe its topology.

(B) Lemma. *Let*

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

be the Laplacian on $\mathbb{R}^n$, *and define* $K = I - \Delta$. *Then for any* t *we have*

$$\langle K^t u, v\rangle_s = \langle u, K^t v\rangle_s = \langle u, v\rangle_{s+t}$$

for any trigonometric polynomial u. *Furthermore,* $K^{t/2} : H^{s+t} \to H^s$ *is a bijective isometry; in particular,* $|K^{t/2}u|_s = |u|_{s+t}$.

This is immediate, because

$$Ku(x) = \sum (1 + |k|^2) u_k \exp(i\langle k, x\rangle) \quad \text{if} \quad u = \sum u_k \exp(i\langle k, x\rangle)$$

Proposition.[1] *Let* $s \geq 0$. *The* H^t*-inner product dually pairs* $H^{t-s} \times H^{t+s} \to \mathbb{R}$; *i.e. the continuous linear forms* f *on* H^{t+s} *are uniquely representable as the elements* $v \in H^{t-s}$, *with* $f(u) = \langle u, v\rangle_t$ *for all* $u \in H^{t+s}$; *and every* $v \in H^{t-s}$ *thus represents a continuous linear form on* H^{t+s}. *Furthermore,*

$$|v|_{t-s} = \sup\{\langle u, v\rangle_t : |u|_{t+s} = 1\}$$

The Schwar(t)z inequality shows that $\langle u, v\rangle_t$ is defined for $u \in H^{t+s}$ and $v \in H^{t-s}$.

Proof. First of all, clearly every $v \in H^{t-s}$ defines such a form. Given such a form $f : H^{t+s} \to \mathbb{R}$, there is a unique $\hat{v} \in H^{t+s}$ such that $f(u) = \langle u, \hat{v}\rangle_{t+s}$ for all $u \in H^{t+s}$, and $|\hat{v}|_{t+s} = \sup\{|f(u)| : |u|_{t+s} \leq 1\}$. Let $v = K^s \hat{v} \in H^{t-s}$, whence $|v|_{t-s} = |\hat{v}|_{t+s}$. Then $\langle u, v\rangle_t = \langle u, K^s \hat{v}\rangle_t = \langle u, \hat{v}\rangle_{t+s} = f(u)$.

If there were another such $v' \in H^{t-s}$, then $\langle u, v' - v\rangle_t = 0$ for all $u \in H^{t+s}$. Taking $u = K^{-s}(v' - v)$ gives

$$0 = \langle K^{-s}(v' - v), v' - v\rangle_t = |v' - v|^2_{t-s}$$

whence $v' = v$.

(C) Lemma. *If* ϕ *is a smooth function on* T^n, $u \in H^s$, *then* $\phi u \in H^s$ *and* $|\phi u|_s \leq \text{const}|u|_s$. *The constant depends on* ϕ.

[1] We have not yet defined K^t for t not a positive integer. Note that we can define K^t for t not a positive [illegible]

Proof. If $s \geq 0$, then by Proposition 2A

$$|\phi u|_s^2 \leq \text{const} \sum_{|\alpha| \leq s} |D^\alpha(\phi u)|_0^2 \leq \text{const}_\phi \sum_{|\alpha| \leq s} |D^\alpha u|_0^2 \leq \text{const}_\phi |u|_s^2$$

If $-t = s < 0$ then by Proposition 2B and the Schwar(t)z inequality,

$$|\phi u|_{-t} = \sup\{\langle \phi u, v\rangle_0 : |v|_t = 1\} = \sup\{\langle u, \phi v\rangle_0 : |v|_t = 1\} \leq |u|_{-t}|\phi v|_t \leq \text{const}_\phi |u|_{-t}$$

As an application we have the

Proposition. *Let* A *be a smooth differential operator* T^n *of order* r; *i.e.*

$$A = \sum_{|\alpha| \leq r} a_\alpha D^\alpha$$

where $a_\alpha : T^n \to \mathbb{R}$ *are smooth functions. Then* $A : H^s \to H^{s-r}$ *is continuous for all* s.

(D) Theorem (Rellich). *If* $s < t$, *then the injection* $H^t \to H^s$ *is compact and dense.*

Proof. The image is dense because it contains the trigonometric polynomials. If

$$u^j(x) = \sum u_k^j \exp(i\langle k, x\rangle)$$

is a sequence ($j \geq 1$) such that $|u^j|_t \leq M$ for all $j \geq 1$, we shall find a sequence which is H^s-Cauchy.

First of all, we have $|u_k^j| \leq M(1 + |k|^2)^{-t/2}(2\pi)^{-n/2}$. We order $\mathbb{Z}^n$ (denoted by $k_1, k_2, \ldots$) and then select inductively and successively on $k_1, k_2, \ldots$ convergent subsequences of the rows of

$$u_{k_1}^1, u_{k_1}^2, \ldots$$

$$u_{k_2}^1, u_{k_2}^2, \ldots$$

$$u_{k_3}^1, \ldots$$

The diagonal sequence, still called (u_k^j), will be convergent for every $k \in \mathbb{Z}^n$.

Now fix a positive integer N and write $u^j = u_N^j + v_N^j$, where

$$u_N^j(x) = \sum_{|k| \leq N} u_k^j \exp(i\langle k, x\rangle)$$

Then

$$|v_N^i - v_N^j|_s = (2\pi)^n \sum_{|k| \geq N} (1 + |k|^2)^s (u_k^i - u_k^j)(u_{-k}^i - u_{-k}^j)$$

$$= \sum \frac{1}{(1 + |k|^2)^{t-s}} (1 + |k|^2)^t |u_k^i - u_k^j|^2 \leq \frac{1}{(1 + N^2)^{t-s}} |u^i - u^j|_t^2$$

is arbitrarily small for N sufficiently large. Take such N. Then

$$|u_N^i - u_N^j|_s^2 = \left| \sum_{|k| \leqslant N} (u_k^i - u_k^j) \exp\sqrt{-1}\,(k, x) \right|_s^2$$

is arbitrarily small for i, j sufficiently large, since $(u_k^j)_{j=1}$ converge uniformly for $|k| \leqslant N$. It follows that the subsequence (u^j) is H^s-convergent.

3. DIRICHLET'S PROBLEM ON T^n

(A) Let

$$A = \sum_{|\alpha| \leqslant r} a_\alpha D^\alpha$$

be an elliptic differential operator of order r with smooth real periodic coefficients on $\mathbb{R}^n$. That means that its *symbol* $\sigma_A : T^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$\sigma_A(x, \xi) = (-1)^s \sum_{|\alpha| = r} a_\alpha(x) \xi^\alpha$$

is positive definite, where we have written $r = 2s$ and $\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}$. Viewed as a differential operator on T^n, it follows that there is a number $\lambda_0 > 0$ such that $\sigma_A(x, \xi) \geqslant \lambda_0 |\xi|^r$ for all $x \in T^n$, $\xi \in \mathbb{R}^n$. The following result is immediate:

Lemma. *The composition of elliptic operators on* T^n *is elliptic. If* A *is elliptic, then so is its formal adjoint* A^*, *given by*

$$A^* u = \sum_{|\alpha| \leqslant r} (-1)^{|\alpha|} D^\alpha (a_\alpha u)$$

Furthermore, $\langle Au, v\rangle_0 = \langle u, A^* v\rangle_0$ *for all trigonometric polynomials* u, v.

Form $\alpha(u, v) = \langle Au, v\rangle_0$ for all trigonometric polynomials u, v. It is our aim in this section to show that the hypotheses of Theorem 4D of Part I are satisfied, with $V = H^s$, $E = H^0$. Theorem 2D shows that $H^s \to H^0$ is a compact dense injection. Since $A : H^s \to H^{-s}$ is continuous by Proposition 2C, we have $|\alpha(u, v)| \leqslant |\langle Au, v\rangle_0| \leqslant |Au|_{-s} |v|_s \leqslant \text{const } |u|_s |v|_s$ for all $u, v \in H^s$.

Proposition (Gårding's inequality). *If* A *is an* r^{th}*-order smooth elliptic differential operator on* T^n, *then there are numbers* $c > 0$, $\lambda_0 > 0$ *such that*

$$\langle Au, u\rangle_0 + \lambda_0 |u|_0^2 \geqslant c|u|_s^2$$

for all $u \in H^s$. *Again* $r = 2s$. This will be proved in Section 3C.

Example: The operator $K^s = (1 - \Delta)^s$ is elliptic of order 2s. Lemma 2B shows that $\langle K^s u, u\rangle_0 = \langle u, u\rangle_s$, that Gårding's inequality is verified in a very strong sense.

(B) Before proving Gårding's inequality, we need to know that in $\mathbb{R}^n$ there are sufficiently many smooth functions to separate closed sets.

Lemma. *Let* $0 \leqslant a \leqslant b$. *Then there is a smooth function* $\phi : \mathbb{R} \to \mathbb{R}$ *such that* $\phi(t) = 0$ *if* $t \leqslant a$, $\phi(t) = 1$ *if* $t \geqslant b$ *and* $0 < \phi(t) < 1$ *otherwise.*

In fact, define $\phi_0(t) = \exp(-1/(t-a)(b-t))$ for $a < t < b$, and 0 elsewhere. Then set

$$\varphi(t) = \int_{-\infty}^{t} \phi_0(s)\,ds \Big/ \int_{-\infty}^{+\infty} \phi_0(s)\,ds$$

Proposition. *If* C_0, C_1 *are disjoint non-void, closed subsets of* $\mathbb{R}^n$, *then there is a smooth function* $\phi : \mathbb{R}^n \to \mathbb{R}$ *such that* $\phi(x) = 0$ *if* $x \in C_0$; $\phi(x) = 1$ *if* $x \in C_1$; *and* $0 \leqslant \phi(x) \leqslant 1$ *for all* $x \in \mathbb{R}^n$.

Proof. If C_1 is the closed disc centred at $0 \in \mathbb{R}^n$ and of radius $a > 0$, and C_0 is the complement of the open disc centred at 0 and of radius $b > a$, then set $\phi(x) = 1 - \psi(|x|)$, where ψ is defined as in the Lemma.

In the general case we start by taking an open covering $(U_k)_{k \geqslant 1}$ of C_1 by discs such that the family $(\bar{U}_k)_{k \geqslant 1}$ is locally finite and each $\bar{U}_k \cap C_0 = \emptyset$. Let $\phi_{1k} : \mathbb{R}^n \to \mathbb{R}$ be a smooth function such that $\phi_{1k}(x) > 0$ if $x \in U_k$, and $\phi_{1k}(x) = 0$ if $x \in \mathbb{R}^n - U_k$. Define $\phi_1 : \mathbb{R}^n \to \mathbb{R}$ by

$$\phi_1(x) = \sum_{k \geqslant 1} \phi_{1k}(x)$$

Thus $\phi_1(x) > 0$ if $x \in C_1$, whence there is a neighbourhood U of C_1 in which $\phi_1 > 0$. Set $C_1^* = \mathbb{R}^n - U$, $C_0^* = C_1$. These are disjoint closed subsets of $\mathbb{R}^n$, and we repeat the construction, giving a smooth function $\phi_2 : \mathbb{R}^n \to \mathbb{R}$ such that $\phi_2(x) > 0$ for $x \in \mathbb{R}^n - U$ and $\phi_2(x) = 0$ if $x \in C_0^*$.

Finally, the function

$$x \to \phi(x) = \phi_1(x)/[\phi_1(x) + \phi_2(x)]$$

satisfies the requirements of the proposition.

Definition. The *support of a function* ϕ is the closure $\{x : \phi(x) \neq 0\}$; denote it by $\text{spt}(\phi)$.

The following result is a special case of a theorem proved later.

Lemma. *Let* $(U_k)_{1 \leqslant k \leqslant m}$ *be a finite cover of* T^n *by open sets. Then there are smooth functions* $\phi_k : T^n \to I = [0, 1]$ *for* $1 \leqslant k \leqslant m$ *such that*

$$\sum_{k=1}^{m} \phi_k^2 = 1$$

and $\text{spt}(\phi) \subset U_k$.

Proof. First of all, construct by induction an open covering $(V_k)_{1 \leqslant k \leqslant m}$ of T^n such that each $\bar{V}_k \subset U_k$. Use it and the preceding proposition to define a smooth function $\psi_k : T^n \to \mathbb{R}$ such that

$\psi_k(x) > 0$ if $x \in V_k$ and $\mathrm{spt}(\psi_k) \subset U_k$. Then

$$x \to \sum_{k=1}^{m} \psi_k^2(x)$$

is a smooth positive function on T^n, whence the function

$$x \to \phi_k(x) = \psi_k(x) \Big/ \left[\sum_{k=1}^{m} \psi_k^2(x)\right]^{1/2}$$

satisfies the requirements of the lemma.

Similarly for the following result, needed in Section 5C.

Lemma. *Let* $(U_j)_{j \geq 0}$ *be a locally finite open cover of the open set* U *in* $\mathbb{R}^n$. *Then there are smooth functions* $(\phi_j)_{j \geq 0}$ *on* U *such that* $\mathrm{spt}(\phi_j) \subset U_j$ *for all* $j \geq 0$, $\sum_{j \geq 0} \phi_j(x) = 1$ *for all* $x \in U$, *and each* $0 \leq \phi_j(x) \leq 1$.

(C) We now proceed to the proof of Proposition 3A for all trigonometric polynomials $u(x) = \sum u_k \exp(i\langle k, x\rangle)$.

Case 1. All coefficients a_α are constant functions, and $a_\alpha = 0$ for $|\alpha| < r$. Then

$$\langle Au, u\rangle_0 = \left\langle \sum_k \sum_{|\alpha|=r} a_\alpha (ik)^\alpha u_k \exp(i\langle k, x\rangle)\ ,\ \sum_k u_k \exp(i\langle k, x\rangle)\right\rangle_0$$

But $a_\alpha(ik)^\alpha = (-1)^s a_\alpha k$, whence

$$\langle Au, u\rangle_0 = (-1)^s \sum_k \sum_{|\alpha|=r} a_\alpha k^\alpha |u_k|^2 = \sum_k \sigma_A(x, k)|u_k|^2 \geq \sum_k \lambda_0 |k|^r |u_k|^2$$

$$= \lambda_0 \sum_k [1 + |k|^r]|u_k|^2 - \lambda_0 |u|_0^2$$

Consider the function $f(x) = \lambda_0(1 + x^{2s})/(1 + x^2)^s$. Then there is a number such that $0 < c \leq f(x)$ for all positive $x \in \mathbb{R}$; in particular, $\lambda_0(1 + |k|^r) \geq c(1 + |k|^2)^s$. Therefore,

$$\langle Au, u\rangle_0 \geq c \sum_k (1 + |k|^2)^s |u_k|^2 - \lambda_0 |u|_0^2$$

$$\geq c|u|_s^2 - \lambda_0 |u|_0^2$$

Case 2. The coefficients $a_\alpha = 0$ for $|\alpha| < r$, and $A = A_0 + A_1$, where A_0 has constant coefficients,

$$A_1 = \sum_{|\alpha|=r}' b_\alpha D^\alpha$$

with all $|b_\alpha(x)| \leq \eta$ for some sufficiently small $\eta > 0$; the size of η will be determined in Cases 3, 4.

We apply integration by parts over T^n to obtain $\langle A_1 u, u\rangle_0 = I_1 + I_2$, where

$$I_1 = \sum' \int b_{\beta\gamma} D^\beta u D^\gamma u$$

is the collection of all terms with $|\beta| = |\gamma| = s$;

$$I_2 = \sum' \int b_{\beta\gamma} D^\beta u D^\gamma u$$

where $|\beta| + |\gamma| < r$ and $|\beta| \leq s$, $|\gamma| \leq s$. Now in I_1 the $b_{\beta\gamma}$ are just relabellings of the b_α, whence by Proposition 2A we have

$$|I_1| \leq \eta \left|\sum' \int D^\beta u D^\gamma u\right| \leq \eta\, \mathrm{const}|u|_s^2$$

Similarly, by the second and third lemmas in Section 2A, for every $\epsilon > 0$,

$$|I_2| \leq \mathrm{const}\,|u|_s |u|_{s-1} \leq \epsilon\ \mathrm{const}\,|u|_s^2 + \epsilon^{-2s+1}\mathrm{const}\,|u|_0^2$$

Suppose $c > 0$ and $\lambda_0 > 0$ are constants for Gårding's inequality for A_0. Because the preceding estimates for $|I_1|$ and $|I_2|$ are valid for any $\eta > 0$, $\epsilon > 0$, we can redefine $c > 0$ and $\lambda_0 > 0$:

$$\langle A_0 u, u\rangle_0 \geq c|u|_s^2 - \lambda_0|u|_0^2 + I_1 + I_2$$

$$\geq c|u|_s^2 - \lambda_0|u|_0^2 - |I_1| - |I_2| \geq c'|u|_s^2 - \lambda_0'|u|_0^2$$

and the proposition follows for this case.

Case 3. $A = A_0 + A_1 + A_2$, with A_0, A_1 as in Case 2, and

$$A_2 = \sum_{|\alpha|<r}' c_\alpha D^\alpha$$

Then arguing as in Case 2,

$$\langle A_2 u, u\rangle_0 = \sum' \int c_{\beta\gamma} D^\beta u D^\gamma u$$

summed over all β, γ such that $|\beta| \leq s$, $|\gamma| \leq s$, $|\beta| + |\gamma| < r$. Again, for all $\epsilon > 0$

$$|(A_2 u, u)_0| \leq \text{const}\, |u|_s |u|_{s-1}$$

$$\leq \epsilon\, \text{const}\, |u|_s^2 + \epsilon^{-2s+1}\, \text{const}\, |u|_0^2$$

whence we can also absorb this contribution into Gårding's inequality for $A_0 + A_1$.

Case 4. Thus far we have proved the proposition for those elliptic operators A with nearly constant coefficients. We now prove the general case by cutting up T^n into small domains on which A is nearly constant. Let $(U_i)_{1 \leq i \leq m}$ be a finite open disc cover of T^n such that in each U_i every a_α with $|\alpha| = r$ has oscillation $< \eta$, with η chosen as in Cases 2, 3. Let $(\phi_i)_{1 \leq i \leq m}$ be functions on T^n chosen in the second lemma of Section 3B. Now write

$$(Au, u)_0 = \sum_{j=1}^{m}{}' \phi_j^2 (Au, u)_0 = \sum_{j=1}^{m}{}' (\phi_j Au, \phi_j u)_0 = \sum_{j=1}^{m} (A(\phi_j u), \phi_j u)_0 + R$$

We can apply Case 3 to the summands in the right member, to obtain

$$(Au, u)_0 \geq c \sum_{j=1}^{m}{}' |\phi_j u|_s^2 - \lambda_0 \sum_{j=1}^{m}{}' |\phi_j u|_0^2 + R$$

But $|\phi_j u|_0 \leq |u|_0$, and using Proposition 2A we can find $a > 0$, $b > 0$ such that

$$\sum_{j=1}^{m} |\phi_j u|_s^2 \geq a|u|_s^2 - b|u|_0^2$$

Thus by adjusting c and λ_0 if necessary, $(Au, u)_0 \geq c|u|_s^2 - \lambda_0 |u|_0^2 + R$. The term R may involve derivatives of u through order $< r$, and through integration by parts we can express it in the form

$$R = \sum{}' \int c_{\beta\gamma} D^\beta u D^\gamma u$$

summed over all $|\beta| \leq s$, $|\gamma| \leq s$, $|\beta| + |\gamma| < r = 2s$. Once again, for any $\epsilon > 0$ we have

$$|R| \leq \epsilon\, \text{const}\, |u|_s^2 + \epsilon^{-2s+1}\, \text{const}\, |u|_0^2$$

By choosing ϵ sufficiently small we can absorb R to obtain $(Au, u)_0 \geq c|u|_s^2 - \lambda_0 |u|_0^2$. This completes the proof.

4. REGULARITY OF SOLUTIONS

(A) Let $C^s(T^n)$ be the Banach space of functions on T^n having continuous derivatives of order $\leq s$, with norm

$$|u|_{C^s(T^n)} = \sup\{|D^\alpha u(x)| : x \in T^n \quad \text{and} \quad |\alpha| \leq s\}$$

Clearly $C^s(T^n) \subset H^s(T^n)$ for all integers $s \geq 0$. Let $C^\infty(T^n) = \cap\{C^s(T^n) : s \geq 0\}$, the vector space of smooth functions on T^n.

Theorem (Sobolev). *If $t > n/2 + s$, then we have a continuous injection $H^t(T^n) \to C^s(T^n)$. In particular, considering this as an inclusion, we have $H^\infty(T^n) = C^\infty(T^n)$.*

Proof. In case $s = 0$, if $u(x) = \sum u_k \exp(i\langle k, x\rangle)$ is a trigonometric polynomial, then for all $x \in T^n$

$$|u(x)|^2 \leq \left(\sum{}' |u_k|^2\right)^2 \leq \left\{\sum{}' (1 + |k|^2)^t |u_k|^2\right\} \left\{\sum{}' (1 + |k|^2)^{-t}\right\} = |u|_t^2 \sum (1 + |k|^2)^{-t}$$

But $t > n/2$ implies that the sum is convergent, whence $|u(x)| \leq \text{const}\, |u|_t$. Thus the Fourier series representation of any $u \in H^t$ converges uniformly, whence its limit is continuous. In case s is any positive integer such that $t > n/2 + s$ and $|\alpha| \leq s$, then $|D^\alpha u(x)| \leq \text{const}\, |D^\alpha u|_{t-s} \leq \text{const}\, |u|_t$, which proves the theorem in general.

(B) Let A be an elliptic operator of order $r = 2s$ on T^n.

Lemma. *Define $A_\lambda = A + \lambda$. Let t be an integer. Then there are numbers $c > 0$ and $\lambda_0 > 0$ such that $(A_\lambda u, u)_t \geq c|u|_{t+s}^2$ for all $u \in H^{t+s}$ and $\lambda \geq \lambda_0$.*

Proof. If t is positive and $K = I - \Delta$, then $K^t A$ and AK^t are elliptic of order $2(t + s)$. Gårding's inequality shows the existence of c and λ_0 such that

$$(A_\lambda u, u)_t = (K^t A_\lambda u, u)_0 \geq c|u|_{t+s}^2 - \lambda_0 |u|_0^2 + \lambda |u|_t^2$$

$$\geq c|u|_{t+s}^2 + (\lambda - \lambda_0)|u|_0^2$$

$$\geq c|u|_{t+s}^2 \quad \text{if} \quad \lambda \geq \lambda_0$$

If t is negative, then

$$(A_\lambda u, u)_t = (A_\lambda K^{-t}(K^t u), K^t u)_0 \geq c|K^t u|_{-t+s}^2 - \lambda_0 |K^t u|_0^2 + \lambda \langle u, K^t u\rangle_0$$

$$\geq c|u|_{t+s}^2 - \lambda_0 |u|_{2t}^2 + \lambda |u|_t^2$$

$$\geq c|u|_{t+s}^2 \quad \text{if} \quad \lambda \geq \lambda_0$$

Lemma. *There is a unique continuous linear map $A^* : H^{t+s} \to H^{t-s}$ such that for all $u, v \in H^{t+s}$ we have $(Au, v)_t = (u, A^* v)_t$.*

Proof. Fix v and consider $f : H^{t+s} \to \mathbb{R}$ defined by $f(u) = (Au, v)_t$. Then $|f(u)| \leq |Au|_{t-s} |v|_{t+s} \leq \text{const}\, |u|_{t+s} |v|_{t+s}$. By Proposition 2B there is a unique $A^* v \in H^{t-s}$ for which $f(u) = (u, A^* v)_t$ for all $u \in H^{t+s}$, and A^* is bounded on H^{t+s}.

Proposition. *There is a number $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$ the map $A_\lambda : H^{t+s} \to H^{t-s}$ is bijective, with universal bound independent of $\lambda \geq \lambda_0$.*

Proof. First of all, $|A_\lambda u|_{t-s} |u|_{t+s} \geq |(A_\lambda u, u)_t| \geq c|u|_{t+s}^2$, whence $|A_\lambda u|_{t-s} \geq c|u|_{t+s}$ for all $u \in H^{t+s}$ and $\lambda \geq \lambda_0$. By Part I, Lemma 1D, $R_\lambda = A_\lambda H^{t+s}$ is closed in H^{t-s} and A_λ is injective.

If R_λ is a proper closed subspace, there is a non-zero $w \in H^{t+s}$ for which $\langle A_\lambda u, w\rangle_t = \langle u, A_\lambda^* w\rangle_t = 0$ for all $u \in H^{t+s}$. Therefore $A_\lambda^* w = 0$; but our choice of λ shows that $\langle v, A_\lambda^* u\rangle_t = \langle A_\lambda u, u\rangle_t = c|u|^2_{t+s}$, so that A_λ^* is injective; i.e. $v = 0$, a contradiction.

Theorem. *If* $v \in H^{-\infty}(T^n)$ *and* $Av \in H^t(T^n)$, *then* $v \in H^{t+r}(T^n)$. *In particular, if* $Av \in H^\infty(T^n)$ *then* v *is smooth.*

Proof. Assume $v \in H^p = H^p(T^n)$ and set $u = Av$. Then $u + \lambda v \in H^{\min(t,p)}$. But $v = A_\lambda^{-1}(u + \lambda v) \in H^{\min(t+r,p+r)}$ if λ is suitably large. Repeating the process, $u + \lambda v \in H^{\min(t,p+r)}$ whence $v \in H^{\min(t+r,p+2r)}$ for suitably large λ. The process can be continued until we find $v \in H^{t+r}$.

Remark. It can be shown using this result that if the coefficients of A are analytic on T^n and if Av is analytic, then so is v.

(C) Finally we are in the position to apply the results of Part I, in particular Theorem 4D.

Theorem. *Let* A *be a smooth* r^{th}*-order elliptic operator on* T^n $(r = 2s)$. *Then*
(1) A *maps* $H^{t+s}(T^n)$ *onto* $H^{t-s}(T^n) \cap K(A^*)^\perp$, *with kernel* $K(A)$ *for all* t.
(2) $K(A)$ *and* $K(A^*)$ *are subspaces of* $C^\infty(T^n)$ *of the same finite dimension.*
(3) *For every* t *there is a number* $\lambda_t > 0$ *such that if* $\lambda \geq \lambda_t$ *then* $A_\lambda = A + \lambda: H^{t+s}(T^n) \to H^{t-s}(T^n)$ *is a continuous bijection. In particular,* $A_\lambda: C^\infty(T^n) \to C^\infty(T^n)$ *is a bijection.*
(4) $\dim K(A_\lambda) > 0$ *for at most a countable number of* λ *with no finite accumulation point.*

Proof. We define $\alpha: H^s \times H^s \to \mathbb{R}$ by $\alpha(u, v) = \langle Au, v\rangle_0$. Then α is a continuous coercive bilinear form on H^s, whence α satisfies the hypotheses of Part I, Theorem 4D; if $\alpha_\lambda(v, y) = \langle A_\lambda v, y\rangle_0$, then $\alpha_\lambda(v, y) = \langle u, y\rangle_0$ for all $y \in H^s$ implies that $A_\lambda v = u$ by Proposition 2B. That the solutions actually lie in the designated spaces H^{t+s} follows from Theorem 4B. The statement (4) is a consequence of the observation $K(A_\lambda) = K(I - (\lambda_0 - \lambda)G_{\lambda_0})$; for $G_{\lambda_0}: H^0 \to H^0$ is compact, so that we can apply Part I, Proposition 3E.

5. ZERO BOUNDARY VALUES IN $\mathbb{R}^n$

Our next step is to make certain minor modifications to obtain corresponding results in Euclidean space. The Dirichlet problem in its general form is a theory paying special attention to boundary value assignments; our applications go in another direction, and we shall consider only the elementary case of zero boundary values (taken in the Hilbert space sense of belonging to the space $H_0^1(U)$ below).

It would be possible to imitate rather closely the existence and regularity theory of Sections 2–4, starting with Hilbert spaces $H_0^t(U)$ defined through the Fourier transform in $\mathbb{R}^n$ for $t \geq 0$ by the norm:

$$|u|_t^2 = \int_{\mathbb{R}^n} (1 + |\xi|^2)^t |\hat{u}(\xi)|^2 \, d\xi \quad \text{for all} \quad t \in \mathbb{R}^n$$

We shall not do that; rather, we shall use the results just obtained to derive the analogous results in U.

(A) Let U be a bounded open set in $\mathbb{R}^n$. For smooth function $u, v: \overline{U} \to \mathbb{R}$ we define the scalar product for positive integers s:

$$\langle u, v\rangle_{H^s(U)} = \sum_{|\alpha| \leq s} \int_U D^\alpha u(x) D^\alpha v(x) \, dx$$

We let $H^s(U)$ be the completion of the vector space of smooth function on U in that inner product.

Definition. A *test function in* U is a smooth function on $\mathbb{R}^n$ whose support is contained in U. Let $H_0^s(U)$ be the closure in $H^s(U)$ of the space of test functions in U. We observe that $H_0^0(U) = H^0(U)$; however, $H_0^s(U)$ is a proper subspace of $H^s(U)$ for $s > 0$.

Proposition. **(Poincaré's inequality).** *Consider on* $H_0^s(U)$ *the inner product*

$$\langle\langle u, v\rangle\rangle_{H_0^s(U)} = \sum_{|\alpha| = s} \binom{s}{\alpha} \langle D^\alpha u, D^\alpha v\rangle_0$$

where $\binom{s}{\alpha} = s!/\alpha_1! \ldots \alpha_n!$ *and* $\langle D^\alpha u, D^\alpha v\rangle_0 = \langle D^\alpha u, D^\alpha v\rangle_{H^0(U)}$. *This is equivalent to that given on* $H_0^s(U)$ *considered as a subspace of* $H^s(U)$.

Proof. Since the test functions are dense in $H_0^s(U)$ it suffices to work with these. First of all, it is clear that

$$\|u\|^2_{H_0^s(U)} \leq \text{const}\, |u|^2_{H^s(U)}$$

To prove an opposite inequality we start with

$$|u|^2_{H^s(U)} = \sum_{t=0}^{s} \sum_{|\alpha| = t} |D^\alpha u|_0^2 \leq \text{const} \sum_{t=0}^{s} \|u\|^2_{H_0^t(U)}$$

It therefore suffices to prove

$$\|u\|^2_{H_0^t(U)} \leq \text{const}\, \|u\|^2_{H_0^s(U)} \quad \text{if} \quad t \leq s$$

By iteration we reduce the problem to that of proving

$$|u|_0^2 \leq \text{const} \left|\frac{\partial u}{\partial x_i}\right|_0^2$$

For that we observe that

$$u(x) = \int_{-\infty}^{x_i} \frac{\partial u(x)}{\partial x_i} \, dx_i$$

since u has compact support; it follows that

$$|u(x)|^2 \leq \text{const} \int_{-\infty}^{x_i} \left|\frac{\partial u}{\partial x_i}\right|^2 dx_i$$

and the desired inequality follows by integration.

(B) The spaces $H_0^s(U)$ have many of the properties of $H^s(T^n)$, by virtue of the following lemma.

Let C be a closed cube in $\mathbb{R}^n$ whose interior contains $\bar{U}$. Without loss of generality we can suppose that C is the period domain defining T^n.

Lemma. *For each* $u \in H_0^s(U)$ *define* $\tilde{u} = u$ *in* U, 0 *in* $C - U$ *and extend* $\tilde{u}$ *over* $\mathbb{R}^n$ *by periodicity. Then* $u \to \tilde{u}$ *defines an inclusion* $H_0^s(U) \to H^s(T^n)$, *and* $|u|_{H_0^s(U)} = |\tilde{u}|_{H^s(T^n)}$ (= $|\tilde{u}|_s$ in the notation of Section 3).

Proof. Given $u \in H_0^s(U)$, let $(\phi_j)_{j \geq 1}$ be a sequence of test functions such that $|\phi_j - u|_{H_0^s(U)} \to 0$. View each ϕ_j as defined on T^n; then

$$|\phi_j - \tilde{u}|_{H^s(T^n)} = |\phi_j - u|_{H^s(C)} \to 0$$

and the lemma follows.

We can now transfer some of our results from T^n to U.

Theorem (Rellich). *If* $s < t$, *the injection* $H_0^t(U) \to H_0^s(U)$ *is compact and dense.*

Let $C^s(U)$ denote the vector space of C^s-functions on U.

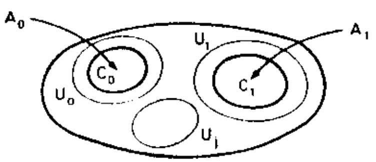
Theorem (Sobolev). *If* $t > n/2 + s$ *then we have an injection* $H_0^t(U) \to C^s(U)$. *In particular,* $H_0^\infty(U)$ *consists of smooth functions in* U.

(C) We shall write $U \Subset W$ to mean that $\bar{U} \subset W$.

Lemma. *Given disjoint closed sets* C_0 *and* C_1 *in an open set* U *of* $\mathbb{R}^n$ *and* r^{th}*-order elliptic operators*

$$A_j = \sum_{|\alpha| \leq r} a_{j\alpha} D^\alpha$$

defined in two disjoint open subsets U_j *of* U *with* $C_j \subset U_j$ $(j = 0, 1)$. *Then there exists an* r^{th}*-order elliptic operator in* U *extending these in* C_j



Proof. Construct a locally finite open covering $(U_j)_{j \geq 0}$ of U such that $U_j \cap C_i = \emptyset$ for $j \neq i$. Let $(\phi_j)_{j \geq 0}$ be a smooth partition of unity with $0 \leq \phi_j(x) \leq 1$ and each $\operatorname{spt}\phi_j \subset U_j$ as in the third lemma of Section 3B; furthermore, note that $\phi_j | C_j \equiv 1$ for $j = 0, 1$. For each $j \geq 2$ let

$$A_j = \sum_{|\alpha| \leq r} a_{j\alpha} D^\alpha$$

be a smooth r^{th}-order elliptic operator in U_j; e.g. we can take for A_j the iterated Laplacian $\Delta^{r/2}$. Then define A in U by

$$A = \sum_{j \geq 0} \sum_{|\alpha| \leq r} \phi_j a_{j\alpha} D^\alpha$$

Clearly A is well defined, and has the symbol

$$\sigma_A(x, \xi) = (-1)^{r/2} \sum_{|\alpha| = r} \sum_{j \geq 0} \phi_j(x) a_{j\alpha}(x) \xi$$

In particular, for each $x \in U$ all $\phi_j(x) \geq 0$ and some $\phi_k(x) > 0$, whence A is elliptic throughout U. Also, $A = A_j$ in C_j by our choice of covering.

Corollary. *If* A *is an elliptic operator in* W *with* $U \Subset W \Subset \operatorname{Int} C$, *then there is an extension* $\tilde{A}$ *of* A *in* U *to a periodic elliptic operator with* C *as the period domain.*

Proof. Take $C_0 = \bar{U}$ and C_1 = a closed neighbourhood of bdy C which does not meet W. Then use the lemma to extend A in C_0 and $\Delta^{r/2}$ in C_1.

(D) If A is an elliptic operator in $W \Subset U$ of order $r = 2s$, then we define $\alpha(u, v)$ for test functions in U by

$$\alpha(u, v) = (Au, v)_0 = \sum \int_U b_{\beta\gamma} D^\beta u D^\gamma v \, dx$$

where the sum extends over all β, γ with $|\beta| \leq s$, $|\gamma| \leq s$. By using the right member to define α we have a unique extension of α as a continuous bilinear form on $H_0^s(U)$; thus there is a number $a > 0$ such that

$$|\alpha(u, v)| \leq a |u|_{H_0^s(U)} |v|_{H_0^s(U)}$$

for all $u, v \in H_0^s(U)$.

Theorem (Gårding's inequality). *If* A *is an elliptic operator of order* $r = 2s$ *in an open set* W *such that* $U \Subset W \Subset \operatorname{Int} C$, *then there are numbers* $c > 0$ *and* $\lambda_0 > 0$ *such that*

$$\alpha(u, u) + \lambda_0 |u|_0^2 \geq c|u|_{H_0^s(U)}^2 \quad \textit{for all} \quad u \in H_0^s(U)$$

First of all, it suffices to verify this for test functions in U. We extend A to a periodic operator $\tilde{A}$ in C by Corollary 5C, and then apply Proposition 3A.

(E) Next we come to the problem of regularity.

Theorem. *Let* $v \in H^0(U)$, *and suppose that for some integer* $t \geq 0$ *we have* $Av \in H^t(U)$. *Then for all open sets* U_1 *with* $U_1 \Subset U$ *we have* $v \in H^{t+r}(U_1)$. *If* $Av \in H^\infty(U)$, *then* v *is smooth in* U.

Proof. Let U_2, U_3 be open sets for which $U_1 \Subset U_2 \Subset U_3 \Subset U$. Choose a test function ζ in U_3 which is 1 on U_2. Set $Av = u$. Then

$$\langle \zeta u, w\rangle_{H^0(U)} = \langle Av, \zeta w\rangle_{H^0(U)} = \langle v, A^*(\zeta w)\rangle_{H^0(U)} = \langle \zeta v, A^* w\rangle_{H^0(U)} + \langle v, A_1^* w\rangle_{H^0(U)}$$

for all periodic smooth functions w in $\mathbb{R}^n$ with period domain C, where A_1^* is a smooth differential operator of order $\leqslant r-1$ whose coefficients have supports in U_3. It follows that

$$\langle \zeta u, w\rangle_{H^0(C)} = \langle v, A^* w\rangle_{H^0(C)} + \langle v, A_1^* w\rangle_{H^0(C)}$$

Let η be a test function in U which is 1 on U_3, and define $\widetilde{v}$

$$\widetilde{v} = \begin{cases} v \text{ in } U \\ 0 \text{ in the rest of } C \end{cases}$$

similarly for the definition of $\widetilde{u} = \eta u$. Extend ζ and A_1^* to $\widetilde{\zeta}$ and $\widetilde{A}_1^*$ in C by defining them to be zero in $C - U_3$.

Let $\widetilde{A}$ be an extension of A to a periodic elliptic operator with period domain C, by Corollary 5C.

If $\widetilde{A}_1$ denotes the $H^0(C)$-adjoint of $\widetilde{A}_1^*$,

$$\langle \zeta\widetilde{u}, w\rangle_{H^0(C)} = \langle \widetilde{A}(\zeta\,\widetilde{v}), w\rangle_{H^0(C)} + \langle \widetilde{A}_1\widetilde{v}, w\rangle_{H^0(C)}$$

for all w, so that $\zeta\widetilde{u} = \widetilde{A}(\zeta\,\widetilde{v}) + \widetilde{A}_1(\widetilde{v})$. But $\zeta\widetilde{u} - \widetilde{A}_1(\widetilde{v}) \in H^{1-r}(T^n)$ and $\widetilde{A}$ is elliptic on T^n, whence $\zeta\widetilde{v} \in H^1(T^n)$ by Theorem 5B. In particular, $v \in H^1(U_2)$.

We now repeat the process, starting with an open set U_2' for which $U_1 \Subset U_2' \Subset U_2$; we find that $\zeta\widetilde{u} - \widetilde{A}_1(\widetilde{v}) \in H^{2-r}(T^n)$, so that $v \in H^2(U_2')$. We can continue in this way until $v \in H^{t+r}(U_2'')$ for some open set U_2'' containing U_1. The final statement of the theorem follows from Sobolev's theorem.

(F) Taken together, these results yield a conclusion similar to that of Theorem 4C, with T^n replaced by U.

First of all, let $H^t_{loc}(U)$ be the vector space of functions in U which have all derivatives of orders $\leqslant t$ square integrable on every open $U_1 \Subset U$.

Theorem. *Let* A *be an elliptic operator in an open set* $W \supset U$ *of order* $r = 2s$. *Then*

(1) A *maps* $H^s_0(U) \cap H^{t+r}_{loc}(U)$ *onto* $H^t_{loc}(U) \cap K^1(A^*)$ *with kernel* $K(A) \subset C^\infty(U)$.

(2) $\dim K(A) = \dim K(A^*) < \infty$.

(3) There is a $\lambda_0 > 0$ *such that* $A_\lambda : H^s_0(U) \cap H^{t+r}_{loc}(U) \to H^t_{loc}(U)$ *is a bijection for all* $\lambda \leqslant \lambda_0$.

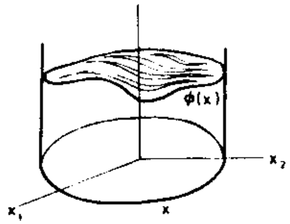
(4) $\dim K(A_\lambda) > 0$ *for at most countably many* λ *having no finite accumulation point.*

Remark. It is quite appropriate that these statements should involve the spaces $H^t_{loc}(U)$ in expressing higher order differentiability; for differentiability is a local concept, whereas belonging to $H^t(U)$ is a global one.

It is possible to obtain more precise interpretations of the assumption of boundary values in case U is a bounded region with smooth boundary. Roughly speaking, solutions $v \in H^s_0(U)$ can then be shown to have all derivatives of order $< s$ vanishing on bdy U; see Bers-John-Schechter [2] and Nirenberg [11]. We shall not pursue that aspect of the theory now.

Remark. Suppose we are given $u \in H^t_{loc}(U)$ and $\phi \in H^{t+r}_{loc}(U)$ and are asked to find $v \in H^{t+r}_{loc}(U)$ such that $Av = u$ in U, with v assuming the boundary values ϕ in the sense that $v - \phi \in H^s_0(U)$. Setting $u' = u - A\phi$, we reduce the problem to solving $Av' = u'$ for $v' \in H^s_0(U) \cap H^{t+r}_{loc}(U)$; for then we can take $v = v' + \phi$ to solve the given problem.

Example. Let $U = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$. Then the negative of the Laplace operator $-\Delta$ is elliptic, with symbol $\sigma_{\Delta}(x, \zeta) = \zeta_1^2 + \zeta_2^2$. Suppose we set $u \equiv 0$ in U, but require that v equals (in the above sense) a given function ϕ on bdy U. The solution describes in $\mathbb{R}^3$ a surface spanning ϕ which is a minimal surface, in the sense that the mean normal curvature is zero.



Remark. Given a subspace H such that $H^s_0(U) \subset H \subset H^s(U)$, we can consider membership in H as expressing boundary conditions on the problem. If we have a form of coerciveness (such as Gårding's inequality) for functions in H, then we can apply the above theory. We have considered the simplest case, with $H = H^s_0(U)$; the opposite extreme $H = H^s(U)$ has been developed by Aronszajn [1], and intermediate cases by Lions [10], Schechter [15], and others.

6. STRONGLY ELLIPTIC SYSTEMS

On T^n let us suppose given for each n-tuple α of natural numbers with $|\alpha| \leqslant r$ a smooth map $a_\alpha : T^n \to M(p, p)$, the vector space of real $p \times p$ matrices. If $C^\infty(T^n, \mathbb{R}^p)$ denotes the vector space of smooth maps $T^n \to \mathbb{R}^p$, then the differential operator

$$A = \sum_{|\beta| \leqslant r} a_\beta D^\beta$$

is an endomorphism of $C^\infty(T^n, \mathbb{R}^p)$. We define its symbol

$$\sigma_A : T^n \times \mathbb{R}^n \to M(p, p)$$

by

$$\sigma_A(x, \xi) = (-1)^{r/2} \sum_{|\beta| = r} a_\beta(x)\xi^\beta$$

and say that A *is strongly elliptic* if σ_A is positive definite at every point $x \in T^n$. Note that there is a number $\lambda_0 > 0$ such that for all $(x, \xi, \eta) \in T^n \times \mathbb{R}^n \times \mathbb{R}^p$ we have $(r = 2s)$

$$\langle \sigma_A(x, \xi)\eta, \eta\rangle_{\mathbb{R}^p} \geqslant \lambda_0 |\xi|^{2s} |\eta|^2$$

Let $E = H^0 \times \ldots \times H^0$ (p copies), with inner product

$$(u, v)_E = \sum_{i=1}^{p} (u_i, v_i)_0$$

where $u = (u_1, \ldots, u_p)$, $v = (v_1, \ldots, v_p)$. Similarly for $V = H^s \times \ldots \times H^s$. Then we have the bilinear form $\alpha: V \times V \to \mathbb{R}$ defined by

$$\alpha(u, v) = (Au, v)_E$$

As in the case $p = 1$, we again find that α is continuous and coercive. Consequently, *if* $A: H^r(T^n, \mathbb{R}^p) \to H^0(T^n, \mathbb{R}^p)$ *is a strongly elliptic operator, then Theorem 4C is valid with* $H^1 = H^1(T^n, \mathbb{R}^p)$. *Similarly, Theorem 5F is also valid with* $H^1_{loc}(U)$ *replaced by* $H^1_{loc}(U, \mathbb{R}^p)$.

III. DIFFERENTIAL OPERATORS ON VECTOR BUNDLES

We present in Part III an abbreviated treatment of smooth manifolds, vector bundles, and of certain differential operators associated with them. The theory is primarily local, and thus it is most appropriate to formulate the concepts in terms of sheaves. The basic reference for sheaf theory is Godement [5]; the viewpoint in the theory of connections is that of Koszul [9].

1. SHEAVES OF MODULES

(A) Definitions. Let R be a commutative ring with unit. A *sheaf of R-modules* is a triple $(\mathscr{S}, \pi, X)$, where $\mathscr{S}$ and X are topological spaces and $\pi: \mathscr{S} \to X$ is a continuous surjective map, subject to the following conditions:

(1) π is a local homeomorphism; i.e. each $s \in \mathscr{S}$ has an open neighbourhood S such that the restriction $\pi|S$ of π is a homeomorphism of S onto an open neighbourhood of $\pi(s)$.

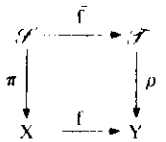
(2) For each $x \in X$ the *stalk* $\mathscr{S}_x = \pi^{-1}(x)$ is an R-module, and the algebraic operations $(s, t) \to s + t$ and $(r, s) \to rs$ are continuous where they are defined, for $r \in R$, $s, t \in \mathscr{S}$.

The totality $\mathscr{S}(X)$ of sections of $(\mathscr{S}, \pi, X)$, — i.e. of continuous maps $\phi: X \to \mathscr{S}$ such that $\pi\phi(x) = x$ for all $x \in X$ — is an R-module, with algebraic operations defined pointwise.

If U is an open subspace of X, then the *restriction* $\mathscr{S}|U$ of $(\mathscr{S}, \pi, X)$ is the sheaf $(\pi^{-1}(U), \pi|\pi^{-1}(U), U)$.

A triple $(\mathscr{S}', \pi', X)$ *is a subsheaf* of $(\mathscr{S}, \pi, X)$ if (1) $\mathscr{S}'$ is an open subspace of $\mathscr{S}$; (2) $\pi' = \pi|\mathscr{S}'$; (3) each stalk $\mathscr{S}'_x$ is a submodule of the stalk $\mathscr{S}_x$.

Given sheaves $(\mathscr{S}, \pi, X)$ and $(\mathscr{T}, \rho, Y)$ and a continuous map $f: X \to Y$, an f-homomorphism is a continuous map $\bar{f}: \mathscr{S} \to \mathscr{T}$ for which the following diagram is commutative, and for which



each restriction $\bar{f}: \mathscr{S}_x \to \mathscr{T}_{f(x)}$ is an R-homomorphism. In particular, for $X = Y$ and f the identity map we have the notion of R-homomorphism $\lambda: \mathscr{S} \to \mathscr{T}$ of sheaves of R-modules over X. Every λ induces a homomorphism $\Lambda: \mathscr{S}(X) \to \mathscr{T}(X)$ of their modules of sections; and Λ *is local*, in the sense that for all $\phi \in \mathscr{S}(X)$ we have $spt(\Lambda\phi) \subset spt(\phi)$. With mild conditions on $\mathscr{S}$ we can prove that every local homomorphism $\mathscr{S}(X) \to \mathscr{T}(X)$ induces a sheaf homomorphism; see Section 2D.

We say that $(\mathscr{S}, \pi, X)$ *is modelled on* $(\mathscr{T}, \rho, Y)$ if every $x \in X$ has a neighbourhood U and homeomorphisms $f: U \to Y$ and $\bar{f}: \mathscr{S}|U \to \mathscr{T}$ such that $\bar{f}$ is an f-isomorphism.

Example. Let $\mathscr{T}$ be the *simple sheaf* $\mathscr{T} = X \times R$, where R is given the discrete topology, and $\rho: \mathscr{T} \to X$ is the projection on the first factor. We shall sometimes write $\mathscr{T} = R$. If $\mathscr{S} \to X$ is modelled on $\mathscr{T}$, with $f: X \to X$ the identity map, then $\mathscr{S}$ is said to be a *locally simple sheaf*. Its components are covering spaces of X.

(B) Let $(\mathscr{S}, \pi, X)$ be a sheaf of R-modules, and $U \supset V$ open subsets of X. Then we have the R-homomorphism

$$r_V^U: \mathscr{S}(U) \to \mathscr{S}(V)$$

defined by restriction. Clearly, if $W \subset V \subset U$ are open, then $r_W^U = r_W^V \cdot r_V^U$, and r_U^U = identity.

Conversely, if for all pairs of open sets $U \supset V$ of a base for the topology of X we have R-modules $\mathscr{S}(U)$ and R-homomorphisms r_V^U defined with the above properties, then this assignment (called a presheaf) defines a sheaf of R-modules over X. In fact, for each point $x \in X$ let $\mathscr{S}_x = \varinjlim \mathscr{S}(U)$, the direct limit taken over the open neighbourhoods of x; define $r_x^U: \mathscr{S}(U) \to \mathscr{S}_x$ as the natural map. Set $\mathscr{S} = \cup\{\mathscr{S}_x : x \in X\}$, and topologize $\mathscr{S}$ by constructing the following base for this topology: for each open U in X and $\phi \in \mathscr{S}(U)$ let $V(U, \phi) = \{r_y^U(\phi) : y \in U\}$. The map $\pi: \mathscr{S} \to X$ is defined by $\mathscr{S}_x \to x$ for all $x \in X$. It is immediate that $(\mathscr{S}, \pi, X)$ satisfies the conditions of a sheaf.

Example 1. Let $\mathscr{R} \to X$ be a sheaf of commutative rings with unit. An $\mathscr{R}$*-module over* X is a sheaf $\mathscr{S} \to X$ of abelian groups such that (1) for every open U in X, $\mathscr{S}(U)$ is an $\mathscr{R}(U)$-module; (2) for every open $V \subset U$ the restriction $\mathscr{S}(U) \to \mathscr{S}(V)$ is a homomorphism of modules compatible with the restrictions $\mathscr{R}(U) \to \mathscr{R}(V)$ of rings of operators.

Example 2. If $\mathscr{S}$ and $\mathscr{T}$ are $\mathscr{R}$-modules over X, their *tensor product* $\mathscr{S} \otimes_{\mathscr{R}} \mathscr{T} \to X$ is the $\mathscr{R}$-module characterized by the assignment $U \to \mathscr{S}(U) \otimes_{\mathscr{R}(U)} \mathscr{T}(U)$ for every open U in X. Note in particular that $\mathscr{R} \otimes_{\mathscr{R}} \mathscr{T} \approx \mathscr{T}$.

Example 3. Let X be a topological space and R the real number field. For each open U in X let $\mathscr{C}(U, R^n)$ be the R-module of continuous maps $U \to R^n$. The resulting sheaf $\mathscr{C}^n = \mathscr{C}_{(X, R^n)} \to X$ is the *sheaf of continuous maps of X into* R^n; it is a $\mathscr{C}^1$-module. Then $\mathscr{C}^n$ is isomorphic to the direct sum $\mathscr{C}^1 \oplus \ldots \oplus \mathscr{C}^1$ (n copies; definition evident), and $\mathscr{C}^n$ is said to be *free of rank* n. A sheaf of $\mathscr{C}^1$-modules is *locally free of rank* n if it is locally isomorphic to $\mathscr{C}^n$.

Example 4. If $\phi: X \to Y$ is a continuous map of toplogical spaces, for each open U in X let $\mathscr{S}(U) = \{\sigma: U \to \mathscr{C}_{(Y,R)} : \eta\sigma(x) = \phi(x)$ for all $x \in U\}$, where $\eta: \mathscr{C}_{(Y,R)} \to Y$ is the sheaf of continuous functions on Y. Thus $\mathscr{S}(U)$ is an R-algebra, and if V is open in U, we have the restriction homomorphism $\mathscr{S}(U) \to \mathscr{S}(V)$. The resulting sheaf is denoted by $\phi^{-1}\mathscr{C}_{(Y,R)} \to X$. For each open U in X we have $\phi^{-1}\mathscr{C}_{(Y,R)}(U)$ identified with a subalgebra of $\mathscr{C}_{(X,R)}(U)$, and thus $\phi^{-1}\mathscr{C}_{(Y,R)}$ as a subsheaf of $\mathscr{C}_{(X,R)}$.

2. VECTOR BUNDLES

For our purposes vector bundles over a space X can be viewed from three standpoints:

(1) As a fibre bundle whose fibres are vector spaces; this is convenient for introducing related structure (e.g. G-structures);

(2) As a locally free sheaf of finite rank; this is appropriate for utilizing local structures such as the differential structure of a manifold;

(3) As a special sort of $\mathscr{C}(R)$-module, where $\mathscr{C}(R)$ is the algebra of continuous functions on X; this viewpoint stresses the formal manipulative aspect of the theory.

(A) Definition. Let X be a topological space. A (real) *vector bundle over* X *of fibre dimension* m is a topological space E together with a continuous surjective map $\xi: E \to X$ such that:

(1) each *fibre* $E_x = \xi^{-1}(x)$ has a structure of a (real) vector space;

(2) each point of X has a neighbourhood U and a homeomorphism (a trivialization) $\rho: U \times R^m \to \xi^{-1}(U)$ such that $\xi\rho(x, v) = x$ for all $(x, v) \in U \times R^m$, and the map $v \to \rho(x, v)$ is linear for all $x \in U$.

If E and F are vector bundles over X, a *homomorphism* $\lambda: E \to F$ is a continuous map such that for each $x \in X$ the restriction $\lambda: E_x \to F_x$ is linear.

(B) Let $\xi: E \to X$ be a vector bundle. Then for any open set U in X the totality of continuous sections $U \to E$ forms a vector space $\underline{E}(U)$; for V open in U the restriction map $r^U_V: \underline{E}(U) \to \underline{E}(V)$ defines a presheaf of vector spaces, whose sheaf (a $\mathscr{C}^1$-module over X) we shall denote by $\underline{E} \to X$. From the definition of vector bundle E it follows that each point of X has a neighbourhood U and m continuous sections $\sigma_i: U \to E$ with $\xi\sigma_i(x) = x$ for all $x \in U$, and each $(\sigma_1(x), ..., \sigma_m(x))$ is a base for E_x. It follows that $\underline{E}$ is locally free of rank m.

Conversely, let $\mathscr{E} \to X$ be a locally free sheaf of rank m. Let $M_x = \{\gamma \in \mathscr{C}^1_x : \gamma(x) = 0\}$. Then the sequence

$$0 \to M_x \mathscr{E}_x \to \mathscr{E}_x \to \mathscr{E}_x / M_x \mathscr{E}_x \to 0$$

of vector spaces is exact. Set $E_x = \mathscr{E}_x / M_x \mathscr{E}_x$, and $E = \cup\{E_x : x \in X\}$, and define the map $\xi: E \to X$ by the condition $E_x \to x$.

If U is an open set in X over which $\mathscr{E}|U$ is isomorphic to $\mathscr{C}_{(U, R^m)}$, then we have sections $\sigma_1, ..., \sigma_m \in \mathscr{E}(U)$ defined so that for each $x \in U$, $r^U_x \sigma_1, ..., r^U_x \sigma_m$ is a base for $\mathscr{E}_x$ over $\mathscr{C}^1_x$. Their images $\sigma_1(x), ..., \sigma_m(x)$ under the coset mapping form a base for E_x over R. Furthermore, the map $\rho: U \times R^m \to \xi^{-1}(U)$ defined by $\rho(x; a_1, ..., a_m) = \sum a_i\sigma_i(x)$ is a bijection; we topologize E by requiring that each ρ be a homeomorphism.

The following consequence is immediate:

Proposition (1). *If* $\xi: E \to X$ *is a vector bundle, then there is a canonical isomorphism of* E *onto the vector bundle constructed from its sheaf* $\underline{E}$ *of sections.*

Proposition (2). *If* $\mathscr{E} \to X$ *is a locally free sheaf of rank* m, *then there is a canonical isomorphism of* $\mathscr{E}$ *onto the sheaf of sections of the vector bundle just constructed from* $\mathscr{E}$.

(C) If E and F are finite-dimensional vector spaces, then we can form their *space*, Hom(E, F) *of linear maps*, their *direct sum* $E \oplus F$, *their tensor product* $E \otimes F$, the p^{th} *tensor power* $\otimes^p E = E \otimes ... \otimes E$ (p copies), the p^{th} *exterior power* $\Lambda^p E$, the p^{th} *symmetric power* $\odot^p E$. We agree that $\otimes^p E = R$ if $p = 0$. Let

$$\otimes E = \sum \otimes^p E, \quad \Lambda E = \sum \Lambda^p E, \quad \odot E = \sum \odot^p E,$$

direct sums in each case. These all have structures of graded algebras over R, whose multiplications pair p- and q-summands to the (p + q)-summand. In particular, viewing $\Lambda^p E$ as the totality of alternating p-linear forms on the *dual space* $E^* = \mathrm{Hom}(E, R)$, we define the product of $\alpha \in \Lambda^p E$ and $\beta \in \Lambda^q E$ by

$$\alpha \wedge \beta(x_1, ..., x_{p+q}) = \sum' \epsilon_\sigma \alpha(x_{\sigma(1)}, ..., x_{\sigma(p)}) \beta(x_{\sigma(p+1)}, ..., x_{\sigma(p+q)})$$

with summation taken over all permutations σ of (1, ..., p + q) and ϵ_σ the sign of the permutation, with $\sigma(1) < ... < \sigma(p)$ and $\sigma(p+1) < ... < \sigma(p+q)$. Thus $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$, and ΛE is an associative, commutative (in the graded sense) graded algebra of $\dim E = 2^n$, where $n = \dim E$.

These constructions are all immediately transferable to sheaves $\underline{E}$ of sections of vector bundles E over X. Thus, for instance, we define the $\mathscr{C}^1$-module $\underline{E} \otimes_{\mathscr{C}^1} \underline{F} \to X$ as in Section 1B; it is an easy matter to verify that it is locally free and corresponds canonically to a vector bundle whose fibre over x is the vector space $E_x \otimes F_x$.

(D) Let X be a normal space, and $\xi: E \to X$ a vector bundle over X with $\mathscr{C}(E) = \underline{E}(X)$ as its $\mathscr{C}^1$-module of sections.

Then $\mathscr{C}(R) = \mathscr{C}(X \times R)$ is the R-algebra of continuous functions on X.

Proposition. *If* E *and* F *are vector bundles over* X, *then there is a natural isomorphism:*

$$\mathscr{C}(\mathrm{Hom}_R(E, F)) \to \mathrm{Hom}_{\mathscr{C}(R)}(\mathscr{C}(E), \mathscr{C}(F))$$

Proof. Given $\lambda \in \mathscr{C}(\mathrm{Hom}_R(E, F))$ and $\phi \in \mathscr{C}(E)$, define $\lambda\phi \in \mathscr{C}(F)$ by $(\lambda\phi)x = \lambda(x)(\phi(x))$ for all $x \in X$. Then $\phi \to \lambda\phi$ is clearly a $\mathscr{C}(R)$-homomorphism. The inverse map is defined as follows. Given a $\mathscr{C}(R)$-homomorphism $\Lambda: \mathscr{C}(E) \to \mathscr{C}(F)$ and $x \in X$, $v \in E_x$, we construct (using the normality of X) $\phi \in \mathscr{C}(E)$ whose support is in some open U over which E is trivial and $\phi(x) = v$. Define $\lambda(x)(v) = (\Lambda\phi)x$. If ϕ' were another such section, then $(\Lambda\phi) = (\Lambda\phi')x$. For if $\sigma_1, ..., \sigma_m \in \underline{E}(U)$ are a set of sections linearly independent at every point $x \in U$, then we can write $\phi = \sum \rho_i\sigma_i$ and $\phi' = \sum \rho'_i\sigma_i$ for suitable $\rho_i, \rho'_i \in \mathscr{C}(R)$ with $\rho_i(x) = \rho'_i(x)$ $(1 \leq i \leq m)$. Then $\Lambda(\phi') = \sum \rho'_i \Lambda(\sigma_i)$, and $(\Lambda\phi')x = \sum \rho'_i(x)\Lambda(\sigma_i) = \sum \rho_i(x)\Lambda(\sigma_i) = (\Lambda\phi)x$. The proposition follows.

Example. If $E^* = \mathrm{Hom}_R(E, R)$ is the dual bundle of E, then $\mathscr{C}(E^*) = \mathrm{Hom}_{\mathscr{C}(R)}(\mathscr{C}(E), \mathscr{C}(R))$. Also, $\mathscr{C}(E) = \mathrm{Hom}_{\mathscr{C}(R)}(\mathscr{C}(E^*), \mathscr{C}(R))$. More generally, if $L^p_R(E, R)$ is the bundle of p-linear maps of $E \times ... \times E \to R$, then $\mathscr{C}(L^p_R(E, R)) = L^p_{\mathscr{C}(R)}(\mathscr{C}(E), \mathscr{C}(R))$.

(E) *Remark.* Any local homomorphism (i.e. support decreasing) $\Lambda: \mathscr{C}(E) \to \mathscr{C}(F)$ defines a sheaf homomorphism $\lambda: \underline{E} \to \underline{F}$.

From the viewpoint of differential operators the following result is useful.

Proposition. *Let* E *and* F *be vector bundles over* X *and* $\Lambda: \mathscr{C}(E) \to \mathscr{C}(F)$ *be a* $\mathscr{C}(R)$ *homomorphism. Let* D *be an* R*-endomorphism of* $\mathscr{C}(R)$, *and suppose* $\Lambda(\gamma\phi) = D(\gamma)\phi + \gamma\Lambda(\phi)$ *for all* $\gamma \in \mathscr{C}(R)$, $\phi \in \mathscr{C}(E)$. *Then* Λ *is local.*

Proof. Let $\phi \in \mathscr{C}(E)$, and let U be an open neighbourhood of a point $x \in X$ such that $\phi|U = 0$. Take a function $\gamma \in \mathscr{C}(R)$ for which $\gamma(x) = 0$ and $\gamma = 1$ in a neighbourhood of $X - U$. Then $\phi = \gamma\phi$, whence $(\Lambda\phi)x = D(\gamma)(x)\phi(x) + \gamma(x)\Lambda\phi(x) = 0$; i.e. $\Lambda(\phi)|U = 0$.

Taking D as the zero homomorphism, we obtain the

Corollary. *Every* $\mathscr{C}(R)$*-homomorphism is local*

3. SMOOTH MANIFOLDS AND VECTOR BUNDLES

(A) A Hausdorff space X is a *topological n-manifold* if it is locally homeomorphic to $\mathbb{R}^n$; thus with every $x \in X$ we have a *chart* (θ, U), consisting of an open neighbourhood U of x and a homeomorphism θ of U onto an open subset of $\mathbb{R}^n$. For such a space the following properties are equivalent:

(1) X is paracompact;
(2) X is metrizable;
(3) X is expressible as a countable union of compact sets;
(4) X has a countable base for its topology.

We assume henceforth that X *has these properties.*

Definitions. Let X be a topological n-manifold. A *differential structure* on X is a subsheaf $\mathscr{D} = \mathscr{D}_{(X,\mathbb{R})}$ of the sheaf $\mathscr{C}_{(X,\mathbb{R})}$ of continuous functions, which is modelled on $\mathscr{D}_{(\mathbb{R}^n,\mathbb{R})}$. We shall call the pair $(X, \mathscr{D})$ *a smooth manifold*, and sometimes denote it by X alone. We shall let $\mathscr{D}(\mathbb{R}) = \mathscr{D}_{(X,\mathbb{R})}(X)$ denote the algebra of all sections of $\mathscr{D}$, called the smooth functions on X. A *smooth map* $f: X \to Y$ of smooth manifolds is a continuous map such that $f^{-1}\mathscr{D}_{(Y,R)}$ is a subsheaf of $\mathscr{D}_{(X,R)}$; this means that for every open V in Y, composition with f defines a homomorphism $\mathscr{D}_{(Y,R)}(V) \to \mathscr{D}_{(X,R)}(f^{-1}(V))$. A *diffeomorphism* is a smooth homeomorphism f whose inverse f^{-1} is also smooth. Note that each chart $\theta: U \to R^n$ is a diffeomorphism.

Remark. It is possible to define a smooth manifold by giving (in place of $\mathscr{D}$) a sheaf of sets on the space X which corresponds to the sheaf of sections of the principal bundle of X. That approach would have the advantage of emphasizing the fibre bundle theoretic aspects of manifolds. See Section 4F.

Remark. Not every compact topological n-manifold admits a differential structure $(n \geqslant 8)$; nor is a differential structure unique $(n \geqslant 7)$.

The following result follows easily from the constructions in Section 3B of Part II:

Proposition. *If* $\mathscr{U} = (U_i)$ *is a locally finite open covering of* X, *then there is a smooth partition of unity subordinate to* $\mathscr{U}$, *i.e. a family* (λ_i) *of smooth functions on* X *such that* $\mathrm{spt}(\lambda_i) \subset U_i$ *for all* i, $\sum_i \lambda_i(x) = 1$ *for all* $x \in X$, *and each* $0 \leqslant \lambda_i(x) \leqslant 1$.

(B) *A smooth vector bundle* $\xi: E \to X$ *of fibre dimension* m is a vector bundle in which E is a smooth manifold, ξ is a smooth map, and all trivializations ρ are smooth. The correspondence between smooth vector bundles and locally free $\mathscr{D}$-modules (i.e. sheaves of $\mathscr{D}$-modules locally isomorphic to $\mathscr{D}^m$) proceeds as in Proposition 2B. In particular, the space $\mathscr{D}(E)$ of smooth sections of E is a $\mathscr{D}(R)$-module.

For any section $\phi \in \mathscr{D}(E)$ we have the notion of its *support*: $\mathrm{spt}(\phi) = \mathrm{Closure}\{x \in X : \phi(x) \neq 0\}$. Then $\mathrm{spt}(\phi + \psi) \subset \mathrm{spt}(\phi) \cup \mathrm{spt}(\psi)$, $\mathrm{spt}(\gamma\phi) \subset \mathrm{spt}(\gamma) \cap \mathrm{spt}(\phi)$ for all $\varphi, \psi \in \mathscr{D}(E)$, $\gamma \in \mathscr{D}(R)$. We shall say that a section ϕ *is compact* if $\mathrm{spt}(\phi)$ is compact. The totality of compact sections is a vector subspace $\mathscr{D}_0(E)$ of $\mathscr{D}(E)$; of course, $\mathscr{D}_0(E) = \mathscr{D}(E)$ if X is compact.

Henceforth we assume that all manifolds, maps and vector bundles are smooth.

Remark. A sheaf is *soft* if every section defined over a closed subset $A \subset X$ can be extended to a section over all X. The sheaf of sections of a smooth vector bundle is always soft. (For proof see Kobayashi-Nomizu: Foundations of Differential Geometry Vol. I, Interscience, p. 58.)

(C) We now construct certain sheaves on X defined in terms of its differential structure. For each open set U of X let $\mathscr{D}_1(U)$ be the algebra of all smooth functions φ in U which admit a smooth extension to an open $U_\varphi \supset U$, which domain is allowed to vary with φ. Let $\mathscr{T}(U)$ be the totality of derivations on $\mathscr{D}_1(U)$, i.e. R-endomorphisms $v: \mathscr{D}_1(U) \to \mathscr{D}_1(U)$ such that $v(\varphi\psi) = v(\varphi)\psi + \varphi v(\psi)$. Thus v is a local homomorphism (it decreases supports, as in Proposition 2E). If V is an open subset of U, we define the restriction map $r_V^U: \mathscr{T}(U) \to \mathscr{T}(V)$ as follows.

Take any $v \in \mathscr{T}(U)$. Then given a $\varphi \in \mathscr{D}_1(V)$, there is (by Part II, Section 3B) a $\varphi_1 \in \mathscr{D}_1(U)$ such that $\varphi_1|V = \varphi$.

Define $w(\varphi) = v(\varphi_1)$ restricted to V. If φ_1' were another such extension, then $(\varphi_1' - \varphi_1)|V = 0$, and because v is local, we find $v(\varphi_1') - v(\varphi_1) = 0$ in V. Thus $w(\varphi)$ is independent of the choice of extension of φ, and we define $r_V^U(v) = w$.

It is clear that we have just defined vector spaces and homomorphisms of a presheaf. The associated sheaf $\mathscr{T} \to X$ is called the *tangent sheaf* of X. This sheaf is locally free of rank n: in fact, for each point of X let (θ, U) be a chart containing it, and define the derivation v_i in $\mathscr{D}(U)$ by

$$(v_i(\phi))x = \left(\frac{\partial}{\partial x_i}(\phi \circ \theta^{-1})\right)\theta(x) \qquad (1 \leqslant i \leqslant n)$$

for $\phi \in \mathscr{D}(U)$. These are easily seen to be a base over $\mathscr{D}(R)$ for $\mathscr{T}(U)$. The vector bundle $\pi: T(X) \to X$ of fibre dimension n associated with $\mathscr{T} \to X$ by Proposition 2B is called the *tangent vector bundle of* X. The fibre $\pi^{-1}(x) = X(x)$ is called the *tangent space of* X at x, and its elements are called the *tangent vectors at* x. These can be identified with those mappings $v: \mathscr{D}(R) \to R$ which are R-linear, vanish on the constant functions, and satisfy $v(\phi\psi) = v(\phi)\psi(x) + \phi(x)v(\psi)$ for all $\phi, \psi \in \mathscr{D}(R)$. Furthermore, the space $\mathscr{D}(T(X)) = \mathscr{T}(X)$ is precisely the Lie algebra over R of all derivations of $\mathscr{D}(R)$ with bracket $[u, v] = u \cdot v - v \cdot u$. These derivations are called the *tangent fields* of X.

By the constructions in Section 2C we can form the vector bundles

$$\otimes^p T(X), \quad \wedge^p T(X), \quad \odot^p T(X),$$

the dual bundle $T^*(X) = \mathrm{Hom}(T(X), R)$ and its powers. The elements in the $\mathscr{D}(R)$-modules $\mathscr{D}(\otimes^p T(X))$, $\mathscr{D}(\otimes^p T^*(X))$ are called p-*contravariant*, p-*covariant tensor fields* on X. Those of $\mathscr{D}(\wedge^p T^*(X))$ are called *exterior* p-*forms* on X. Note that

$$\mathscr{D}(\wedge T^*(X)) = \sum_{p=0}^{n} \mathscr{D}(\wedge^p T^*(X))$$

is a graded commutative algebra. Similarly for the spaces $\mathscr{D}_0(\otimes^p T(X))$, etc., of compact tensor fields. Observe that if X is not compact, then $\mathscr{D}_0(\wedge T^*(X))$ does not have a unit.

If $f: X \to Y$ is a map of manifolds, then f determines an f-homomorphism $T(f): T(X) \to T(Y)$ of the tangent vector bundles, which at each point $x \in X$ is the *differential* $f_*(x): X(x) \to Y(f(x))$, given for each $v \in X(x)$ by the function which assigns to each $\psi \in \mathscr{D}_{(Y,R)}(Y)$ the number $(w(\psi))x = v(\psi \circ f)x$. It is clear that w has the properties of a tangent vector at $f(x)$, and we define $f_*(x)v = w$. The map f therefore induces f-homomorphisms:

$$\otimes^p T(f): \otimes^p T(X) \to \otimes^p T(Y), \qquad \otimes^p T^*(f): \otimes^p T^*(Y) \to \otimes^p T^*(X)$$

(with evident interpretation of the term f-homomorphism in the second case). f induces homomorphisms of the spaces of sections of the covariant bundles; not of the contravariant bundles.

In particular, we have the algebra homomorphism

$$\Lambda(f):\mathscr{D}(\Lambda T^*(Y)) \to \mathscr{D}(\Lambda T^*(X))$$

which on the p-components is given by

$$((\Lambda^p f)\varphi)x(u_1, \ldots, u_p) = \varphi(f(x))(f_*(x)u_1, \ldots, f_*(x)u_p)$$

for all $\varphi \in \mathscr{D}(\Lambda^p T^*(Y))$; $u_1, \ldots, u_p \in X(x)$.

(D) For each open set U in X let $\mathscr{T}_p(U)$ be the submodule of $\mathrm{Hom}_R(\mathscr{D}_1(U), \mathscr{D}_1(U))$ generated by the monomials $v_1 \circ \ldots \circ v_k$ $(1 \leq k \leq p)$, where each v_i is a derivation on $\mathscr{D}_1(U)$. Just as in the construction in (C), we have a sheaf $\mathscr{T}_p \to X$ defined, called the *sheaf of* p^{th}*-order contact vectors of* X (or *sheaf of differential operators of order* p). Again, $\mathscr{T}_p \to X$ is locally free of rank

$$\nu(n, p) = \sum_{k=1}^{p} \binom{n+k-1}{k}$$

because in a chart (θ, U) the iterated partial derivatives $D^\alpha (0 < |\alpha| \leq p)$ form a base over $\mathscr{D}(R)$. Thus we have a vector bundle $\pi : T_p(X) \to X$ canonically associated.

Its dual bundle $J^p(X) = \mathrm{Hom}_R(T_p(X), R) \to X$ is called the *bundle of p-jets of functions on* X. Note that the fibre of $J^p(X)$ over $x \in X$ is $\mathscr{D}(R)/Z_x^p$, where $Z_x^p = \{\phi \in \mathscr{D}(R) : \text{all derivatives of orders} \leq p \text{ of } \phi \text{ are } 0 \text{ at } x\}$. The natural map $j^p : \mathscr{D}(R) \to \mathscr{D}(J^p(X))$ such that $j^p(\phi)$ assigns to each $x \in X$ its coset in $\mathscr{D}(R)/Z_x^p$ is called the p^{th} *jet extension.*

Again, if $f : X \to Y$ is a map of manifolds, we have induced f-homomorphisms $T_p(f) : T_p(X) \to T_p(Y)$ and $J^p(f) : J^p(Y) \to J^p(X)$.

(E) *Exercise.* Throughout the past three sections we have constructed many functors on various categories; e.g.:

(1) $\mathscr{C}$ mapping the category of sheaves of R-modules over a space to the category of R-modules over a space to the category of R-modules. Also $\mathscr{D}$.
(2) The functor mapping vector bundles over a space X to sheaves of R-modules over X; its inverse functor (which is exact, in a sense easily made precise).
(3) The various products $\otimes$, Λ, $\odot$, in the category of vector bundles over X.
(4) T_p mapping the category of smooth manifolds to the category of smooth vector bundles.

Organize these sections categorically, and discuss the relations between these functors.

4. CERTAIN OPERATORS ON VECTOR BUNDLES

(A) Definitions. Let $\xi : E \to X$ be a vector bundle over X. A *linear connection* on E is a map

$$\nabla : \mathscr{D}(T(X)) \times \mathscr{D}(E) \to \mathscr{D}(E)$$

written $\nabla(u, \phi) = \nabla_u \phi$, such that:

(1) the map $u \to \nabla_u \phi$ is $\mathscr{D}(R)$-linear for each $\phi \in \mathscr{D}(E)$;
(2) the map $\phi \to \nabla_u \phi$ is R-linear for each $u \in \mathscr{D}(T(X))$;
(3) $\nabla_u(\gamma\phi) = (\nabla_u \gamma)\phi + \gamma \nabla_u \phi$ for each $\phi \in \mathscr{D}(E)$, $\gamma \in \mathscr{D}(R)$ where $\nabla_u \gamma = u(\gamma)$. We shall refer to $\nabla_u \phi$ as the *covariant derivative of* ϕ *with respect to* u.

The *curvature* of ∇ is the map

$$R : \Lambda^2 \mathscr{D}(T(X)) \times \mathscr{D}(E) \to \mathscr{D}(E)$$

defined by

$$R(u_1, u_2)\phi = \nabla_{u_1} \nabla_{u_2} \phi - \nabla_{u_2} \nabla_{u_1} \phi - \nabla_{[u_1, u_2]} \phi$$

$$= -R(u_2, u_1)\phi$$

The curvature is $\mathscr{D}(R)$-linear in each variable u_1, u_2, ϕ. The *covariant differential* is the map

$$\nabla : \mathscr{D}(E) \to \mathscr{D}(T^*(X) \otimes E) = \mathscr{D}(\mathrm{Hom}(T(X), E))$$

given by $(\nabla \phi)u = \nabla_u \phi$. If ∇^E and ∇^F are connections on the vector bundles E and F, we define the connection ∇ on $E \oplus F$ by

$$\nabla_u(\phi \oplus \psi) = \nabla_u^E(\phi) \oplus \nabla_u^F(\psi) \qquad \text{for } u \in \mathscr{D}(T(X)),\ \phi \in \mathscr{D}(E),\ \psi \in \mathscr{D}(F)$$

Also the connection ∇ on $E \otimes F$, characterized by

$$\nabla_u(\phi \otimes \psi) = \nabla_u^E(\phi) \otimes \psi + \phi \otimes \nabla_u^F(\psi)$$

In particular, we have induced connections on $\otimes^p E$, $\Lambda^p E$, $\odot^p E$. Also the connection ∇^* on the dual bundle E^*, given by

$$\nabla_u^*(\psi)v = \nabla_u(\psi v) - \nabla_u(\psi v) - \psi \nabla_u(v) \qquad \text{for } \psi \in \mathscr{D}(E^*),\ v \in \mathscr{D}(E)$$

where again $\nabla_u(\psi v) = u(\psi v)$. More generally, define the connection ∇ on the space $L^p(E, F)$ $= \mathrm{Hom}(\otimes^p E, F)$ of p-linear maps $E \to F$ by

$$(\nabla_u \phi)(v_1, \ldots, v_p) = \nabla_u^F(\phi(v_1, \ldots, v_p)) - \sum_{i=1}^{p} \phi(v_1, \ldots, \nabla_u^E(v_i), \ldots, v_p)$$

Similarly for $\mathrm{Hom}(\Lambda^p E, F)$, $\mathrm{Hom}(\odot^p E, F)$.

Remark. Throughout this section it would be possible to generalize the notions and results, replacing $\mathscr{D}(T(X))$ by $\mathscr{D}(S)$, where S is a vector sub-bundle of T(X) which is completely integrable, i.e. such that $\mathscr{D}(S)$ is a Lie subalgebra of $\mathscr{D}(T(X))$. By Frobenius' theorem these are just the leaved structures on X.

(B) Definition. If $\xi : E \to X$ is a vector bundle, the *Lie derivative* is the map

$$\theta : \mathscr{D}(T(X)) \times L^p(\mathscr{D}(T^*(X)), \mathscr{D}(E)) \to L^p(\mathscr{D}(T^*(X)), \mathscr{D}(E))$$

defined by

$$(\theta_u \alpha)(\phi_1, \ldots, \phi_p) = \sum_{i=1}^{p} \alpha(\phi_1, \ldots, \theta_u(\phi_i), \ldots, \phi_p) \qquad \text{for } \phi_i \in \mathscr{D}(T^*(X))$$

where $(\theta_u\phi)v = \phi([u, v])$ for $v \in \mathscr{D}(T(X))$. Similarly for alternating and symmetric p-linear α. The Lie derivative is R-linear in each variable and satisfies $\theta_u(\gamma\alpha) = (\nabla_u\gamma)\alpha + \gamma\theta_u\alpha$ for $\gamma \in \mathscr{D}(R)$, $\alpha \in L^p(\mathscr{D}(T^*(X)), \mathscr{D}(E))$; however, $\theta_{\gamma u}(\alpha) \neq \gamma\theta_u(\alpha)$ in general, so that θ is not usually a connection.

Now suppose ∇ is a connection on $\xi : E \to X$. Then we can define the *dual Lie derivative (relative to ∇)*:

$$\theta : \mathscr{D}(T(X)) \times L^p(\mathscr{D}(T(X)), \mathscr{D}(E)) \to L^p(\mathscr{D}(T(X)), \mathscr{D}(E))$$

by

$$\theta_u(\alpha)(u_1, \ldots, u_p) = \nabla_u\alpha(u_1, \ldots, u_p) - \sum_{i=1}^{p} \alpha(u_1, \ldots, [u, u_i], \ldots, u_p)$$

Similarly for alternating and symmetric p-linear α.

(C) If $\xi : E \to X$ is a vector bundle, let $\mathscr{A}^p(E) = AL^p(\mathscr{D}(T(X)), \mathscr{D}(E))$ be the space of alternating p-linear maps from $\mathscr{D}(T(X)) \to \mathscr{D}(E)$; we agree that $\mathscr{A}^0(E) = \mathscr{D}(E)$, and that $\mathscr{A}^p(E) = 0$ for $p < 0$ and for $p > n$. Thus $\mathscr{A}(E) = \sum_p \mathscr{A}^p(E)$ is a graded commutative vector space over R with only finitely many non-zero summands. We call the elements of $\mathscr{A}^p(E)$ the *E-valued differential p-forms on* X.

Definition. Let ∇ be a connection on $\xi : E \to X$. The *exterior differential (relative to ∇)* is the map $d : \mathscr{A}^p(E) \to \mathscr{A}^{p+1}(E)$ defined by

$$(d\phi)(u_1, \ldots, u_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \nabla_{u_i}\phi(u_1, \ldots, \hat{u}_i, \ldots, u_{p+1})$$

$$+ \sum_{i<j} (-1)^{i+j}\phi([u_i, u_j], u_1, \ldots, \hat{u}_i, \ldots, \hat{u}_j, \ldots, u_{p+1})$$

where $u_i \in T(X)$.

Given $u \in \mathscr{D}(T(X))$, let us define the *interior product* $i_u : \mathscr{A}^p(E) \to \mathscr{A}^{p-1}(E)$ by $(i_u\phi)(u_1, \ldots, u_{p-1}) = \phi(u, u_1, \ldots, u_{p-1})$.

The following formal computations are left as an exercise.

Proposition.

$$\theta_u i_v - i_v\theta_u = i_{[u,v]}$$

$$i_u i_u = 0$$

$$\theta_u = d i_u + i_u d$$

$$\theta_u\theta_v - \theta_v\theta_u = \theta_{[u,v]} + R(u, v) \quad \textit{for all} \quad u, v \in \mathscr{D}(T(X))$$

In particular,

$$\theta_u i_u = i_u\theta_u$$

(D) Let E_1, E_2, F be vector bundles over X with linear connections $\nabla^1, \nabla^2, \nabla$ respectively. Suppose we are given a bilinear pairing $E_1 \times E_2 \to F$, written $(v_1, v_2) \to v_1 \cdot v_2$, such that $\nabla_u(\phi_1 \cdot \phi_2) = (\nabla^1_u\phi_1)\cdot\phi_2 + \phi_1\cdot(\nabla^2_u\phi_2)$ for all $u \in \mathscr{D}(T(X)), \phi_i \in \mathscr{D}(E_i)$. Then we have the induced bilinear pairing

$$\wedge : \mathscr{A}^p(E_1) \times \mathscr{A}^q(E_2) \to \mathscr{A}^{p+q}(F)$$

relative to which

$$\nabla_u(\phi \wedge \psi) = (\nabla^1_u\phi) \wedge \psi + \phi \wedge (\nabla^2_u\psi)$$

$$\theta_u(\phi \wedge \psi) = (\theta_u\phi) \wedge \psi + \phi \wedge (\theta_u\psi)$$

$$i_u(\phi \wedge \psi) = (i_u\phi) \wedge \psi + (-1)^p\phi \wedge (i_u\psi)$$

$$d(\phi \wedge \psi) = (d\phi) \wedge \psi + (-1)^p\phi \wedge (d\psi) \qquad \text{(proofs by induction).}$$

For instance, taking $E_1 = \mathrm{Hom}_R(E, E)$, $E_2 = E = F$ and the natural pairing $\mathrm{Hom}_R(E, E) \times E \to E$, we have the

Proposition. *For any connection ∇ on* E,

$$\theta_u d - d\theta_u = i_u R \qquad \text{i.e.} \qquad \theta_u d(\phi) - d\theta_u(\phi) = (i_u R)\phi$$

$$d^2 = R \qquad \text{i.e.} \qquad d^2\phi = R \wedge \phi$$

$$dR = 0$$

(E) *Example 1.* $E = X \times R$. Then $\nabla_u\phi = u(\phi) = d\phi \cdot u$ is a connection. Its curvature $R(u, v) = 0$ for all $u, v \in \mathscr{D}(T(X))$, by definition of the bracket $[u, v]$. The Lie derivative $\theta_u(v) = [u, v]$; for $\phi \in \mathscr{D}(T^*(X))$ we have $(\theta_u\phi)v = d(\phi\cdot v)u - \phi([u, v])$. The product formula for the exterior differential in $\mathscr{A}^p(R)$ is obtained through the exterior product $\wedge : \wedge^p T^*(X) \times \wedge^q T^*(X) \to \wedge^{p+q}T^*(X)$. In particular,

(1) $d\phi$ is the differential of $\phi \in \mathscr{D}(R)$;
(2) $d(\phi \wedge \psi) = d\phi \wedge \psi + (-1)^p\phi \wedge d\psi$ for all $\phi \in \mathscr{A}^p(R)$, $\psi \in \mathscr{A}^q(R)$;
(3) $d^2\phi = 0$ for all $\phi \in \mathscr{A}^p(R)$. In fact, these properties characterize d.

Note also that

$$\theta_u d = d\theta_u$$

$$\theta_u\theta_v - \theta_v\theta_u = \theta_{[u,v]}$$

Example 2. Let $E_1 = E$, $E_2 = E^*$, $F = X \times R$ with the natural pairing. Then given any connection ∇ on E, the dual connection and the differential (on functions) are related in the desired way — by definition of dual connection:

$$d(\phi \cdot \psi)u = (\nabla_u\phi)\cdot\psi + \phi\cdot(\nabla_u\psi) \quad \text{for} \quad \phi \in \mathscr{D}(E), \quad \psi \in \mathscr{D}(E^*)$$

Example 3. Take $E = T(X)$ (or more generally any completely integrable sub-bundle). A connection ∇ on $T(X)$ is called a *connection on* X. A special phenomenon occurs because of the existence of the identity map $\alpha : \mathscr{D}(T(X)) \to \mathscr{D}(T(X))$. Its exterior differential

$$d\alpha(u_1, u_2) = \nabla_{u_1}\alpha(u_2) - \nabla_{u_2}\alpha(u_1) - \alpha([u_1, u_2])$$

The map $T : \mathscr{D}(T(X)) \times \mathscr{D}(T(X)) \to \mathscr{D}(T(X))$ given by

$$T(u_1, u_2) = \nabla_{u_1}(u_2) - \nabla_{u_2}(u_1) - [u_1, u_2] = -T(u_2, u_1)$$

is called *the torsion of the connection* on X, and ∇ is said to be *symmetric* if $T = 0$.

Bianchi identities:

(1) $(dT)(u, v, w) = R(u, v)w + R(v, w)u + R(w, u)v$;

(2) $(\nabla_u R)(v, w) + (\nabla_v R)(w, u) + (\nabla_w R)(u, v) = 0$ for all $u, v, w \in \mathscr{D}(T(X))$, if $T = 0$.

The connection ∇ induces a connection ∇^p on $\otimes^p T(X)$; the tensor pairing $\otimes^p T(X) \times \otimes^q T(X) \to \otimes^{p+q} T(X)$ has the required derivation property $\nabla_u^{p+q}(\phi \otimes \psi) = (\nabla_u^p \phi) \otimes \psi + \phi \otimes (\nabla_u^q \psi)$.

The *covariant differential* on X is the map $\mathscr{D}(\otimes^p T^*(X)) \to \mathscr{D}(\otimes^{p+1} T^*(X))$ characterized by

$$(\nabla\phi)(v_0, \ldots, v_p) = \sum_{i=0}^{p} \nabla_{v_i}\phi(v_0, \ldots, \hat{v}_i, \ldots, v_p)$$

Proposition. *Let ∇ be a symmetric connection on* X *and ∇^E a connection on the vector bundle* $\xi : E \to X$. *Then we have the following formula for the exterior differential* $d : \mathscr{A}^p(E) \to \mathscr{A}^{p+1}(E)$:

$$(d\phi)(u_1, \ldots, u_{p+1})$$
$$= \sum_{i=1}^{p+1} (-1)^{i+1} \left[\nabla^E_{u_i}\phi(u_1, \ldots, \hat{u}_i, \ldots, u_{p+1}) - \sum_{j=1}^{p+1} \phi(u_1, \ldots, \hat{u}_j, \ldots, \nabla_{u_i}(u_j), \ldots, u_{p+1})\right]$$

(If the right member is denoted by $(d'p)(u_1, \ldots, u_{p+1})$, then $i_u d' + d' i_u = \theta_u$. But the exterior differential is the unique operator with that property.)

Example 4. A p^{th}-order linear connection on X is a connection ∇ on $T_p(X)$ such that $\nabla_u \phi = u \cdot \phi$ for all $u \in \mathscr{D}(T(X))$, $\phi \in T_{p-1}(X)$. The operators constructed above can be viewed in this context.

(F) For each open set U of the n-manifold X let $\mathscr{S}(U)$ be the totality of charts with domain U. If V is open in U we define $\mathscr{S}(U) \to \mathscr{S}(V)$ by restriction, whence we have the sheaf (of sets) $\mathscr{S} \to X$.

Let G be a subgroup of the general linear group L_n of linear automorphisms of R^n. Say that $\theta, \theta' \in \mathscr{S}(U)$ are *G-equivalent* if $\theta' \circ \theta^{-1}$ has its differential belonging to G for every point of U. Let $\mathscr{S}(U)/G$ denote these equivalence classes and $\mathscr{S}/G \to X$ the corresponding sheaf.

Definition. An *(integrable)* G-*structure on* X is a section of $\mathscr{S}/G$.

For instance, *an orientation of* X is an L_n^+-structure, where $L_n^+ = \{\lambda \in L_n : \det(\lambda) > 0\}$; *an oriented manifold* is a manifold with a particular orientation. A *Riemannian structure on* X is an O_n-structure, where O_n is the subgroup of orthogonal matrices (λ^{-1} = transpose λ); a *Riemannian manifold* is a manifold with a particular Riemannian structure. Most of differential geometry centres around the theorv of G-structures.

Remark. The viewpoint in the preceding definition admits a far-reaching generalization, as emphasized by Spencer [19]. Let $\mathscr{M}$ be a category of topological spaces and continuous maps, closed under unions and restrictions to open sets. Given any topological space X, we define the sheaf $\mathscr{S} \to X$ whose space $\mathscr{S}(U)$ of sections over an open set U of X is the totality of homeomorphisms θ of U onto an object $\theta(U)$ of $\mathscr{M}$. Say $\theta, \theta' \in \mathscr{S}(U)$ are equivalent (written $\theta \sim \theta'$) if $\theta' \circ \theta^{-1}$ is a map of $\mathscr{M}$; let $\mathscr{S}(U)/(\sim)$ denote the space of these equivalence classes. An $\mathscr{M}$*-structure on* X is a section of the sheaf

$$\mathscr{S}/(\sim) \to X$$

For instance, taking for $\mathscr{M}$ the totality of domains of R^n, whose maps are the diffeomorphisms of one domain to another, an $\mathscr{M}$-structure is a differential structure.

Remark. A large class of vector bundles can be constructed as follows: Let $\Theta \in \mathscr{S}/G(X)$ be a G-structure on X, and define the sheaf $\mathscr{P} \to X$ whose space of sections over U is $\mathscr{P}(U) = \{(\theta' \circ \theta^{-1})_* : U \to G \text{ with } \theta, \theta' \in \Theta(U)\}$.

If F is a finite-dimensional vector space on which G operates on the left (i.e. F is a left G-module), then G operates on each $\mathscr{P}(U) \times F$ by $((\theta' \cdot \theta^{-1})_* f) \cdot g = ((\theta' \cdot \theta^{-1})_* g, g^{-1} f)$; clearly $(\theta \cdot \theta^{-1})_* \cdot g = (\theta \cdot \theta^{-1} \cdot g)_*$, so that the operation is well defined. Let $\mathscr{P}(U) \times_G F$ be the orbit space, and $\mathscr{P} \times_G F \to X$ the resulting sheaf. This is easily verified to be a sheaf of vector spaces, locally free of rank = dim F.

5. RIEMANNIAN STRUCTURES

In this section we present briefly the few ideas that we shall need from Riemannian geometry: Riemannian bundles, dual differential forms, integration of scalar densities, the Levi-Cività connection.

(A) Let E be a Euclidean n-dimensional vector space with inner product $\langle\,\rangle$ or $\langle\,\rangle_E$. We have the canonical isometric isomorphism $P : E \to E^*$ given by $P(x)y = \langle x, y\rangle = P(y)x$.

If E and F are both Euclidean spaces, then we define inner products on the direct sum $E \oplus F$ and the tensor product $E \otimes F$, characterized by

$$\langle x_1 \oplus y_1, x_2 \oplus y_2\rangle = \langle x_1, x_2\rangle_E + \langle y_1, y_2\rangle_F$$

$$\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle = \langle x_1, x_2\rangle_E \times \langle y_1, y_2\rangle_F$$

respectively. In particular, we have an induced inner product in each $\otimes^p E$, $\otimes^p E^*$.

Introduce the projection map $A : \otimes^p E \to \wedge^p E$ by

$$A(x_1 \otimes \ldots \otimes x_p) = \sum_\sigma e_\sigma x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(p)}/p!$$

summed over all permutations σ of $(1, \ldots, p)$, where e_σ is the sign of σ. The elements $x_1 \wedge \ldots \wedge x_p = A(x_1 \otimes \ldots \otimes x_p)$ generate $\wedge^p E$. We define the inner product on $\wedge^p E$ by setting

$$\langle x_1 \wedge \ldots \wedge x_p, y_1 \wedge \ldots \wedge y_p\rangle = \langle A(x_1 \otimes \ldots \otimes x_p), A(y_1 \otimes \ldots \otimes y_p)\rangle$$

$$= \langle x_1 \otimes \ldots \otimes x_p, A(y_1 \otimes \ldots \otimes y_p)\rangle$$

$$= \det\langle x_i, y_j\rangle$$

Similarly for the inner product on $\odot^p E$.

(B) **Definition.** An *orientation* in an n-dimensional vector space E is a choice of ray in the one-dimensional space $\Lambda^n E$; if E is Euclidean, this is equivalent to a choice of $\alpha_0 \in \Lambda^n E$ with $|\alpha_0| = 1$, using the norm in $\Lambda^n E$ induced from the inner product in E.
The duality isomorphism $\mathscr{D}: \Lambda^p E^* \to \Lambda^{n-p} E$ is defined by $\mathscr{D}(\varphi)\psi = (\varphi \wedge \psi)\alpha_0$ for all $\psi \in \Lambda^{n-p}E^*$. The *star isomorphism* $*: \Lambda^p E^* \to \Lambda^{n-p}E^*$ is the composition $* = P\mathscr{D}$, where $P: \Lambda^{n-p}E \to \Lambda^{n-p}E^*$ is the isomorphism of Euclidean spaces induced from $P: E \to E^*$.

Then $*$ *is self adjoint*: $(*\varphi, \psi) = (-1)^{p(n-p)}(\varphi, *\psi)$ for all $\varphi \in \Lambda^p E^*$, $\psi \in \Lambda^{n-p}E^*$; namely,

$$(P\mathscr{D}(\varphi), \psi) = \mathscr{D}(\varphi)\psi = (\varphi\wedge\psi)\alpha_0 = (-1)^{p(n-p)}\mathscr{D}(\psi)\varphi = (-1)^{p(n-p)}(P\mathscr{D}(\psi), \varphi)$$

Next, $*$ *is an isometry*: $(*\varphi, *\psi) = (\varphi,\psi)$; namely, P is an isometry, and so is $\mathscr{D}$ (as we can see by introducing a base in E).

It follows that $*$ *is an involution*: $**\varphi = (-1)^{p(n-p)}\varphi$ for all $\varphi \in \Lambda^p E^*$. The number $1 \in \Lambda^0 E^*$ is mapped to $*1 \in \Lambda^n E^*$, characterized by $(*1)\alpha_0 = 1$. The $*$ operation does not preserve the algebra structure in ΛE^*; however, for all $\varphi, \psi \in \Lambda^p E^*$ we have

$$\varphi\wedge(*\psi) = (\phi,\psi)*1 = \psi\wedge(*\varphi)$$

because

$$(\varphi\wedge(*\psi))\alpha_0 = \mathscr{D}(\varphi)P\mathscr{D}(\psi) = (\mathscr{D}\varphi, \mathscr{D}\psi) = (\varphi,\psi)$$

If E and F are both Euclidean spaces and E is oriented, we extend the definition of the star isomorphism to $*: \Lambda^p E^* \otimes F \to \Lambda^{n-p}E^*\otimes F^*$ by setting $*(\varphi\otimes y) = (*\varphi)\otimes Py$. The above properties of $*$ continue to hold; for instance, using the natural pairing $F\times F^* \to R$ to define the exterior product,

$$(\varphi\otimes y)\wedge(*(\psi\otimes z)) = (\varphi\wedge(*\psi))(yP(z)) = (\varphi,\psi)(y,z)*1 = (\varphi\otimes y, \psi\otimes z)*1$$

(C) **Definition.** Let $\xi: E\to X$ be a vector bundle. A *Riemannian metric* in E is an element $g \in \mathscr{S}(\odot^2 E^*)$ such that each g(x) is positive definite; i.e. each g(x) is a symmetric bilinear form on the fibre E_x, and we require that g(x) be a Euclidean structure on E_x. Write $g(x)(u,v) = (u,v)_x$. A Riemannian metric in E determines the bundle isomorphism $P: E\to E^*$.

A *Riemannian structure* on the manifold X is a Riemannian metric in the tangent bundle $T(X)\to X$; we shall call X a *Riemannian manifold* if it has a Riemannian structure. That this definition is equivalent to the one given in Section 4F (in fact, the assertion that there is a natural bijective correspondence between the Riemannian metrics on T(X) and the O_n-structures on X) is left as an exercise (of which we shall not make use).

An application of a partition of unity on X yields the

Lemma. *Every vector bundle admits a Riemannian metric.*

Remark. Suppose that X is a Riemannian n-manifold. Then the n-covectors of length one form an orientable submanifold $\tilde X$ of the total space of $\pi: \Lambda^n T^*(X)\to X$, and $\pi: \tilde X \to X$ is a two-leaved covering map. Over each orientable component of X there are precisely two components of $\tilde X$, which (viewed as sections over X) determine the orientations of X.

(D) We now extend the definition of the star isomorphism to vector bundles. That concept requires some sort of orientation which can be achieved either by: (1) restricting attention to oriented manifolds; or (2) twisting the differential forms. We introduce the machinery for (2), which should not obscure the path of a reader wishing to follow (1).

Definition. Let X be an n-manifold. The *bundle of twisted real numbers of* X is the vector bundle of fibre dimension 1 constructed as in Section 4F through the action $L_n\times R\to R$ given by $(\lambda, r)\to \operatorname{sign}(\det\lambda)r$. To conform with our convention of writing R for the trivial line bundle $X\times R$ we let $\tilde R\to X$ denote the bundle of twisted real numbers. If X is orientable, then a choice of orientation determines a bundle isomorphism $\tilde R\to R$; X is non-orientable if and only if $\tilde R\to X$ is a non-trivial bundle. (The twisted real numbers are a special case of what in topology is called a system of local coefficients.)

Given a vector bundle $\xi: E\to X$ we shall speak of $E\otimes\tilde R\to X$ as the *twisted bundle* of E and of its sections $\varphi\in\tilde{\mathscr{S}}(E) = \mathscr{S}(E\otimes\tilde R)$ as *twisted sections* of E. In particular, we have the notion of a twisted p-form (called a p-form of odd kind by de Rham). A twisted n-form is usually called a *scalar density* on X.

Exercise. Define the notion of an *oriented map* $f: X\to Y$ from one manifold to another. Describe the behaviour of twisted forms under an oriented map.

Definition. Let X be Riemannian n-manifold, and $\xi: E\to X$ a vector bundle with Riemannian metric. We define the *star isomorphism*

$$*: \Lambda^p T^*(X)\otimes E \to \Lambda^{n-p}T^*(X)\otimes\tilde R\otimes E^*$$

as follows. If X is oriented then we define $*$ by using the definition in (B) on the fibres. If X is not oriented then we take any chart (θ, U) at x and define

$$*: \Lambda^p X^*(x)\otimes E_x \to \Lambda^{n-p}X^*(x)\otimes E_x^*$$

using the orientation in U provided by θ. Another chart (θ', U') at x defines another $*' = \operatorname{sign}(\theta'\cdot\theta^{-1})*$, where $\operatorname{sign}(\theta'\cdot\theta^{-1})$ is the sign of the Jacobian of $\theta'\cdot\theta^{-1}$ at x; it follows that the star of an E-valued p-form is well defined as an E^*-valued twisted (n-p)-form.

From the natural pairings of bundles

$$R\times\tilde R\to\tilde R,\ \tilde R\times\tilde R\to R,\ E\times E^*\to R$$

we obtain pairings of the type

$$\wedge: (\Lambda^p T^*(X)\otimes E)\times(\Lambda^q T^*(X)\otimes\tilde R\otimes E^*)\to\Lambda^{p+q}T^*(X)\otimes\tilde R$$

In particular, we have

$$(\Lambda^p T^*(X)\otimes E)\times(\Lambda^p T^*(X)\otimes E)\to\Lambda^n T^*(X)\otimes\tilde R$$

given by

$$(\varphi,\psi)\to\varphi\wedge*\psi = \varphi *\psi$$

Definition. The *volume density* of the Riemannian n-manifold X is the twisted n-form $*1$. If X is oriented then we view $*1$ as an n-form. Note that if (θ, U) is a chart at $x\in X$ and $e^{1\cdots n} = e^1\wedge\ldots\wedge e^n$ is the unit n-covector of R^n defining its orientation, then $*1(x)$ has the representation $|\Lambda^n\theta(x)e^{1\cdots n}|$.

(E) Let X be any n-manifold.

Definition. Given any compact twisted n-form $w\in\mathscr{D}_0(\Lambda^n T^*(X)\otimes\tilde R)$, we define its *integral* $\int_X w$ as follows.

Suppose first that spt(w) is contained in a chart (θ, U) of X. Then θ orients U, and $(\Lambda^n \theta^{-1})w$ is a compact n-form on R^n. We define

$$\int_X w = \int_{R^n} (\Lambda^n \theta^{-1})w$$

If (θ', U') is another chart containing spt(w), then $\theta' \cdot \theta^{-1}$ is a diffeomorphism $\theta(U \cap U') \to \theta'(U \cap U')$. By the transformation of the integral formula

$$\int_{R^n} (\Lambda^n \theta^{-1})w = \mathrm{sign}(\theta' \cdot \theta^{-1}) \int_{R^n} (\Lambda^n \theta'^{-1})w$$

where $\mathrm{sign}(\theta' \cdot \theta^{-1})$ is the sign of the Jacobian of $\theta' \cdot \theta^{-1}$. It follows that the value of the integral is independent of the choice of chart.

For any compact w let (γ_i) be a finite partition of unity whose supports are contained in charts whose union covers spt(w). We define

$$\int_X w = \sum_i \int_X \gamma_i w$$

If (β_j) were another such partition of unity, then we mix the partitions:

$$\sum_i \int_X \gamma_i w = \sum_i \int_X \sum_j \beta_j \gamma_i w = \sum_j \int_X \sum_i \gamma_i \beta_j w = \sum_j \int_X \beta_j w$$

Thus the value of the integral is independent of the choice of partition.

Remark. If X is oriented, then we can restrict attention to those charts preserving orientation, so that every sign $(\theta' \cdot \theta^{-1}) = 1$ in the transformation of integral formula. Thus we obtain the definition of the integral over X if $w \in \mathscr{D}_0(\Lambda^n T^*(X))$.

Theorem. *If* w *is a compact twisted* (n−1) *form on* X, *then* dw *is also twisted, and* $\int_X dw = 0$.

Proof. It suffices to suppose that spt(w) is contained in a chart; for if (γ_i) is a partition of unity on X whose supports are contained in charts whose union covers spt(w), then

$$\sum d(\gamma_i w) = \Big(\sum_i d\gamma_i\Big) \wedge w + \sum_i \gamma_i \wedge dw = dw$$

Thus we have reduced the theorem to that of a compact (n − 1) form on R^n, an elementary case.

(F) Suppose now that ∇ is a connection in the bundle $\xi : E \to X$; in particular, ∇ induces a connection in E^* and in $\odot^2 E^*$.

Definition. *A Riemannian bundle* $\xi : E \to X$ consists of a pair (∇, g) on E, where ∇ is a connection, g is a Riemannian metric, and $\nabla_u g = 0$ for all $u \in \mathscr{D}(T(X))$, i.e.

$$\nabla_u\langle\phi, \psi\rangle = \langle\nabla_u \phi, \psi\rangle + \langle\phi, \nabla_u \psi\rangle$$

where once again we have written

$$\nabla_u\langle\phi, \psi\rangle = u(\langle\phi, \psi\rangle)$$

It is natural to determine when a Riemannian metric on $\xi : E \to X$ induces a Riemannian bundle structure. There are two standard situations: the fundamental theorem of Riemannian geometry which follows, and a class of examples of holomorphic vector bundles.

Theorem (Levi-Cività). *If* g *is a Riemannian metric on* $T(X) \to X$, *then there is a unique symmetric connection* ∇ *on* T(X) *such that* $\nabla g = 0$.

Proof. First of all, if (∇, g) is any Riemannian bundle structure on T(X) and we calculate

$$g(\nabla_u v, w) + g(\nabla_v w, u) - g(\nabla_w u, v)$$

then we find the identity

$$2g(\nabla_u v, w) = ug(v, w) + vg(w, u) - wg(u, v) - g(u, [v, w]) + g(v, [w, u]) + g(w, [u, v])$$
$$- g(u, T(v, w)) + g(v, T(w, u)) + g(w, T(u, v))$$

for all $u, v, w \in \mathscr{D}(T(X))$. In particular, if ∇ is symmetric the last three terms vanish; because g is non-singular, we see that $\nabla_u v$ is completely determined by g, thereby giving both the uniqueness of ∇ and its existence (with the required properties).

Remark. Observe that this theorem is valid for *pseudo-Riemannian metrics on* X, i.e. for $g \in \mathscr{D}(\odot^2 T^*(X))$ such that each g(x) is non-singular at every $x \in X$.

6. DIFFERENTIAL OPERATORS

The viewpoint (using covariant differentials) in this section is that of Singer [18].

(A) Let X be a Riemannian n-manifold with Levi-Cività connection D as in Theorem 5F. Let $\xi : E \to X$ be a vector bundle with connection ∇. We take the tensor product connection on $(\otimes^k T^*(X)) \otimes E$ as in Section 4A, and still denote it by ∇. The covariant differential $\nabla : \mathscr{D}(E) \to \mathscr{D}(T^*(X) \otimes E)$ can then be iterated:

$$\nabla^k : \mathscr{D}(E) \to \mathscr{D}(\otimes^k T^*(X) \otimes E)$$

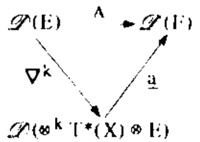
We set ∇^0 = identity on $\mathscr{D}(E)$.

If $\eta : F \to X$ is another vector bundle, we define the *coefficient bundle* $\mathrm{Hom}(\odot^k T^*(X), \mathrm{Hom}(E, F)) = \mathrm{Hom}(\odot^k T^*(X) \otimes E, F)$. We shall sometimes treat its elements as belonging to $\mathrm{Hom}(\otimes^k T(X) \otimes E, F)$ using the canonical projection $\otimes^k T^*(X) \to \odot^k T^*(X)$.

As in Proposition 2D we have the identification:

$$\mathscr{D}(\mathrm{Hom}_R(\otimes^k T^*(X) \otimes E, F) = \mathrm{Hom}_{\mathscr{D}(R)}(\mathscr{D}(\otimes^k T^*(X) \otimes E), \mathscr{D}(F))$$

Definition. *A differential operator from E to F homogeneous of order* k is a homomorphism $A:\mathscr{D}(E)\to\mathscr{D}(F)$ which factor through ∇^k for some coefficient section $\underline{a}$:



If we replace $\otimes^k T^*(X)$ by

$$\sum_{p=0}^{k}\otimes^p T^*(X)$$

and ∇^k by

$$\sum_{p=0}^{k}\nabla^p$$

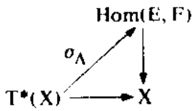
we obtain the notion of *differential operator from* E *to* F *of order* k. The totality of differential operators of all orders forms a vector subspace Diff(E, F) of $\mathrm{Hom}_R(\mathscr{D}(E),\mathscr{D}(F))$ and a $\mathscr{D}(R)$-submodule, filtered by $\mathrm{Diff}^k(E,F)$, the space of operators of orders $\leq k$.

It is important to note that differential operators are local, i.e. $\mathrm{spt}(A\varphi)\subset\mathrm{spt}(\varphi)$ for all $\varphi\in\mathscr{D}(E)$; in fact that property goes a long way toward characterizing differential operators in the space $\mathrm{Hom}_R(\mathscr{D}(E),\mathscr{D}(F))$.

(B) Definition. Suppose that $A\in\mathrm{Diff}^k(E,F)$ has the form

$$A=\sum_{p=0}^{k}\underline{a}_p\nabla^p$$

with leading coefficient $\underline{a}_k\neq 0$. We define the *symbol* of A as the map $\sigma_A:T^*(X)\to\mathrm{Hom}(E,F)$ covering the projection $\pi:T^*(X)\to X$ given by $\sigma_A(w)=(-1)^{k/2}\underline{a}_k(\otimes^k w)$ for all $w\in T^*(X)$:



We make no comment that the factor $(-1)^{k/2}$ may take us momentarily out of the real domain.

Property 1. The composition $B\circ A$ of differential operators is a differential operator, and $\sigma_{B\circ A}=\sigma_B\circ\sigma_A$, where the right member is induced by the bundle pairing $\mathrm{Hom}(F_1,F_2)\times\mathrm{Hom}(E,F_1)\to\mathrm{Hom}(E,F_2)$.

Property 2. If $A, B, A+B\in\mathrm{Diff}^k(E,F)$ but are not in $\mathrm{Diff}^{k-1}(E,F)$, then $\sigma_{A+B}=\sigma_A+\sigma_B$.

Property 3. Each $u\in\mathscr{D}(TX)$ defines an element in $\mathrm{Diff}^1(E,E)$ by ∇_u. The evaluation $\sigma(w)$ on $w\in T^*(X)$ of its symbol is scalar multiplication in $E_{\pi(w)}$ by $\sqrt{-1}\,w\cdot u$.

Definition. We say that A *is elliptic* if fibre dim E = fibre dim F, and each $\sigma_A(w):E_{\pi(w)}\to F_{\pi(w)}$ is an isomorphism.

(C) **Proposition.** For each $A\in\mathrm{Diff}^k(E,F)$ *there is a unique* $A^*\in\mathrm{Diff}^k(F^*,E^*)$ *such that for each* $\varphi\in\mathscr{D}(E)$, $\psi\in\mathscr{D}(F^*)$ *there is a twisted form* $\tau\in\tilde{\mathscr{D}}(\Lambda^{n-1}T^*(X))$ *satisfying*

$$(A\varphi)\cdot\psi*1-\varphi\cdot(A^*\psi)*1=d\tau$$

Furthermore, $\mathrm{spt}(\tau)\subset\mathrm{spt}(\varphi)$, so that if $\varphi\in\mathscr{D}_0(E)$ has compact support we have by Theorem 5E

$$\int_X(A\varphi)\cdot\psi*1=\int_X\varphi(A^*\psi)*1$$

Again, if X is oriented, then $\tau\in\mathscr{D}(\Lambda^{n-1}T^*(X))$.

A^* is called *the formal adjoint* of A; compare Lemma 3A of Part II. Note that its uniqueness is an immediate consequence of the integral formula.

Proof. If $A\in\mathrm{Diff}^0(E,F)=\mathscr{D}(\mathrm{Hom}(E,F))$, then we define A^* at each $x\in X$ as the algebraic adjoint $A_x^*:F_x^*\to E_x^*$ of A_x. Then $(A\varphi)\cdot\psi=\varphi\cdot(A^*\psi)$ at every point.

Consider next the special case that E = F and $A=\nabla_u$ for some $u\in\mathscr{D}(T(X))$, i.e. $A=\underline{a}\nabla$, where $\underline{a}:\mathscr{D}(T^*(X)\otimes E)\to\mathscr{D}(E)$ is given by $\underline{a}(w\otimes\varphi)=(w\cdot u)\varphi$. Then there is a unique $f_u\in\mathscr{D}(R)$ such that $d(*P^{-1}u)=f_u*1$. We define $A^*=-\nabla_u^*-f_u$; then

$$(A\varphi)\cdot\psi-\varphi\cdot(A^*\psi)=u(\varphi\cdot\psi)+f_u(\varphi\cdot\psi)$$

whence

$$(A\varphi)\cdot\psi*1-\varphi\cdot(A^*\psi)*1=d(\varphi\cdot\psi)\wedge *P^{-1}u+(\varphi\cdot\psi)\wedge d(*P^{-1})u$$

$$=d[(\varphi\cdot\psi)\wedge *P^{-1}u]$$

The last equality follows from the derivation formulas of Part III, Section 4D.

For the next step we observe that if A, B are differential operators satisfying our first identity (for some τ_A,τ_B) then so is $B\circ A$, with $(B\circ A)^*=A^*\circ B^*$ (and $\tau_A+\tau_B$). Thus all linear combinations of compositions of 0^{th}-order operators and ∇^p satisfy the identity. But these include all differential operators, whence the identity is verified in general.

Corollary. *If* E *and* F *are Riemannian bundles, then*

$$\int_X(A\varphi)*\psi=\int_X\varphi*(A^*\psi)$$

for any $\varphi\in\mathscr{D}(E)$, $\psi\in\mathscr{D}(F)$, *at least one of which is compact.* Note that we are interpreting $A^*\in\mathrm{Diff}(F,E)$, as we shall do consistently in case of Riemannian bundles.

Corollary. *If* $A \in \mathrm{Diff}^k(E, F)$, *then* $\sigma_{A^*} = (-1)^k \sigma_A^*$, *where the right member assigns to each* $w \in T^*(X)$ *the adjoint homomorphism of* $\sigma_A(w)$. It is sufficient to prove this for operators of order 0 and 1, where it is trivial.

Definition. If $\xi : E \to X$ is a Riemannian bundle, then $A \in \mathrm{Diff}^{2s}(E, E)$ is *strongly elliptic* if σ_A is positive definite at every point, i.e. for every non-zero $w \in T^*(X)$ and non-zero $v \in E_{\pi(w)}$ we have $\langle \sigma_A(w) v, v \rangle_{\pi(w)} > 0$. Note that the order of A is necessarily even if A is strongly elliptic; see Section 6A of Part II.

Proposition. *If* E *and* F *are Riemannian bundles with the same fibre dimension, the* $A \in \mathrm{Diff}^s(E, F)$ *is elliptic if and only if* $(-1)^s A^* A \in \mathrm{Diff}^{2s}(E, E)$ *is strongly elliptic.*

This is immediate, because $\sigma_{A^*A} = \sigma_{A^*}\sigma_A = (-1)^s \sigma_A^* \sigma_A$.

(D) Let (θ, U) be a chart on X and $\rho : \xi^{-1}(U) \to U \times R^p$ a trivialization of $\xi : E \to X$ over U. For each $\varphi \in \mathscr{D}(E)'$ let $\bar{\varphi} : \theta(U) \to R^p$ be defined by $\bar{\varphi}(x) = \pi\rho\varphi\theta^{-1}(x)$, where $\pi : U \times R^p \to R^p$ is projection onto the second factor. We shall use the same notation for local representations for $\psi \subset \mathscr{D}(F)$, relative to a trivialization $\sigma : \eta^{-1}(U) \to U \times R^q$.

Proposition. *An* R-*homomorphism* $A : \mathscr{D}(E) \to \mathscr{D}(F)$ *is a* k^{th}-*order differential operator if and only if each point of* X *is contained in a chart* (θ, U) *over which there are trivializations of* E *and* F, *such* A *has the representation:*

$$\overline{A\varphi}(x) = \sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha \bar{\varphi}(x)$$

for suitable smooth maps $a_\alpha : \theta(U) \to M(p, q)$, *the vector space of* $p \times q$ *matrices. Furthermore,*

$$\sigma_A(w) = (-1)^{k/2} \sum a_{i_1 \ldots i_k} w_{i_1} \cdots w_{i_k}$$

where $(w_1, \ldots, w_n)$ *represents* w *in* (θ, U).

Proof. First of all, if ∇ and $\tilde{\nabla}$ are any connections on E then we can find $a_j \in \mathscr{D}(\otimes^j T^*(X) \otimes E, F)$ for $0 \leq j < k$ such that

$$\underline{a}\nabla^k = \underline{a}\tilde{\nabla}^k + \sum_{j=0}^{k-1} a_j \tilde{\nabla}^j$$

Namely, for $k = 1$, $\nabla - \tilde{\nabla} = w$ assigns to each $\varphi \in \mathscr{D}(E)$ an element $w(\varphi) \in \mathscr{D}(\mathrm{Hom}(T(X), E))$, and furthermore w is $\mathscr{D}(R)$-linear in φ, so that $a_0 = \underline{a}w$; the general case follows by induction. In particular, if a homomorphism A can be expressed as a polynomial in ∇ (some connection E), then it can be expressed in terms of any other $\tilde{\nabla}$. The co-ordinate expression for the symbol will follow [illegible]

To prove the necessity we take any chart (θ, U) and trivializations ρ, σ of the bundle restrictions $E|U$, $F|U$ respectively. Define the connection:

$$\tilde{\nabla} : \mathscr{D}(T(X)|U) \times \mathscr{D}(E|U) \to \mathscr{D}(E|U)$$

by

$$\tilde{\nabla}_u \varphi = \sum_{\alpha=1}^{n} u^\alpha \frac{\partial \varphi}{\partial x^\alpha}$$

Thus ∇ and $\tilde{\nabla}$ are two connections on $E|U$ each expressible in terms of the other. The restrictions of the coefficients $\underline{a}_j$ of A to U have (using the trivializations ρ, σ) the co-ordinate representatives $(\underline{a}_j(x))_{i_1 \ldots i_j} \in M(p, q)$. Composing these gives the desired representation of $A\bar{\varphi}$ in U.

To prove the sufficiency, let $\mathscr{U} = (\theta_i, U_i)$ be a locally finite covering of X by charts over which A has the stated co-ordinate representation. Let (γ_i) be a partition of unity subordinate to $\mathscr{U}$, and $\tilde{\nabla}_i$ the above connection in each (θ_i, U_i). Then

$$\nabla = \sum_i \gamma_i \tilde{\nabla}_i$$

is a connection on E in terms of which we can express A.

Remark. The proposition shows that the concepts of differential operators, their symbols, and the notion of ellipticity are independent of choice of Riemann structure on X, and of Riemannian metric and connection on the bundle E. Of course the adjoint A^* of A will depend on metrics. An alternative formalism (not involving Riemann structures) which is co-ordinate free is given in Ref.[20], characterizing the k^{th}-order differential operators as those homomorphisms which factor through the k^{th} jet extension $j^k : \mathscr{D}(E) \to \mathscr{D}(J^k(E))$, where $J^k(E) \to X$ is the vector bundle of k-jets of sections of E.

IV. THE EXISTENCE THEOREM AND APPLICATIONS

1. THE EXISTENCE THEOREM

(A) Let X be a Riemannian n-manifold, and $\xi : E \to X$ a Riemannian vector bundle over X. We take the tensor product Riemannian structure in each bundle $\otimes^k T^*(X) \otimes E$, and let $\langle\,\rangle_x$ denote indifferently the inner product in $X(x)$, $X^*(x)$, E_x and in their tensor products.

If $\varphi, \psi \in \mathscr{D}_0(E)$ are compact sections we define the number

$$\langle \varphi, \psi \rangle_0 = \int_X \langle \varphi, \psi \rangle * 1$$

where $\langle\varphi,\psi\rangle$ denotes the function $x \to \langle\varphi(x),\psi(x)\rangle_x$ on X. Then for each integer $r \geq 0$ let

$$\langle\varphi,\psi\rangle_r = \sum_{k=0}^{r} \langle\nabla^k\varphi, \nabla^k\psi\rangle_0$$

These are clearly inner products on $\mathscr{D}_0(E)$, and we let $H_0^r(E)$ denote the indicated completion.

Suppose henceforth that X *is compact*, and write $H^r(E) = H_0^r(E)$. Then we have the following properties:

Rellich's theorem. *If* $s < t$, *then the injection* $H^t(E) \to H^s(E)$ *is compact and dense.* That the map is dense is clear, since $\mathscr{D}(E)$ is dense in both. To see that it is compact, let (φ_i) be any bounded sequence in $H^t(E)$, and take $\gamma \in \mathscr{D}(R)$ with $\mathrm{spt}(\gamma)$ contained in some chart (θ, U). Then $(\gamma\varphi_i)$ can be viewed as a sequence in $H_0^t(U)$, so that we can appeal to Section 5B of Part II to extract a subsequence converging in $H^s(E)$. If (γ_p) is a finite squared partition of unity

$$\sum_p \gamma_p^2 = 1$$

then we construct a subsequence of (φ_i) such that for each p the sequence $(\gamma_p\varphi_i)$ converges in $H^s(E)$.

Let $C^s(E)$ denote the Banach space of sections of E of class C^s, with obvious norm.

Sobolev's theorem. *If* $t > n/2 + s$, *then we have a continuous injection* $H^t(E) \to C^s(E)$. In particular, $H^\infty(E) = \cap\{H^t(E) : t \geq 0\} = \mathscr{D}(E)$. This follows at once from Section 5B of Part II.

Remark. It is an elementary matter to check that the Hilbertian structure of $H^r(E)$ is independent of the Riemannian structures of X and E. In fact, let $\mathscr{U} = (\theta_i, U_i, \rho_i, \gamma_i)$ be a finite system consisting of a covering of X by charts (θ_i, U_i), trivializations $\rho_i : \xi^{-1}(U_i) \to U_i \times \mathbb{R}^m$, and a squared partition of unity (γ_i) with each $\mathrm{spt}(\gamma_i) \subset U_i$. Then we define an inner product on $\mathscr{D}(E)$ by

$$\langle\langle\varphi,\psi\rangle\rangle_r = \sum_i \langle\overline{\gamma_i\varphi}, \overline{\gamma_i\psi}\rangle_{H_0^r(\mathbb{R}^n)}$$

where $\bar{\varphi}$ is defined as in Section 6D (Part III). Any such inner product is easily seen to be equivalent to that given using the iterated (through r) covariant differentials defined by Riemannian structures.

(B) *If* $A \in \mathrm{Diff}^r(E, F)$, *then* A *has a unique extension to a continuous linear map* $A : H^{s+r}(E) \to H^s(F)$ *for all* $s \geq 0$. This is clear because the coefficients in the representation $A = \sum a_k \nabla^k$ are smooth, whence there is a constant such that

$$|A\varphi|_s \leq \mathrm{const}|\varphi|_{s+r} \quad \text{for all} \quad \varphi \in \mathscr{D}(E)$$

Suppose now that F = E and r = 2s; then A can be represented (not uniquely, in general)

$$A = \sum_k C_k \circ B_k$$

where order $B_k \leq s$, order $C_k \leq s$. Then for any $\varphi, \psi \in \mathscr{D}(E)$ we have by Corollary 6C (Part III)

$$\langle A\varphi,\psi\rangle_0 = \sum_k \langle B_k\varphi, C_k^*\psi\rangle_0$$

Define

$$\alpha : H^s(E) \to \mathbb{R} \quad \text{by} \quad \alpha(\varphi) = \sum_k \langle B_k\varphi, C_k^*\varphi\rangle_0$$

On the dense subspace $\mathscr{D}(E) \subset H^s(E)$ we have the equality $\alpha(\varphi) = \langle A\varphi, \varphi\rangle_0$, so that α is defined independently of the representation in terms of operators of orders $\leq s$.

Theorem (Gårding's inequality). *If* $A \in \mathrm{Diff}^{2s}(E, E)$ *is strongly elliptic, then* α *is coercive on* $H^s(E)$, *i.e. there are numbers* $\lambda_0 > 0$, $c > 0$ *such that* $\alpha(\varphi) + \lambda_0|\varphi|_0^2 \geq c|\varphi|_s^2$ *for all* $\varphi \in H^s(E)$.

Proof. First of all, since X is compact there exists $\lambda_0 > 0$ such that $\langle\sigma_A(x,\xi)v, v\rangle_x \geq \lambda_0|\xi|^{2s}|v|^s$ for all $\xi \in X^*(x)$, $v \in E_x$, and $x \in X$. Secondly, it suffices to verify the inequality for smooth sections. Thirdly, Theorem 5D of Part II shows that the inequality is valid (for suitable λ_0, c) for all φ having support in a sufficiently small chart. Finally, to establish the inequality in general we proceed as in Case 4 of Section 3C of Part II. Let (γ_i) be a squared partition of unity with sufficiently small supports, and write

$$\langle A\varphi, \varphi\rangle_0 = \sum \langle A(\gamma_i\varphi), \gamma_i\varphi\rangle_0 + R$$

Now Gårding's inequality is valid for the sections $\gamma_i\varphi$. Also, for any $\epsilon > 0$ we have $|R| \leq \epsilon\,\mathrm{const}|\varphi|_s^2 + \epsilon^{-2s+1}\mathrm{const}|\varphi|_0^2$; by choosing ϵ sufficiently small we can again absorb R to obtain the desired inequality for all $\varphi \in \mathscr{D}(E)$.

(C) We are now in position to apply Theorem 4D of Chapter I. Furthermore, given $\psi \in H^t(E)$, if $\varphi \in H^s(E)$ satisfies $\alpha(\varphi, \zeta) = \langle\phi, \zeta\rangle_0$ for all ζ, then $\varphi \in H^{t+r}(E)$ by Theorem 5E of Chapter II, because differentiability is a local matter.

In summary, we have the fundamental

Theorem. *Let* $A \in \mathrm{Diff}^{2s}(E, E)$ *be strongly elliptic. Then*

(1) A *maps* $\mathscr{D}(E)$ *onto* $K^\perp(A^*)$ *in* $\mathscr{D}(E)$ *with kernel* $K(A)$;

(2) $\dim K(A) = \dim K(A^*) < \infty$;

(3) There is a $\lambda_0 > 0$ *such that* $A_\lambda = A + \lambda : \mathscr{D}(E) \to \mathscr{D}(E)$ *is a bijection for all* $\lambda \geq \lambda_0$;

(4) $\dim K(A_\lambda) > 0$ *for at most countably many* λ *with no finite accumulation point.*

An immediate consequence is the theorem of the Introduction.

Corollary. *Let* $A \in \mathrm{Diff}^s(E, F)$ *be elliptic. Then for any* $\psi \in \mathscr{D}(F)$ *there is a* $\varphi \in \mathscr{D}(E)$ *such that* $A\varphi = \psi$ *if and only if* $\psi \in K^\perp(A^*)$. *Furthermore,* $\dim K(A) < \infty$, $\dim K(A^*) < \infty$.

Proof. The last statement follows because $A^*A = B$ is strongly elliptic, and $K(A) = K(B)$ since $\langle B\varphi, \varphi\rangle_0 = \langle A\varphi, A\varphi\rangle_0$. The necessity in the first statement is also clear. To prove the sufficiency, take $\psi \in K^\perp(A^*)$; then $A^*\psi$ is orthogonal to $K(B^*) = K(A)$. The fundamental theorem asserts the existence of $\varphi \in \mathscr{D}(E)$ such that $A^*A\varphi = A^*\psi$; i.e. $A\varphi - \psi \in K(A^*)$. On the other hand, both ψ and $A\varphi$ belong to $K^\perp(A^*)$, so that $A\varphi - \psi = 0$.

Remark. In general the index $(A) = \dim K(A) - \dim K(A^*)$ of an elliptic operator is not zero: see Section 2 below. Its evaluation in terms of differential invariants (in particular, the characteristic classes of X, E, F) was a problem formulated by Gelfand and solved in complete generality by Atiyah-Singer [20].

Remark. In the preceding Corollary three integers appear: dim X, fibre dim E = fibre dim F, and order A = s. The existence of an elliptic operator in such a setting with prescribed order and/or with prescribed analytic properties implies drastic topological restrictions on X, E, F. Very little is known about this basic question (but see Ref. [21]).

Extensive exercise. Throughout the last three Parts we have restricted attention to elliptic operators on closed manifolds (or to 0 boundary values in case of a bounded domain in Euclidean space). It would be a most instructive project now to take a compact Riemannian n-manifold with boundary (a smooth Riemannian (n − 1)-manifold) and re-examine our steps with an eye towards: (1) formulating the concept of elliptic boundary value problems; (2) developing the required version of Gårding's inequality so that an analogue of Theorem 1C and its Corollary is obtained. Hints and special cases can be obtained from various sources mentioned in the Bibliography.

2. HODGE'S THEOREM

(A) Suppose that X is a compact Riemannian n-manifold, and $\xi: E \to X$ a Riemannian bundle with connection ∇. Then as in Sections 4C and D of Part III, using the natural pairing $E \times E^* \to \mathbb{R}$ we have the exterior differential $d: \mathscr{A}^p(E) \to \mathscr{A}^{p+1}(E)$ defined on E-valued p-forms on X, and

$$d(\varphi \wedge * \psi) = d\varphi \wedge * \psi + (-1)^p \varphi \wedge d * \psi$$

for all $\varphi \in \mathscr{A}^p(E)$, $\psi \in \mathscr{A}^q(E)$; similarly for twisted E-valued forms. In particular, replacing (p, q) by (p − 1, p) we have

$$\int_X d(\varphi * \psi) = 0$$

so that

$$\int_X d\varphi \wedge * \psi = (-1)^p \int_X \varphi \wedge (d * \psi) = (-1)^{np+n+1} \int_X \varphi * (* d * \psi)$$

i.e.

$$\langle d\varphi, \psi \rangle_0 = \langle \varphi, (-1)^{np+n+1} * d * \psi \rangle_0$$

Thus we obtain the

Lemma. *The formal adjoint* $d^*: \mathscr{A}^p(E) \to \mathscr{A}^{p-1}(E)$ *of* d *relative to* $\langle\,\rangle_0$ is given by $d^*\varphi = (-1)^{np+n+1} * d * \varphi$. Furthermore, $* d^*\varphi = (-1)^p d * \varphi$, $* d\,\varphi = (-1)^{p+1} d^* * \varphi$ *for* $\varphi \in \mathscr{A}^p(E)$, *where in both right members* d *and* d^* *are constructed in* E^*.

Definition. The *Dirichlet integral* in $\mathscr{A}(E)$ is the positive quadratic function $D: \mathscr{A}(E) \to \mathbb{R}$ given by

$$D(\varphi) = \tfrac{1}{2}(|d\varphi|_0^2 + |d^*\varphi|_0^2)$$

Its Laplace operator $\Delta: \mathscr{A}^p(E) \to \mathscr{A}^p(E)$ is the second-order linear differential operator $\Delta = dd^* + d^*d$. A form $\varphi \in \mathscr{A}^p(E)$ is *harmonic* if $\Delta\varphi = 0$. Let $\mathscr{H}^p(X, E) = \mathrm{Ker}\{\Delta: \mathscr{A}^p(E) \to \mathscr{A}^p(E)\}$; similarly for the space $\tilde{\mathscr{H}}^p(X, E)$ of twisted harmonic E-valued p-forms on X.

We have $\langle \Delta\varphi, \psi \rangle_0 = \langle \varphi, \Delta\psi \rangle_0$, so that Δ *is symmetric;* and $\langle \Delta\varphi, \varphi \rangle_0 = |d\varphi|_0^2 + |d^*\varphi|_0^2 \geq 0$, whence Δ is positive. From that we obtain the

Lemma. $\varphi \in \mathscr{A}(E)$ *is harmonic if and only if both* $d\varphi = 0$ and $d^*\varphi = 0$.

Note that the directional derivative of D at φ in the direction ψ

$$\frac{d}{dh}[D(\varphi + h\psi)]_{h=0} = \langle \Delta\varphi, \psi \rangle_0$$

so that $\Delta\varphi$ serves as the gradient of D at φ; *the critical points of* D *in* $\mathscr{A}(E)$ *are just the harmonic forms*, and these are the absolute minima for D.

From the relations $* dd^* = d^*d *$ and $* d^*d = dd^* *$ we find $* \Delta = \Delta *$, where in the right member Δ operates in E^*; that implies the important

Duality theorem. *Let* X *be a compact Riemannian n-manifold, and* $\xi: E \to X$ *a Riemannian vector bundle. Then for each* p $(0 \leq p \leq n)$ *we have the involutory isomorphism:*

$$* \tilde{\mathscr{H}}^p(X, E) \to \mathscr{H}^{n-p}(X, E^*)$$

Remark. It would seem that the sequence of eigenvalues of Δ would be useful in studying the topological and differential geometric properties of $\xi: E \to X$. However, beyond knowledge of their asymptotic distribution and in spite of many attempts, practically nothing is known about that sequence even for the case $E = X \times \mathbb{R}$ and $A = \Delta$.

(B) For each point $x \in X$ and $w \in X^*(x)$ we use the interior product (certainly a local operator) of Section 4C of Part III to define the endomorphism:

$$i_w = i_{p^{-1}(w)}: \wedge^p X^*(x) \otimes E_x \to \wedge^{p-1} X^*(x) \otimes E_x$$

Letting ϵ_w its adjoint:

$$\langle \epsilon_w \varphi, \psi \rangle_x = \langle \varphi, i_w \psi \rangle_x$$

for all $\varphi \in \wedge^p X(x) \otimes E_x$, $\psi \in \wedge^{p+1} X^*(x) \otimes E_x$, we find that $\sigma_{d+d^*}(w) = \sqrt{-1}(\epsilon_w + i_w)$, so that $d + d^*$ *is an elliptic operator* on E-valued differential forms on X. Also,

$$\langle \sigma_\Delta(w)\varphi, \varphi \rangle = \langle (\sigma_d \sigma_{d^*}(w) + \sigma_{d^*}\sigma_d(w))\varphi, \varphi \rangle = \langle (\epsilon_w i_w + i_w \epsilon_w)\varphi, \varphi \rangle = \langle 2|w|^2 \varphi, \varphi \rangle$$

whence Δ *is strongly elliptic.* We can now apply Theorem 1C and its Corollary to assert the existence of harmonic E-valued forms.

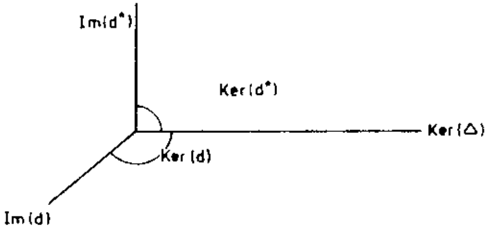
Remark. As we have emphasized all along, both existence and differentiability in Theorem 1C are based on Gårding's inequality, which in the case of the present section takes the form

$$D(\varphi) + \lambda_0 |\varphi|_0^2 \geq c|\varphi|_1^2 \quad \text{for all } \varphi \in \mathscr{A}(E)$$

If we were interested only in harmonic forms we could produce a special simplified proof (avoiding the Fourier transform, and making use of the special character of the Laplacian) of that estimate. That is the basis of the proof of Hodge's theorem (for the special case of E the trivial line bundle and X oriented, but these are superficial simplifications) given by Morrey-Eells (Section 5 of [13]).

(C) Now specialize to $E = X \times \mathbb{R}$ with its connection given as in Example 1 of Section 4E of Part III. Let us write $\mathscr{A}^p = \mathscr{A}^p(E)$ and $\mathscr{H}^p = \mathscr{H}^p(E)$. Then the exterior differential $d : \mathscr{A}^p \to \mathscr{A}^{p+1}$ has square zero, as does its adjoint $d^* : \mathscr{A}^p \to \mathscr{A}^{p-1}$. It follows that $\mathrm{Im}(d) = d\mathscr{A}$ is orthogonal to $\mathrm{Ker}(d^*)$; and that $\mathrm{Im}(d^*) = d^*\mathscr{A}$ is orthogonal to $\mathrm{Ker}(d)$. By Theorem 1C we obtain the

Theorem. $\mathscr{A} = \mathrm{Im}(d) \oplus \mathscr{H} \oplus \mathrm{Im}(d^*)$. *Also,* $H^0 = dH^1 \oplus \mathscr{H} \oplus d^*H^1$.



Exercise. Formulate and prove an analogous theorem for a differential operator B with $B^2 = 0$. Note that by Proposition 4D of Part III we obtain such differential operators whenever $\xi : E \to X$ is a Riemannian bundle with curvature zero.

Exercise. Study the relations between harmonic E-valued forms and polyharmonic E-valued forms (i.e. those which belong to the kernel of some iterated Laplacian Δ^k). Compare these last relations with those obtained from studying the extremals of the iterated Dirichlet integral

$$D^k(\phi) = \tfrac{1}{2}|(d+d^*)^k\phi|_0^2$$

Definition. For any n-manifold X its *de Rham cohomology* is the quotient

$$\mathscr{R} = \mathrm{Ker}(d : \mathscr{A} \to \mathscr{A})/\mathrm{Im}(d : \mathscr{A} \to \mathscr{A})$$

It is clear from the formula $d(\varphi \wedge \psi) = d\varphi \wedge \psi + (-1)^p \varphi \wedge d\psi$ that the denominator is a bilateral ideal in the numerator, whence the exterior product induces an associative, commutative (in the graded sense), graded algebra structure

$$\mathscr{R} = \sum_{p=0}^{n} \mathscr{R}^p$$

in the de Rham cohomology.

Corollary (Hodge's theorem). *If X is a compact Riemannian n-manifold then in every de Rham cohomology class there is precisely one harmonic form.*

In particular, each $\dim \mathscr{R}^p = \dim \mathscr{H}^p$ is finite, and is called the p^{th} *Betti number* $\beta_p(X)$ *of* X. Also, $\mathscr{R}^p = 0$ for $p > n$; and if X is furthermore orientable, then by the duality theorem 2A we have the isomorphism $* : \mathscr{H}^p \to \mathscr{H}^{n-p}$, so that $\beta_p(X) = \beta_{n-p}(X)$ for all $0 \leq p \leq n$. This relation, when coupled with the homology properties below, is a special case of Poincaré's duality theorem.

Exercise. Suppose X is connected and compact. Then $\beta_n(X) = 1$ or 0 depending on whether X is orientable or not.

(D) We now assume familiarity with the definitions and elementary properties of the singular homology space

$$H(X) = \sum_{p \geq 0} H_p(X)$$

and the singular cohomology algebra (with cup product)

$$H^*(X) = \sum_{p \geq 0} H^p(X)$$

with real coefficients; also the fact that on X the cohomology based on smooth singular chains and that based on continuous singular chains are canonically isomorphic as algebras (the isomorphism is induced from the inclusion map of the chain spaces).

Given any $\varphi \in \mathscr{A}^p$ and any smooth p-chain c, we have the integral $\int_c \varphi$ defined. If $\varphi', \varphi \in \mathrm{Ker}(d : \mathscr{A}^p \to \mathscr{A}^{p+1})$ and $\varphi' - \varphi = d\psi$, and similarly if c', c are smooth singular p-chains such that $\partial c' = 0 = \partial c$ and $c' - c = \partial b$ (here ∂ denotes the singular chain boundary operator), then by Stokes' theorem

$$\int_{c'} \varphi' = \int_c \varphi$$

Thus the integral $\int_c \varphi$ depends only on the homology class of c and the de Rham cohomology class of φ. That number is called the *period of* φ on c. Now $H^p(X) = \mathrm{Hom}_{\mathbb{R}}(H_p(X), \mathbb{R})$, and the integral permits us to view the elements of $\mathscr{R}^p$ as linear forms on $H_p(X)$, i.e. the integral induces a linear map

$$I : \mathscr{R} \to H^*(X)$$

De Rham's theorem asserts that I *is an isomorphism of algebras.* For a sheaf theoretic proof (which is in essence a general uniqueness theorem for cohomologies on X) see Hirzebruch (Ref.[8], Section 2). Since $H^*(X)$ is a homology (in particular, a topological) invariant of X, it follows that $\beta_p(X)$ *is independent of both the Riemannian and the differential structures of* X.

We can put Hodge's theorem in a more classical form as follows. Let $c_1, \ldots, c_\beta$ ($\beta = \beta_p(X) < \infty$) be a set of chains on X with all $\partial c_i = 0$ whose homology classes form a base for $H_p(X)$.

Given any set $\pi_1, \ldots, \pi_\beta$ of β real numbers, *Hodge's theorem asserts that there is one and only one harmonic p-form* φ *having these periods*

$$\int_{c_i} \varphi = \pi_i \qquad (1 \leq i \leq \beta)$$

Remarks. If ϕ and ψ are harmonic forms on X, they represent de Rham cohomology classes, as does their exterior product $\phi \wedge \psi$. However, in general $\phi \wedge \psi$ is not itself harmonic. Give an example.

If X and Y are both compact Riemannian manifolds, and $f : X \to Y$ a smooth map, then f induces a degree preserving homomorphism of algebras:

$$\mathcal{R}(f) : \mathcal{R}(Y) \to \mathcal{R}(X)$$

In general, however, f does not map harmonic forms into harmonic forms. Even fibre maps (respecting Riemannian structures) do not behave well. But study the special case of projection of a Riemannian product onto one of its factors.

(E) *Example 1.* Let X be oriented, and define

$$E = \sum_{p \geq 0} \Lambda^{2p} T^*(X), \qquad F = \sum_{p \geq 0} \Lambda^{2p+1} T^*(X)$$

These have the same fibre dimension, because

$$\sum_{p=0}^{n} (-1)^p \binom{n}{p} = 0$$

The elliptic operator $A = d + d^*$ maps forms of even degree into forms of odd degree, and A^* does the opposite. Then

$$\text{index } (d + d^*) = \sum_{p \geq 0} (\beta_{2p}(X) - \beta_{2p+1}(X)) = \text{Euler characteristic of X}$$

Suppose for simplicity that X is oriented.

Example 2. The elements of $\mathscr{A}^0$ are the real-valued functions φ on X, and $d\varphi = 0$ if and only if φ is a constant. Since $d^*|\mathscr{A}^0 = 0$, we find that *the harmonic 0-forms are just the constant functions.* (This fact reflects the maximum principle for the second-order elliptic operator Δ on a manifold without boundary.) By duality, the harmonic n-forms are just the constant multiples of the volume form $*1$.

Example 3. Let E be an n-dimensional vector space, and g, g' two *conformally related* inner products, i.e. there is a number γ such that $g' = \exp(2\gamma)g$; this is merely a convenient way of writing a strictly positive multiplicative factor. If $\alpha_0, \alpha_0' \in \Lambda^n E$ define the same orientation of E and are of norm one relative to g, g' respectively, then $\alpha_0' = \exp(-n\gamma)\alpha_0$; thus if $\mathscr{D}$ and $\mathscr{D}'$ are the duality isomorphisms, we have $\mathscr{D}' = \exp(-n\gamma)\mathscr{D}$. Similarly, if $P, P' : E \to E^*$ are the isomorphism defined by g, g' respectively, then $P'(x)y = g'(x, y) = \exp(2\gamma)P(x)y$. It follows that for $\Lambda^{n-p}E \to \Lambda^{n-p}E^*$ we have $P' = \exp(2(n-p)\gamma)P)$, whence for $\Lambda^p E^* \to \Lambda^{n-p}E^*$ we have the star isomorphism

$$*' = \exp((n-2p)\gamma)*$$

Thus if n = 2p, the automorphism * of $\Lambda^p E^*$ is conformal invariant

Suppose that X has dimension n = 2p. Then $* : \mathscr{H}^p \to \mathscr{H}^p$ is an automorphism depending only on the conformal equivalence class of the Riemannian metric; it is symmetric if p is even, and skew symmetric if p is odd. We conclude:

(1) If dim X = 4r, then the signature of * (= number of positive eigenvalues minus the number of negative eigenvalues) is a significant invariant of X. (It admits an expression in terms of the characteristic classes of X (see Hirzebruch [8]).

(2) If dim X = 4r + 2, then $\beta_{2r+1} \equiv 0(2)$.

These properties are especially interesting for complex manifolds of complex dimension one, for it is known in that case that the complex structure defines a conformal equivalence class of Riemannian metrics, whence *the entire space* $\mathscr{H}^1$ *depends only on the complex structure of* X.

Example 4. Let $\pi : \hat{X} \to X$ be a finite regular covering of X. Then a Riemannian metric on X induces a Riemannian metric on $\hat{X}$, and $\pi^* : \mathscr{H}^p(X) \to \mathscr{H}^p(\hat{X})$ is injective. Thus $\beta_p(\hat{X}) \geq \beta_p(X)$ for all $0 \leq p \leq n$. Betti number relations between $\hat{X}$ and X can be given by properties of harmonic forms, and the action of the subgroup of the fundamental group of X corresponding to the covering space $\hat{X}$. We suggest that as an exercise, starting with a reinterpretation of the pleasant paper of Eckmann [3].

For the two-leaved covering $\tilde{X} \to X$ of any compact manifold, we have powerful cohomology relations between $\tilde{X}$ and X (for any system of local coefficients) given by its Gysin sequence (the fibre is a 0-sphere); see Thom (Ref.[22], Chapter I,III).

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