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Variational techniques in geometry

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VARIATIONAL TECHNIQUES IN GEOMETRY

(some preliminary informal notes)

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A. Introduction.

In these lectures, we shall be mainly concerned with "differential calculus" on the space of Riemannian metrics on a given manifold, and later on on the space of connections on a vector bundle..
The motivation to introduce this technique in Riemannian Geometry is threefold.

i) First, the linearized version of a Riemannian geometric problem, when it makes sense, can give some insight and help to prove or disprove some conjecture.

Example : Riemannian manifolds all of whose geodesics are closed and have the same length (see [B1] for detailed information about this famous problem).

Try to find a one-parameter family $g(t)$ of such metrics on S^2 by conformal perturbation of the standard metric. Setting $g'(0) = fg(0)$, a straightforward computation (ir [B1], p. 151) shows that the integral of f along any geodesic, that is along any great circle, must vanish. Using eigenfunctions of the Laplace operator, it follows that f must be odd (see [B1], p. 123 for various proofs of this fact). This is a mild condition, but it means that such a deformation does not go down to $\mathbb{R}P^2$.

In fact, this rough infinitesimal approach predicts the good results. On one hand, for any smooth odd function on S^2 there actually exists such a family of metrics with $g'(0) = fg(0)$ (Guillemin, 1976, providing the analytical tools to an idea of Funk, 1913, see [B1], ch. 4). On the other hand, on $\mathbb{R}P^2$ (Green, 1962, cf. [B1], ch. V), and even on $\mathbb{R}P^n$ (Berger, 1976, see [B1, Appendix) a Riemannian metric all of whose geodesics are closed is isometric to the standard metric.

ii) Some Riemannian geometric problems appear as an equation whose unknown is the metric.

Example : prescribing the scalar curvature of a compact manifold.
That is, solve the equation $Scal(g) = f$, for a given f. A complete answer

has been given by J. Kazdan and F. Warner ([K-W]). The map $g \longrightarrow Scal(g)$ from the space of positive definite two-forms to the space of smooth functions turns out to be generically a submersion (lemma 2 of [K-W]), or [B3], 4.36), and this fact is very helpful.

Another example : the existence of Einstein metrics.

That is, the equation $Ric(g) = kg$, k constant. Very little is known, when the problem is set in such a general way. Anyhow, it can be proved by linearization (cf. [B3], 4. 62 and ch. 12) and we shall see later that the space of solutions has finite dimension. (This why the existence problem is so difficult).

iii) the equations of General Relativity have a variational interpretation. In the empty space, the action is just the integral of the scalar curvature.

In the sequel, after discussing the difference between *Riemannian metrics* and *Riemannian structures*, we shall compute the first variation of the basic Riemannian invariants (connection, curvatures). The *total scalar curvature* of a compact Riemannian manifold, that is the functional

$$S(g) = \int_M Scal(g)\, v_g$$

is especially interesting : if we restrict ourselves to metrics with prescribed volume, the critical points are just the Einstein metrics. Therefore, it is natural to compute the second variation of S for an Einstein metric (like in the case of the energy for geodesics). But we shall see that the situation is more complicated : the indice and the co-indice are both infinite. However, the infinite dimensional positive and negative spaces have a nice geometric interpretation.

Now, we come to the space of connections on a vector bundle. Such spaces have been considered by particle physicists for some fifteen years (they speak if connections as "gauge potentials"), as configuration spaces for weak and strong interactions. In this situation, the field is given by the curvature of the connection and the Lagrangian is the integrated square norm of the curvature. The corresponding variational equation is the Yang-Mills equation. We shall derive it later on. It turned out in the beginning of the eighties that a thorough study of the geometry of the space of some special solutions of the Yang-Mills equation leads to very striking results in low dimensional topology (see the introductions of [L] and [F-U]). This is a good reason to give an account of some basic facts about the space of connections and the Yang Mills functional. In fact, there are so many similarities between this situation and the Riemannian one that studying both may help to understand what is going on.

B. Riemannian metrics, Riemannian structures.

When studying the set of Riemannian metrics on a manifold M, one must work modulo the action of the diffeomorphism group of M. This simple remark has important consequences that we shall explain below. More-over, we shall meet later a completely analogous situation, with the space of connections on a given vector bundle, which must be studied modulo the gauge group of the bundle (see for instance [L], pp. 23).

B1. The action of the diffeomorphism group.

Given a manifold M and a diffeomorphism ϕ of M, any Riemannian metric g on M is isometric to $\phi^* g$. This is just the definition of an isometry ! If we look at g in local coordinates, looking $\phi^* g$ amounts to take other coordinates.

Examples. i) One of the basic theorems of Riemannian Geometry says that a Riemannian metric with vanishing sectional curvature is locally isometric to $\mathbb{R}^n$ or that $g = \sum(dx^i)^2$ *in some coordinate system.*

ii) All the simply connected Riemannian manifolds with constant curvature -1 are isometric, but there are several classical "models" of the hyperbolic space. Of course, they are related in the way explained above.

Now, denote by $\mathcal{M}$, (or $\mathcal{M}_M$ if there is no ambiguity) the space of Riemannian metrics on M. The pull-back gives a right action of $Diff(M)$ on $\mathcal{M}$, and two metrics lie in the same orbit if and only if they are isometric.

Definition. The set of *Riemannian structures* on M is the orbit space $\mathcal{M}/Diff(M)$.

$\mathcal{M}$ is an open cone in the space $\Gamma(S^2M)$ of smooth symmetric 2-forms for the compact open topology, and can therefore be considered as a manifold modelled on that space.

From now on, M will be assumed to be compact.

With some more work, it is possible to make $Diff(M)$ a manifold modelled on the space of smooth vector fields, with the compact open topology. (However, it is necessary to introduce metrics and diffeomorphisms of class H^s $(s > n/2)$ so that the inverse function theorem may be used in the Banach spaces context (cf. for instance [E], §2)). One has for instance the following

Proposition. (D. Ebin, [E], §6) *For any metric g, the orbit $Diff(M).g$ is a smooth closed submanifold of $\mathcal{M}$.*

Afterwards, it can be proved -see B.2 - that $\mathcal{M}/Diff(M)$ is a manifold *in the neighborhood of any Riemannian structure which has no isometries.* (Indeed, non trivial isometries provide fixed points for the action of $Diff(M)$ on $\mathcal{M}$. Although a reasonable description of $\mathcal{M}/Diff(M)$ which take these singularities into account is available (see [E] again, §2 and 7), it will be more simple for us to work with $\mathcal{M}$, and remember that everything which has a Riemannian geometric meaning must be equivariant with respect to $Diff(M)$.

B.2. The tangent space to the space of Riemannian metrics.

As we have seen, $\mathcal{M}_M$ is open in $\Gamma(S^2M)$ and we can set

$$T_g\mathcal{M}_M = \Gamma(S^2M)$$

In a more down to earth way, taking M compact for simplicity, notice that, for any $g \in \mathcal{M}$ and $h \in \Gamma(S^2M)$, $g + th \in \mathcal{M}$ for t small enough. Therefore an h in $\Gamma(S^2M) = T_g\mathcal{M}O$ shall be called an *(infinitesimal) deformation.* There are three interesting subspaces of $T_g\mathcal{M}$.

i) the *conformal* deformations, that is the space $C^\infty(M).g$.

ii) the *traceless* deformations (they preserve the volume element, see below).

iii) the *trivial* deformations.

Definition. A *trivial* deformation of g is a symmetric two-tensor with can be written as L_Xg where X is a vector field on M.

Namely, taking the local flow ϕ_t of X, we get a curve $t \longrightarrow \phi_t^*g$ of metrics which are isometric to g, and

$$\frac{d}{dt}\phi_t^*g_{|t=0} = L_Xg$$

In other words, the space of trivial deformation is the tangent space at g to the orbit of $Diff(M)$.

Now, recall that, denoting by D the Levi-Civita connection of g, we have

$$L_Xg(u,v) = g(D_uX,v) + g(D_vX,u)$$

Instead of L_Xg, we shall consider $\frac{1}{2}L_Xg$ and denote it δ_g^*X (or δ^*X if there is no ambiguity). This notation will be explained presently.

Since any h in $\Gamma(S^2M)$ can be written as

$$h - (1/n)(tr\ h)g + (1/n)(tr\ h)g$$

conformal and traceless deformations give a splitting of $\Gamma(S^2M)$. But it would be much more interesting to have a splitting involving $Im\delta^*$, so that we get rid of the trivial deformations.

Definition. The *divergence* δ of a symmetric two-tensor h is the one-form $- tr_{12}Dh$ (we trace with respect to the first two indices).

In local coordinates, $\delta h_i = -\sum g^{jk} D_j h_{ki}$. By the way, notice that this definition makes sense for any tensor field, and generalizes the co-differential of exterior forms.

Now, recall that on the space of tensors of a given type on a Riemannian manifold M, we have the *local scalar product* $g(s,t)$ (take the scalar product fiberwise) and the *integral scalar product*

$$<s,t>_g = \int_M g(s,t)\ v_g$$

for compactly supported tensors.

Theorem (Berger-Ebin, cf. [B 3]), 4.5.. *If M is compact, the space $T_g\mathcal{M}$ = $\Gamma(S^2M)$ admits the splitting*

$$\Gamma(S^2M) = Im\ \delta_g^* \oplus Ker\ \delta_g$$

which is orthogonal with respect to the integral scalar product defined by g.

The proof relies on standard elliptic theory for manifolds.

Indeed, δ^* can be seen to have injective symbol, and is the *formal adjoint* of δ. Namely, for any compactly suported h in $\Gamma(S^2M)$ and X in $\Gamma(TM)$,

$$<\delta h\ ,\ X>_g = <h,\ \delta^* X>_g$$

as can be seen from Stokes formula.

(With the Riemannian metric, we have identified vector fields and one-forms) □.

The geometric meaning of this decomposition is the following : take g without infinitesimal isometries. Then there is a neighborhood U of 0 in $\delta^{-1}(0)$ and a neighborhood V of the identity in $Diff(M)$ such that the map

$$(h,\ \phi) \longrightarrow \phi^*(g + h)$$

is a diffeomorphism of UxV onto a neighborhood of g in $\mathcal{M}$ (compare with the preceding paragraph, and see [E] §7.)

By the way, $\Gamma(S^2M)$ is a prehilbert space for the integral scalar product, and $< .,\ . >_g$ depens smoothly on g. Therefore, $\mathcal{M}$ appears to be naturally equipped with a so-called *weak Riemannian metrics*. This simply means that $< , >_g$ is non degenerate ; it is not required that any continuous linear form on $\Gamma(S^2M)$ should be given by the scalar product with an element of $\Gamma(S^2M)$.

B.3. Riemannian functionals ; gradients.

Definition. A *Riemannian functional* on $\mathcal{M}$ is a map F from $\mathcal{M}$ into $\mathbb{R}$ such that $F(\phi^* g) = F(g)$ for any ϕ in $Diff(M)$ and g in $\mathcal{M}$. (In other words, F goes down to the set of Riemannian structures).

Examples. We shall be mainly concerned with the *volume*

$$V(g) = \int_M v_g$$

and the *total scalar curvature*

$$S(g) = \int_M Scal(g) v_g$$

of a compact manifold M.

(Trivial) counter-example. Fix $m \in M$. The functional $g \longrightarrow Scal(g)(m)$ is not Riemannian.

Definition. A functional F is differentiable if it is differentiable for $\mathcal{M}$ equipped with some C^k or H^k norm (if this works with different norms, the differential is clearly the same).

For example, V and S are clearly differentiable (take the C^2 norm). Using well known properties of derivatives of the determinant, we see that

$$V'(g).h = (1/2) \int_M trh\ v_g$$

We shall see later on that

$$S'(g).h = < Ric - (1/2)Scal\ g,\ h >_g.$$

The followng property is both trivial and important.

Proposition. *If a Riemannian functional is differentiable, then for any vector field* X, $F'(g).\delta^* X = 0$.

Proof. Take the flow ϕ_t of X, write that $F(\phi_t^*(g)) = F(g)$ and take the derivative for $t = 0$. □

For the functional V, we get that for any X, the integral of δX vanishes. This property is neither new nor difficult, but it should be noted that we can prove it in that way.

Definition. A functional F is said to have a (smooth) *gradient* if there exists a (smooth) symmetric two-form *gradF* such that

$F'(g).h = < grad\ F,\ h >_g$

for any $h \in \Gamma(S^2M)$.

For example, $gradV = g/2$, and $gradS = Ric - (1/2)Scal\ g$.

Then a straightforward consequence of the preceding property is the following

Corollary. *If a Riemannian functional F has a (smooth) gradient, then* $\delta_g(grad\ F) = 0$.

Proof. For any vector field X, we have

$$F'(g).\delta^*X = 0 = \langle grad\ F, \delta^*X \rangle = \langle \delta(grad\ F), X \rangle$$

(by taking the Stokes formula). This forces $\delta(grad\ F)$ to vanish.□

Remark. This property is true for any manifold, compact or not. Just take compactly supported vector fields. It should also be pointed out that it is completely elementary, in the sense that it does not use Berger-Ebin decomposition.

Example : $\delta(Ric - (1/2)Scal\ g) = 0$, that is

$$\delta(Ric) + (1/2)dScal = 0.$$

Usually, this is proved by tracing the second Bianchi identity. In fact, it was first pointed out by Hilbert that this identity can be proved by using the action of the diffeomorphism group (Hilbert spoke of "invariance under change of coordinates").

B.3. Vector fields on the space of Riemannian metrics

Since $\mathcal{M}_M$ is an open set in $\Gamma(S^2M)$, a vector field on $\mathcal{M}_M$ is just a smooth map of $\mathcal{M}_M$ into $\Gamma(S^2M)$. Since some differential calculus is available on this space, we would like to go a bit further and integrate differential equations.

Examples. i) the vector field $g \longrightarrow fg$ (where f is a given smooth function) admits the flow $\Phi_t(g) = exp(tf)g$.

ii) the vector field $g \longrightarrow \delta_g^*X$ (where X is a given vector field on M) admits the flow $\Phi_t(g) = \phi_t^*(g)$, where ϕ_t is just the flow of X.

Now, these examples are not very interesting, not only because they are trivial. In fact, the data f and X have no geometric Riemannian meaning. It should be clear from the preceeding discussion that we should consider *Diff(M)-equivariant maps* of $\mathcal{M}_M$ into $\Gamma(S^2M)$. The first non trivial example is given by the map $g \longrightarrow Ric(g)$. The corresponding differential equation on $\mathcal{M}_M$ is just

$$\partial g_t/\partial t = Ric(g_t)$$

That is, we have a second order non-linear system of partial differential equation (and this is the simplest situation that we can get, since there are no differential Riemannian invariants of order 1).

It turns out that the vector field $g \longrightarrow -Ric(g)$ has a local flow (R. Hamilton, cf. [B 3], ch. 5 ; in fact $+Ric(g)$ has no flow). We shall not go in this direction, which involves difficult analytical techniques (Nash-Moser theorem). It should also be noted (and this is why this problem was studied) that $Ric(g)$ is roughly the gradient of the total scalar curvature S, and taking the flow of $-Ric(g)$ amounts to a "steepest descent" method. Indeed, Hamilton proved that e.g. if $dim M = 3$ and $Ric(g) > 0$, this flow does converge towards a critical point for S, that is an Einstein metric. In that case the differential calculus on $\mathcal{M}$ does not provide the solution, but provides a guideline to set good questions.

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Most of the topics of our lectures are covered by [B3], especially ch. I, § K, and ch. IV, § A,B,C,G. The basic facts of Yang-Mills theory we shall be concerned with can be found in [L], ch. II and III, and in [B-L 2]. See also the expository article [B-L 1].

The reference [B 2] (especially ch. 9, 10, 11, 13, 16 and the appendix) gives elementary properties of four-dimensional manifolds which are considered as "well-known" by the experts in Yang-Mills theory.

[H] was not quoted in the text and will not be used. It gives a very neat and detailed account of infinite dimensional differential calculus, with numerous geometric examples.

The conference of Yang at the Chern Symposium (in a book published by Springer in the M.S.R.I. series, I am sorry I have not the precise reference by now) gives physical motivations in a way a mathematician can understand.

Suggestions for further readings :a) about analysis on the space of Riemannian metrics : [B 3], ch. 5 and 12 (prescribing the Ricci curvature, moduli of Einstein manifolds).

b) about Yang-Mills theory : [B-L 2] (the title is quite explicit), [L], [F-U] (beautiful accounts of Donaldson theory.)