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Manifolds of negative curvature (I)

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Manifolds of Negative Curvature

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We are interested in the intrinsic geometry of Riemannian manifolds. (All the manifolds are assumed to be smooth, connected, complete Riemannian.) A fundamental aspect of the intrinsic geometry is the (local or global) trigonometry of a space. This will be seen for example in section 1 below, where simply connected spaces of non-positive curvature are discussed. In section 2 we discuss symmetric spaces, in particular the space $O(p,q)/O(p)\times O(q)$. In section 3 we turn to global problems of spaces of non-positive curvature.

Section 1. Hadamard manifolds

Throughout this section, M is connected, complete, and $K_M \leq 0$.

1.1 Let γ be a geodesic in M and Y a Jacobi-field along γ. At places where Y does not vanish we have

$$\|Y\|'' = \left(\frac{\langle Y', Y\rangle}{\|Y\|}\right)' = \frac{(\langle Y'', Y\rangle + \langle Y', Y'\rangle)\|Y\| - \langle Y', Y\rangle^2 \cdot \frac{1}{\|Y\|}}{\|Y\|^2}$$

$$\geq \langle R(Y,\dot\gamma)\dot\gamma, Y\rangle \cdot \frac{1}{\|Y\|} \geq 0,$$ so $\|Y\|$ is convex!

In particular, if $Y(0) = 0$, then $\|Y(t)\| \geq t \cdot \|Y'(0)\|$ for all $t \geq 0$. Hence for $p \in M$

$$\|d\exp_p|_v \cdot w\| \geq \|w\|$$

and therefore $\exp_p$ has maximal rank everywhere and increases lengths of curves.

1.2 (Cheeger-Ebin, Lemma 1.32) If $f: N \to M$ is length preserving, then f is a covering map. That is, for each $p \in M$ there is a neighborhood U of p in M such that $f^{-1}(U) = \bigcup_\alpha U_\alpha$, $U_\alpha \cap U_\beta = \emptyset$ for $\alpha \neq \beta$ and $f|U_\alpha$ is a diffeomorphism onto U.

Remark. True without assuming $K \leq 0$.

1.3 Theorem (Hadamard - Cartan). For any $p \in M$, $\exp_p : M_p \to M$ is the universal covering. Hence $\pi_i(M) = 0$ for $i \geq 2$ and the homotopy type of M is determined by $\pi_1(M)$ (M is a $K(\pi, 1)$).
If M is simply connected, then $\exp_p$ is a diffeomorphism. Hence $\mathrm{inj}(p) = \mathrm{conj}(p) = \infty$ for all $p \in M$ (uniqueness of geodesic connections).

Proof. We proved already that $\exp_p$ has maximal rang everywhere. Hence $\tilde{g} = \exp_p^*(g)$, g the Riemannian metric on M, makes $N = M_p$ into a Riemannian manifold. N is complete since - by the definition of $\exp_p$ - the geodesics through 0_p are the lines $t \mapsto t \cdot v$, $t \in \mathbb{R}$, $v \in N$, and these are defined on all of $\mathbb{R}$. Now apply 1.2.

The simply connected spaces will be called Hadamard manifolds. From now on until the end of the section we will always assume that M is a Hadamard manifold.

1.4 ("Inverse Toponogov", special case) M Hadamard, $p \in M$, $v, w \in M_p \setminus \{0_p\}$, $\sphericalangle(v,w) = \alpha$. For $q = \exp_p v$, $r = \exp_p w$,

$$(*) \qquad \operatorname{dist}(q,r) \geq \|v - w\| \quad \left(= \left(\|v\|^2 + \|w\|^2 - 2\|v\|\|w\| \cos\alpha \right)^{1/2} \right);$$

and equality holds iff $\exp_p$ maps the (Euclidean) triangular surface in M_p spanned by 0_p, v, and w isometrically and totally geodesic into M, that is, "p, q, and r span a flat triangle".

The inequality $(*)$ is also called the <u>cosine-inequality</u>.

An equivalent inequality is the following (M Hadamard):

Let (p,q,r) be a geodesic triangle in M and let $\alpha_1, \alpha_2, \alpha_3$ be its interior angles. Then

$$(**) \qquad \alpha_1 + \alpha_2 + \alpha_3 \leq \pi$$

and equality holds iff p, q, and r span a flat triangle.

1.5 ("Convexity") M Hadamard; γ_1, γ_2 geodesics in M. Then $\operatorname{dist}(\gamma_1(t), \gamma_2(t))$ is convex in t. In particular, if $C \subset M$ is closed and convex, then $\operatorname{dist}(\gamma(t), C)$ is convex in t, for any geodesic γ in M.

1.6 <u>Lemma.</u> M Hadamard, $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ a geodesic quadrilateral - that is, γ_i is a geodesic segment such that the end point of γ_i is the initial point p_{i+1} of γ_{i+1} (i mod 4) - and let α_i be the angle subtended by γ_i and γ_{i-1} at p_i, $\alpha_i = \measuredangle_{p_i}(p_{i-1}, p_{i+1})$. Then

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \leq 2\pi$$

and equality holds iff for one (and hence any) i, $\exp_{p_i}^{-1}(\gamma_1 \cup \dots \cup \gamma_4)$ is a Euclidean quadrangle in a 2-dim. subspace of M_{p_i} and $\exp_{p_i}$ maps the quadrangular surface bounded by $\exp_{p_i}^{-1}(\gamma_1 \cup \dots \cup \gamma_4)$ isometrically and totally geodesic into M, that is, $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ span a flat quadrangle.

<u>Proof.</u> Consider the triangles (p_1, p_2, p_3) with interior angles $\beta_1, \beta_2, \beta_3$ and (p_1, p_4, p_3) with interior angles $\beta_1', \beta_4', \beta_3'$. Then $\alpha_1 \leq \beta_1 + \beta_1'$, $\alpha_2 = \beta_2$, $\alpha_3 \leq \beta_3 + \beta_3'$, $\alpha_4 = \beta_4'$. Hence $(**)$ above implies the inequality. Equality implies that (p_1, p_2, p_3) and (p_1, p_4, p_3) span flat triangles. Furthermore, since then $\alpha_1 = \beta_1 + \beta_1'$, they fit smoothly together.

1.7 We say that normal geodesics γ_1, γ_2 in M (M Hadamard) are <u>parallel</u> if $\operatorname{dist}(\gamma_1(t), \gamma_2(t))$ is uniformly bounded in t for all $t \in \mathbb{R}$. This is equivalent to $\operatorname{dist}(\gamma_1(t), \gamma_2) \equiv a = \text{constant}$.

<u>Proposition.</u> Suppose γ_1 and γ_2 are parallel and γ_2 is not an (orientation preserving) reparametrization of γ_1. Then $\operatorname{dist}(\gamma_1(t), \gamma_2) \equiv a > 0$ and there is a totally geodesic isometric imbedding of $\mathbb{R} \times [0, a]$ into M such that γ_1 corresponds to $\mathbb{R} \times \{0\}$ and γ_2 to $\mathbb{R} \times \{a\}$.

<u>Proof.</u> Let $s < t$, $p_1 = \gamma_1(s)$ and $q_1 = \gamma_1(t)$. Let p_2 be the point on γ_2 closest to p_1 and q_2 the point closest to q_1. Then $\measuredangle_{p_2}(p_1, q_2) = \pi/2$ and $\measuredangle_{q_2}(q_1, p_2) = \pi/2$. But also $\measuredangle_{p_1}(p_2, q_1) = \pi/2$ because otherwise there would be a point p on γ_1 closer to p_2 than p_1. But then

$$d(p, \gamma_2) \le d(p, p_2) < d(p_1, p_2) = a = d(p_1, \gamma_2) = d(p, \gamma_2),$$

a contradiction. Hence (p_1, p_2, q_2, q_1) span a flat quadrilateral.

1.8 We say that normal geodesics γ_1, γ_2 in M (M Hadamard) are <u>asymptotic</u> if $\operatorname{dist}(\gamma_1(t), \gamma_2(t))$ is uniformly bounded in t for all $t \geq 0$.

(i) This defines an equivalence relation on the set of all normal geodesics in M. The equivalence class of a normal geodesic γ is denoted $\gamma(\infty)$, the one of γ^{-1} is $\gamma(-\infty)$. The set of all equivalence classes is denoted $M(\infty)$, and we set $\overline{M} = M \cup M(\infty)$.

(ii) If γ_1 and γ_2 are normal geodesics and γ_2 is obtained from γ_1 by an orientation preserving reparametrization, then γ_1 and γ_2 are asymptotic, that is, $\gamma_1(\infty) = \gamma_2(\infty)$.

(iii) If γ_1 and γ_2 are asymptotic normal geodesics and if γ_1 and γ_2 have a point in common, then γ_2 is obtained from γ_1 by an orientation preserving reparametrization.

<u>Proposition 1.</u> M Hadamard, γ normal geodesic in M, $p \in M$. There is exactly one (normal) geodesic σ in M such that $\sigma(0) = p$ and $\gamma(\infty) = \sigma(\infty)$. If $t_n \to \infty$ and σ_n is the normal

geodesic from p to $\sigma(t_n)$, then $\dot{\sigma}_n(0) \to \dot{\sigma}(0)$, that is, $\sigma_n \to \sigma$.

This is a direct consequence of convexity (1.5).

Now let $p \in M$, $q \in \overline{M} \setminus \{p\}$. Then there is a unique (normal) geodesic γ_{pq} in M such that $\gamma_{pq}(0) = p$ and $\gamma_{pq}(t) = q$ for some $t \in (0, \infty]$ ($t = \mathrm{dist}(p,q)$ if $q \in M$ and $t = \infty$ if $q \in M(\infty)$). Now let v be a unit tangent vector at p. Then

$$C(v, \varepsilon) = \{ q \in \overline{M} \setminus \{p\} \mid \measuredangle(\dot{\gamma}_{pq}(0), v) < \varepsilon \}$$

is called a <u>cone</u>, p is its <u>vertex</u>, the geodesic ray determined by v its <u>axis</u>, and its <u>width</u> is ε.

<u>Proposition 2.</u> The open subsets of M together with the cones $C(v, \varepsilon)$, v a unit tangent vector of M, $\varepsilon > 0$, are a base for a topology of $\overline{M}$, and $\overline{M}$ together with this topology is homeomorphic to the closed unit disc in Euclidean space E^d, $d = \dim(M)$. This topology on $\overline{M}$ is called the <u>cone topology</u>.

The proof of Proposition 2 is an easy consequence of (1.4). If D is the closed unit disc in T_pM for some fixed $p \in M$, then an explicit homeomorphism $h : D \to \bar{M}$ carrying $\mathring{D}$ to M and ∂D to $M(\infty)$ is obtained as follows: Choose a homeomorphism $f : [0,1] \to [0,\infty]$ mapping 1 to ∞. Then set

$$h(w) = \begin{cases} \exp(f(\|w\|)) \cdot \frac{w}{\|w\|} & \text{if } w \neq 0 \\ p & \text{if } w = 0 \end{cases}$$

1.9 Let $x \in M(\infty)$. Then it is reasonable to say that the distance of x to any $p \in M$ is ∞, $\operatorname{dist}(p, x) = \infty$. Still one can also define in a reasonable way $\operatorname{dist}(p,x) - \operatorname{dist}(q,x)$, $p, q \in M$. Namely, let v be a unit tangent vector to M (at some point) and γ_v the geodesic determined by v. Assume $\gamma_v(\infty) = x$. We define the <u>Busemann function</u>

(i) $$b_v(p) = \lim_{t \to \infty} \operatorname{dist}(p, \gamma_v(t)) - t, \quad p \in M.$$

This limit exists: for $t < T$ we have by the triangle inequality

$$-\,dist(p,\gamma_v(0)) \le dist(p,\gamma_v(T)) - T \le dist(p,\gamma_v(t)) - t.$$

Then b_v is <u>convex</u>, $b_v(\gamma_v(t)) = -t$, and

(ii) $$|b_v(p) - b_v(q)| \le dist(p,q)$$

since these properties hold for $d(p,\gamma_v(T)) - T$ as a function of p, for all T. If w is a unit tangent vector such that $\gamma_v(\infty) = \gamma_w(\infty)$, then

(iii) $$b_v - b_w = \text{constant}.$$

Thus $x \in M$ defines Busemann function b_v up to a constant and refering to the above "$dist(p,x) - dist(q,x)$" $= b_v(p) - b_v(q)$.

The level surfaces of Busemann functions are called <u>horospheres</u>.

Busemann functions are C^2, see [HI], and $grad\, b_v(p)$ is the unit vector w at q such that $\gamma_w(\infty) = \gamma_v(\infty)$. A useful property is the following, see [BGS, p.27].

<u>Proposition.</u> Let $p \in B_n \subset B_{n+1} \subset \dots \subset M$, B_n open and $\bigcup B_n = M$. Let $f_n : B_n \to \mathbb{R}$ be convex, C^1, $\| grad\, f_n \| \equiv 1$, and $f_n(p) = 0$. Then f_n converges (in the compact-open topology) to a function $f: M \to \mathbb{R}$ iff $grad\, f_n(p)$ converges and then $f = b_v$, $v = \lim_{n\to\infty} grad\, f_n(p)$.

Section 2. Examples of symmetric spaces with non-positive curvature

2.1 A connected Riemannian manifold M is a <u>symmetric space</u> if for each $p \in M$ there is an isometry s_p of M such that $s_p(p) = p$ and $ds_p|_p = -id$. It is easy to see that a symmetric space is complete and homogeneous. Thus they arise as quotients G/K, where G is a transitive group of isometries and K an isotropy group.

<u>Remark.</u> If X, Y are in $\mathfrak{g}$, the Lie-algebra of G, $\mathfrak{g} = T_eG$, then the 1-parameter subgroups of X and Y act as isometries on M and hence generate Killing fields $\tilde{X}, \tilde{Y}$ on M. Then $[\tilde{X}, \tilde{Y}]$ is the Killing field of the 1-parameter subgroup of $-[X,Y]$.

A <u>Riemannian symmetric pair</u> consists of a Liegroup G, an involutive automorphism $\sigma: G \to G$ such that the group F of fixed points of σ is compact, and a compact subgroup K such that $F_0 \subset K \subset F$ ($_0$ indicates the component of the identity) and such that each component of G contains an element of K.

<u>Example.</u> $G = SL(n,\mathbb{R})$, $\sigma(g) = (g^t)^{-1}$, $F = K = SO(n,\mathbb{R})$.

<u>Theorem.</u> Let $M = G/K$, where (G,K,σ) is a Riemannian symmetric pair.

i) M is connected and σ defines an involutive diffeomorphisms s of M leaving the origin $p_0 = [K] \in M$ fixed such that $ds|_{p_0} = -id$ $\quad (s(gK) := \sigma(g)\cdot K)$.

ii) M admits G-invariant Riemannian metrics and is a symmetric space for each such structure such that s is the geodesic symmetry at p_0. Fix such a structure, $\langle\, ,\rangle$.

iii) The Lie algebra $\mathfrak{K}$ of K is the eigenspace of 1 for $d\sigma|_e$. Let $\mathfrak{P}$ be the eigenspace of -1. Then $\mathfrak{g} = \mathfrak{K} + \mathfrak{P}$ and $[\mathfrak{K},\mathfrak{K}] \subset \mathfrak{K}$, $[\mathfrak{K},\mathfrak{P}] \subset \mathfrak{P}$, and $[\mathfrak{P},\mathfrak{P}] \subset \mathfrak{K}$.

iv) Identify $\mathfrak{P}$ and M_{p_0} by sending X to $\frac{d}{dt}\left(e^{tX}\cdot p_0\right)\Big|_{t=0}$. If γ denotes the geodesic in M determined by X, then $\gamma(t) = e^{tX}\cdot p_0$ and the <u>transvection</u> $s_{\gamma(t/2)} \circ s = e^{tX}$.

v) With respect to the above identification of $\mathcal{P}$ and M_{p_0}, we have

$$R(X,Y)Z = -[Z,[Y,X]] = -[[X,Y],Z]$$

$$\mathrm{Ric}(X,Y) = -\tfrac{1}{2}B(X,Y),$$

where B denotes the Killing form of G.

vi) Given $X \in \mathcal{G}$, e^{tX} defines a 1-parameter subgroup of M and thus we obtain a Killing field $\tilde{X}$ of M. If $X \in \mathcal{P}$, then $\tilde{X}$ is an infinitesimal transvection and $\nabla\tilde{X}(p_0) = 0$. If $X \in \mathcal{K}$, then $\tilde{X}(p_0) = 0$ and $\nabla\tilde{X}(p_0)$ is the endomorphism of M_{p_0} corresponding to ad_X on $\mathcal{P}$.

2.2 We say that M (symmetric space) is of <u>compact type</u> if $K_M \geq 0$ and $\mathrm{Ric}_M > 0$. M is of <u>non-compact type</u> if $K_M \leq 0$ and $\mathrm{Ric}_M < 0$. Every simply-connected symmetric space is the Riemannian product of a Euclidean space, a space of compact type and a space of non-compact type (where either factor may be missing).

Proposition. If M is of non-compact type, then M is simply connected.

Proof. If $X \in \mathcal{G}$, then the corresponding Killing field $\tilde{X}$ is a Jacobi field along any geodesic γ in M - it is the variation field of the geodesic variation $\gamma_s(t) = e^{sX} \gamma(t)$ of γ. If $X \in \mathcal{P}$, then $\tilde{X}$ is an infinitesimal transvection.

If M is not simply connected, then there is a geodesic loop γ at p_0, $\gamma(0) = p_0$ and $\gamma(d) = p_0$, $d > 0$. Let $\tilde{X}$ be the infinitesimal transvection defined by γ, that is, $\tilde{X}$ is the Killing field of $s_{\gamma(t/2)} \circ s_{p_0} = e^{tX}$. Since $\tilde{X}(p_0) = \dot{\gamma}(0)$ and $\nabla \tilde{X}(p_0) = 0$ and $\tilde{X} \circ \gamma$ is a Jacobi field along γ, we have $\tilde{X}(\gamma(t)) = \dot{\gamma}(t)$. In particular, $\dot{\gamma}(d) = \dot{\gamma}(0)$, so γ is a closed geodesic.

" $\tilde{X}(\gamma(d)) = \tilde{X}(\gamma(0))$

Since $Ric_M < 0$, there is a vector E orthogonal to $\dot\gamma(0)$ such that $\langle R(E,\dot\gamma)\dot\gamma, E\rangle = -a^2 < 0, a>0$. Let $E(t)$ be the parallel field along γ such that $E(0) = E$. Let $J(t)$ be the Jacobi field along γ such that $J(0) = E$, $J'(0) = 0$. Then $J(t) = \cosh\sqrt{a}\,t \cdot E(t)$ and an argument as above shows that $J = \tilde{Y}\circ\gamma$, where $\tilde{Y}$ is an infinitesimal transvection. But then

$$J(d) = \tilde{Y}(\gamma(d)) = \tilde{Y}(\gamma(0)) = J(0),$$

a contradiction since $\cosh\sqrt{a}\,d > 1$. This completes the proof of the proposition.

<u>Remark.</u> In the first part of the proof we showed that a geodesic loop on a symmetric space must close up smoothly. This implies that $[c] \to [c^{-1}] = [c]^{-1}$ is a group automorphism on $\pi_1(M)$ and shows that $\pi_1(M)$ is abelian, for M a general symmetric space.

2.3 Now let $p, q \geq 1$, $n = p+q$ and consider on $\mathbb{R}^n = \{(x,y) \mid x \in \mathbb{R}^p, y \in \mathbb{R}^q\}$ the quadratic form

$$Q(x,y) = -x^2 + y^2.$$

Denote by $H_{p,q} = H_{p,q}(\mathbb{R})$ the set of all p-dimensional linear subspaces V of $\mathbb{R}^n$ such that $Q|V$ is negative definite. $H_{p,q}$ is an open subset of the Grassmannian $G_{p,q}(\mathbb{R})$. Let $O(p,q)$ be the group of linear transformations of $\mathbb{R}^n$ preserving Q. If $S = \left(\begin{array}{c|c} -I_p & 0 \\ \hline 0 & I_q \end{array}\right)$, $A \in O(p,q)$ iff $A^* S A = S$. $O(p,q)$ operates transitively on $H_{p,q}$: Given $V \in H_{p,q}$, let W be the orthogonal complement of V with respect to Q. Then $V \cap W = \{0\}$ and $Q|W$ is positive definit. Choose a Q-orthonormal base $e_1, \dots, e_p$ of V and a Q-orthonormal basis $f_1, \dots, f_q$ for W, that is, $Q(e_i, e_j) = -\delta_{ij}$, $Q(f_i, f_j) = \delta_{ij}$. Then $A = (e_1, \dots, e_p, f_1, \dots, f)$ is in $O(p,q)$ and maps $\mathbb{R}^p \times \{0\} = V_0$ to V. The isotropy group of V_0 is $O(p) \times O(q)$, thus $H_{p,q} = O(p,q)/O(p) \times O(q)$

The matrix S defines an involution σ on $O(p,q)$: $A \overset{\sigma}{\mapsto} SAS$ and the group F of fixed elements is just $O(p)\times O(q)$. Thus we are just in the situation of the theorem: $G = O(p,q)$, $K = O(p)\times O(q)$, $\sigma(A) = SAS$.

We have $\mathcal{G} = \left\{ \begin{pmatrix} B & D^* \\ D & C \end{pmatrix} \;\middle|\; B \in so(p),\ C \in so(q),\ D \text{ arbitrary } (q\times p)\text{-matrix} \right\}$, $\mathcal{K} = \left\{ \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \;\middle|\; B \in so(p), C \in so(q) \right\}$ and $\mathcal{P} = \left\{ \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} \;\middle|\; D \text{ a } (q\times p)\text{-matrix} \right\}$.

A G-invariant Riemannian metric on $H_{p,q}$ is induced by the Ad_K-invariant bilinear form

$$(E_1, E_2) = \frac{1}{2}\, tr(E_1 \cdot E_2)$$

on $\mathcal{G}$. This form is positive definit on $\mathcal{P}$ and after the identification $\mathcal{P} \cong T_{V_0} H_{p,q}$ gives rise to a scalar product on $T_{V_0} H_{p,q}$. Since $(\ ,\ )$ is Ad_K-invariant, it extends unambiguously to an invariant Riemannian metric on $H_{p,q}$. For $\begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} \in \mathcal{P}$ we have $(D,D) = tr(D^* D)$.

For D a $(q\times p)$-matrix, set $\widetilde{D} = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$. We want to compute the curvatures of the planes containing $\tilde{D}$; more precisely, we want to compute the symmetric operator $R(\cdot, \tilde{D})\tilde{D}$, where R is the curvature tensor of $H_{p,q}$ at V_0 and $\mathcal{P}$ is identified with $T_{V_0}H_{p,q}$. If $\tilde{C} \in \mathcal{P}$, we have $R(\tilde{C}, \tilde{D})\tilde{D} = -[\tilde{D}, [\tilde{D}, \tilde{C}]]$. Now the Lie bracket in matrix groups is just the commutator of the corresponding matrices,

$$\begin{aligned} R(\tilde{C}, \tilde{D})\tilde{D} &= -\tilde{D}\tilde{D}\tilde{C} + \tilde{D}\tilde{C}\tilde{D} + \tilde{D}\tilde{C}\tilde{D} - \tilde{C}\tilde{D}\tilde{D} \\ &= -(DD^*C - 2DC^*D + CD^*D)^{\sim}. \end{aligned}$$

To compute the eigenvalues and eigenvectors of $R(\cdot, \tilde{D})\tilde{D}$ it is convenient to choose D to be in a particular form. Since K acts by isometries, it leaves R invariant. Now the action of K on $T_{V_0}H_{p,q}$ corresponds to the action on $\mathcal{P}$ via Ad. But then it is an easy (exercise) that we can bring D into diagonal form (assume $p \le q$ for ease of notation).

$$D = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_p \\ \hline 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \begin{array}{l} \\ p \\ \\ \\ q-p \ (\ge 0) \\ \\ \end{array}$$

Now let E_{ij} be the matrix with entries 0 except for a 1 on the (i,j)-entry. A straightforward computation yields for $R(\,.\,,\tilde{D})\tilde{D}$:

for $1 \leq j \leq p < i$, $\tilde{E}_{ij}$ is eigenvector, eigenvalue $-a_j^2$.

for $1 \leq i < j \leq p$, $F_{ij} = \tilde{E}_{ij} + \tilde{E}_{ji}$ is eigenvector with eigenvalue $-(a_i - a_j)^2$ and $G_{ij} = \tilde{E}_{ij} + \tilde{E}_{ji}$ with eigenvalue $-(a_i + a_j)^2$. Moreover $\tilde{E}_{ii}$ is in the kernel, $1 \leq i \leq p$. (These vectors span $\mathcal{P}$.)

<u>Remark.</u> That the curvature is ≤ 0 can be seen more easily: $\langle R(\tilde{C},\tilde{D})\tilde{D},\tilde{C}\rangle = -([\tilde{D},[\tilde{D},\tilde{C}]],\tilde{C})$

$$= -\frac{1}{2}\operatorname{tr}([\tilde{D},[\tilde{D},\tilde{C}]]\cdot\tilde{C}) = \frac{1}{2}\operatorname{tr}([\tilde{D},\tilde{C}]\cdot[\tilde{D},\tilde{C}]) \leq 0$$

since $[\tilde{D},\tilde{C}] \in \mathcal{K} \subset so(p+q)$.

The geodesic determined by D as above is $e^{t\tilde{D}}\cdot V$, and

$$e^{t\tilde{D}} = \begin{array}{c@{}l} \left(\begin{array}{ccc|ccc|ccc} \mathrm{ch}\,ta_1 & & 0 & \mathrm{sh}\,ta_1 & & 0 & 0 & \cdots & 0 \\ & \ddots & & & \ddots & & & & \\ 0 & & \mathrm{ch}\,ta_p & 0 & & \mathrm{sh}\,ta_p & & & \\ \hline \mathrm{sh}\,ta_1 & & 0 & \mathrm{ch}\,ta_1 & & 0 & & & \\ & \ddots & & & \ddots & & & & \\ 0 & & \mathrm{sh}\,ta_p & 0 & & \mathrm{ch}\,ta_p & 0 & & 0 \\ \hline 0 & \cdots & & & \cdots & 0 & 1 & & 0 \\ \vdots & & & & & \vdots & & \ddots & \\ 0 & \cdots & & & \cdots & 0 & 0 & & 1 \end{array}\right) & \begin{array}{l} \\ p \\ \\ \\ p \\ \\ \\ q-p \\ \\ \end{array} \\ \begin{array}{ccc} \longleftarrow p \longrightarrow & \longleftarrow p \longrightarrow & \longleftarrow q-p \longrightarrow \end{array} & \end{array}$$

The space $\mathcal{A}$ of all matrices D in the above form is of dimension p and it exponentiates into a flat totally geodesic subspace of $H_{p,q}$. This follows from $e^{t\tilde{D}_1} e^{t\tilde{D}_2} = e^{t\tilde{D}_2} e^{t\tilde{D}_1}$ $\quad ([\tilde{D}_1, \tilde{D}_2] = 0)$.
Thus for $p \le q$, $H_{p,q}$ contains a flat totally geodesic subspace of dimension p.

<u>Remark.</u> The above model $H_{p,q} \subset G_{p,q}$ corresponds to the <u>projective model</u> of hyperbolic space $(p = 1)$. The <u>'Kleinian' model</u> for $G_{p,q}$ is obtained as follows: For $V \in H_{p,q}$ choose a Q-orthonormal base $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \dots, \begin{pmatrix} x_p \\ y_p \end{pmatrix}$ and arrange these vectors as columns in a matrix $\begin{pmatrix} X \\ Y \end{pmatrix}$. Then by definition $-X^*X + Y^*Y = -I_p$, so $X^*X = I_p + Y^*Y$ is positive definite. Hence X is non-singular. Set $Z = Y \cdot X^{-1}$. Then Z is a $(q \times p)$-matrix such that $I_q - Z^*Z > 0$. A different choice of basis for V leads to the same

2. It is easy to see that we obtain a bijection $H_{p,q} \to D_{q,p} = D_{q,p}(\mathbb{R})$, where

$$D_{q,p}(\mathbb{R}) = \{ Z \in M(q \times p; \mathbb{R}) \mid I_q - Z^* Z > 0 \}$$

is a (real) Siegel domain, see [KN], pp. 287-289.