INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
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SMR.404/24

COLLEGE ON DIFFERENTIAL GEOMETRY

(30 October - 1 December 1989)

Convergence of metrics

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These are preliminary lecture notes, intended only for distribution to participants

I From pinching to finiteness and compactness theorems

Cheeger - Weinstein - Gromov - Greene - Wu - Peters - Gao - Anderson

II Proof of a compactness theorem due to M. Anderson

$C^{1+\alpha}$ bounded geometry

$C^{1+\alpha}$ bounded geometry implies compactness

$|\mathrm{Ricci}| \leq 1$ and $\mathrm{inj} \geq \varepsilon$ implies $C^{1+\alpha}$ bounded geometry

Proof of the Cheeger - Gromoll splitting theorem.

Convergence to orbifolds.

III Lower bounds on injectivity radius and applications

Klingenberg and $(\frac{1}{4}-\varepsilon)$- pinching

Cheeger's butterfly Lemma

Heintze - Margulis and $(-1+\varepsilon)$- pinching

Gromov's simplicial volume

IV Collapsing Riemannian manifolds

Gromov - Hausdorff distance

Examples of collapse

Cheeger - Gromov's characterization of collapsable manifolds (dimension 3)

Gromov's Almost Flat Manifolds theorem

Fukaya's description of certain collapses

Pinching theorems, like the sphere theorem of Berger-Klingenberg, can be put under the general heading:

To what extent is a Riemannian metric determined by its curvature?

The question has a local aspect

- existence of local coordinates in which the metric is under control

and also a global aspect

- is the topology of a Riemannian manifold determined by its curvature (and other quantities)

Convergence theorems in Riemannian geometry are concerned with both aspects of the question, and give the following type of answer:

- a certain curvature bound, combined with some global numerical invariants, determines the manifold and its metric up to a bounded error.

I. From pinching to finiteness and compactness theorems

For the proof of the sphere theorem, one shows that a manifold with $K > \frac{1}{4}$ is covered by 2 balls of radius π. To conclude, one needs that these balls be contractible.

$\boxed{\text{Klingenberg 1959}}$: If (M, g) is simply connected, with sectional curvature $\frac{1}{4} \leq K \leq 1$, then the injectivity radius is everywhere $\geq \pi$.

This suggests the following very crude argument:
Assume that all balls in (M, g) of radius $\leq \varepsilon$ are convex. Choose a covering of M by ε-balls such that the concentric $\varepsilon/2$-balls are disjoint.
Consider the nerve of this covering. It is a simplicial complex N with one simplex of dimension k for each collection of $k+1$ balls B_{i_j} in the covering such that $\bigcap_{j=1}^{k+1} B_{i_j} \neq \emptyset$.

<u>Claim</u>: N is homotopy equivalent to M.
The combinatorics of N is bounded by the number of balls in the covering. There is a lower bound depending only on dimension and radius for the volume of balls within the injectivity radius (Croke 1980) Ann. Sci. Ec. Norm. Sup. Paris

So the number of disjoint $\frac{\varepsilon}{2}$-balls is bounded by volume

We obtain

Easy lemma: Given $n, \varepsilon>0$, $V>0$, the number of homotopy types of compact manifolds of dimension n, volume $< V$, convexity radius $\geq \varepsilon$ is finite.

To go from homotopy to topological finiteness is not an easy matter.

Finiteness theorem (Weinstein 1967, Cheeger 1970, Peters 1984)
Given n, $\varepsilon>0$, $D<+\infty$ (resp $V<+\infty$). the number of diffeomorphism types of compact manifolds of dimension n,

sectional curvature $|K| \leq 1$

injectivity radius $\geq \varepsilon$

diameter $\leq D$ (resp volume $\leq V$)

is finite.

In the 1970's, Cheeger, and independantly Gromov, realized that the theorem also provided a control on metrics (this control is needed but not stated in a 1975 paper by Gromov). None of them tried to popularize this idea. The following terminology was introduced by Gromov:
the dilatation of a map $f: X \to Y$ between metric spaces is
$$\sup_{x \neq y} \frac{d(f(x), f(y))}{d(x,y)} =: \operatorname{dil} f$$

Definition. The Lipschitz distance between 2 metric spaces X and Y is the infimum of

$$|\log \operatorname{dil} f| + |\log \operatorname{dil} f^{-1}|$$

over all homeomorphisms $f : X \to Y$.

Compactness theorem (Cheeger? Gromov 1978)
given n, $\varepsilon > 0$, $D < +\infty$, the set of compact Riemannian manifolds with

$$|K| \leq 1$$
injectivity radius $> \varepsilon$
diameter $\leq D$

is "compact" in the Lipschitz topology. The limits are Lipschitz riemannian metrics on smooth compact manifolds.

After the work of DeTurck and Kazdan on local control of the metric by its Ricci curvature (mentionned later) several people realized that the limit metrics could be shown to be more regular.

$C^{1+\alpha}$ compactness theorem (Greene-Wu, Peters 1986).
Under the same bounds as above, we have a $C^{1+\alpha}$ compact set of metrics. I.e., given any sequence in the set, there exists a subsequence (M_j, g_j)

diffeomorphisms $\varphi_j : M_{j_0} \to M_j$

such that the metrics $\varphi_j^* g_j$ are bounded in $C^{1+\alpha}$, any $\alpha < 1$.

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Since 1987, a number of variants of the compactness theorem have appeared (Gao, Yang). Recently, elaborating on Gao's work M. Anderson has found a proof, more direct that previous ones, of a theorem that contains Gromov's. We shall give this proof in detail now.

II Proof of a compactness theorem due to M. Anderson.

Given n, $\varepsilon>0$, $D<+\infty$,

__thm__ the space of compact Riemannian n-manifolds such that

$|\mathrm{Ricci}| \leq 1$

injectivity radius $\geq \varepsilon > 0$

diameter $\leq D$

is $C^{1+\alpha}$ compact, any $\alpha<1$.

The common features of compactness proofs are

1. a local estimate = curvature and injectivity radius bounds directly imply existence of coordinates in which one has good control of the metric.

2. a construction to combine local estimates on a covering by balls, whose combinatorics is bounded by the diameter.

Traditionnally, one says that a Riemannian manifold (or a family of Riemannian manifolds) has bounded geometry if sectional curvature and injectivity radius are bounded.

Indeed, the Rauch - Alexandrov - Toponogov thm implies that exponential coordinates give a (uniformly) Lipschitz map to a euclidean ball, an estimate which is sufficient for many purposes.

But compactness theorems require sharper local estimates, and each proof generates a refined notion of "bounded geometry". One of Anderson's trick is to use a scale homogeneous measurement of bounded geometry.

In Anderson's proof, step 2 is easy and then used in the proof of step 1. Step 1 is indirect and relies on

$\boxed{\text{Cheeger-Gromoll Splitting Theorem}}$ 1971 : If (M,g) is complete, with non negative Ricci curvature, and if (M,g) contains a line (a geodesic that minimizes length between any two of its points), then (M,g) splits along this line, as a product $(M,g) \simeq (N,h) \times (\mathbb{R}, dt^2)$ with N complete, h non negative Ricci curved.

Definition: Fix, once and for all, constants $C > 1$ and α, $0 < \alpha < 1$. For a point x in a Riemannian manifold (M, g) (not necessarily complete), let the $C^{1+\alpha}$-radius of (M, g) at x, denoted by $\rho(x)$, be the largest radius R with the following property:

On the ball $B(x, R)$, there exists a harmonic coordinate system $u : B(x, R) \to \mathbb{R}^n$ onto $B^{\mathbb{R}^n}(0, C^{-1}R)$ such that, if $\tilde{g} = (u^{-1})^* g$, then

$$\tilde{g}(0) = g_0 = \text{euclidean metric}$$

$$C^{-1} g_0 \leq \tilde{g} \leq C g_0 \quad \text{on} \quad B^{\mathbb{R}^n}(0, C^{-1}R)$$

$$R^{1+\alpha} \, \| \tilde{g} \|_{C^{1+\alpha}} \leq C .$$

Remark that if the metric g is multiplied by a constant d^2, then $\rho(x)$ is multiplied by d.

The fact that $\rho(x) > 0$ expresses the existence of local harmonic coordinates, see the proof given in the Einstein manifolds course, or next argument.

Proposition 1: the $C^{1+\alpha}$ radius is lower semi continuous in the $C^{1+\alpha}$ topology on Riemannian manifolds with marked points, ie, if (M_j, g_j, x_j) converge $C^{1+\alpha}$ to (M, g, x) then $\rho(x) \leq \liminf \rho(x_j)$.

Proof: implicit function theorem

By definition, we can see g_j as metrics on a large ball $B(x, R)$ in M, that converge in $C^{1+\alpha}$. Choose $R > \rho(x) > \rho$

Let $u : B(x, \rho(x)) \to \mathbb{R}^n$ be a harmonic embedding. Let

$\mathcal{M} = \{$ metrics on $B(x,\rho)$ $\}$ in $C^{1+\alpha}$ topology

$\mathcal{E} = \{$ embeddings $v : B(x,\rho) \to \mathbb{R}^n$ such that $v = u$ on $\partial B(x,\rho)$ $\}$ in $C^{2+\alpha}$ topology

$\mathcal{G} = \{$ maps $v : B(x,\rho) \to \mathbb{R}^n \}$ in C^α topology.

$\psi : \mathcal{M} \times \mathcal{E} \to \mathcal{G}$, $(h, v) \longrightarrow \Delta_h v$ (laplace operator of h)

then the partial derivative $\frac{\partial \psi}{\partial v}$ at g is Δ_g on $B(x,\rho)$ with Dirichlet boundary conditions; by the maximum principle, it has no kernel. It is self adjoint, thus an isomorphism $T\mathcal{E} \to T\mathcal{G}$. By the implicit function theorem, the solution in $\mathcal{E}$ of $\Delta_h v = 0$ varies smoothly on h.

We obtain, for j large enough, a harmonic embedding $v_j \in C^{2+\alpha}$, $(v_j^{-1})^* g_j$ converges in $C^{1+\alpha}$ to $(u^{-1})^* g$. $\square$

Corollary: $\rho > 0$, ie, local existence of harmonic coordinates for a $C^{1+\alpha}$ metric.

Proof: in some coordinate chart, let $\varphi_t(x) = tx$.

then, if g is $C^{1+\alpha}$, then, as $t \to 0$,

$$\frac{1}{t^2}\varphi_t^* g \longrightarrow \delta, \text{ euclidean metric, in } C^{1+\alpha}$$

so $\liminf_{t \to 0} \frac{1}{t} \rho(\varphi_t^* g, 0) \geq \rho(\delta, 0) = +\infty$

so $\rho(g, 0) = \rho(\varphi_t^* g, 0) > 0$. $\square$

Proposition 2: Given $n, \rho_0 > 0$ the set of complete Riemannian manifolds of dimension n, with marked points, and $C^{1+\alpha}$ radius everywhere $\geq \rho_0$ is $C^{1+\alpha'}$ compact for any $\alpha' < \alpha$.

Remark. Since we deal with complete manifolds, convergence of sequences $\varphi_i^* g_j$ means uniformly in $C^{1+\alpha'}$ on compact sets, and diffeos are pointed.

Proof: Abstract nonsense. If (M_j, g_j) is a sequence of manifolds with $\rho(x) \geq \rho_0$, we can as well assume $\rho_0 = 1$.
We cover M_j by balls B_j^k of radius C^{-1}, such that half concentric balls are disjoint. Let $u_j^k : B_j^k \to \mathbb{R}^n$ be good harmonic coordinates, $\phi_j^{kl} = u_j^l \circ u_j^{k\,-1}$ be defined on $U_j^{kl} = u_j^k(B_j^k \cap B_j^l) \subset \mathbb{R}^n$. We can arrange so that

- the $C^{1+\alpha}$ bounded metrics $\tilde{g}_j^k = (u_j^{k\,-1})^* g_j$ converge in $C^{1+\alpha'}$ to $\tilde{g}^k$
- the approximate balls U_j^{kl} converge to $\tilde{g}^k$-balls U^{kl}
- the diffeomorphisms ϕ_j^{kl}, $C^{2+\alpha}$-bounded, converge in $C^{2+\alpha'}$ to ϕ^{kl} such that $\tilde{g}^k = \phi^{kl*}\tilde{g}^l$ on U^{kl}

and define $M = B^{\mathbb{R}^n}(0, C^{-1}) \times \mathbb{N} / \sim$

where $(z, k) \sim (z', l)$ iff $z \in U^{kl}$, $z' \in U^{lk}$ and $\phi^{kl}(z) = z'$. $\square$

Remark: It is a general fact that [illegible] bounds on metrics $\tilde{g}$ and $\tilde{h}$ on $B^{\mathbb{R}^n}(0,1)$, together with a bound on $\tilde{h}^{-1}$, provide a $C^{k+\alpha}$ bound on any isometry ϕ of $\tilde{h}$ to $\tilde{g}$

Proof: Let $\Gamma^{\tilde{h}}$, $\Gamma^{\tilde{g}}$ be the Christoffel symbols of $\tilde{h}$ and $\tilde{g}$ then the invariance of the Levi-Civita connection,

$$\phi^* D^{\tilde{g}} = D^{\tilde{h}}$$

translates into

$$\phi'' = \phi' \Gamma^{\tilde{g}} - \Gamma^{\tilde{h}}(\phi', \phi').$$

Given that $\|\phi\|_{C^0} \leq 1$, a bootstrap argument, using L^p_k spaces, gives the desired bound. □

Theorem: On the set of complete Riemannian manifolds of dimension n, injectivity radius $\geq \varepsilon$ and Ricci curvature $|Ric| \leq 1$, there is a uniform lowerbound on the $C^{1+\alpha}$ radius.

Proof is by contradiction.

If not, there is a sequence (M_j, g_j) and points x_j such that $\rho(x_j) \to 0$.

Assume that, on M_j, the $C^{1+\alpha}$ radius ρ attains its minimum at x_j

Now inflate M_j: let $h_j = \rho(x_j)^{-2} g_j$ so that the minimal ball $B(x_j, \rho(x_j))$ has now radius 1, and $\rho \geq 1$ everywhere on (M_j, h_j). Apply proposition =

a subsequence converges, i.e., there exist M, with a marked point x, diffeomorphisms φ_j $M \to M_j$ sending x to x_j, such that the metrics $\varphi_j^* h_j$ converge in $C^{1+\alpha'}$, $\alpha' < \alpha$, uniformly on compact sets, to a $C^{1+\alpha'}$ metric h on M.

The rest of the proof goes as follows

1. Show the convergence is in fact in $C^{1+\alpha}$, so that (Prop 1)

$$\rho(h, x) = 1$$

2. Show that (M, h) is C^∞ and Ricci flat.

3. Show that the injectivity radius inject (M, h) is infinite

then one can apply Cheeger-Gromoll's splitting theorem: all geodesics in (M, h) are lines, so (M, h) is isometric to Euclidean space, and so

$$\rho(h, x) = +\infty,$$

a contradiction.

Proof of 2. As we see in the proof of Proposition 2,
the metric h_j read in harmonic coordinates, ie

$$\tilde{h}_j = (v_j^{-1})^* h_j$$

converges in $C^{1+\alpha}$ to h read in harmonic coordinates

$$\tilde{h} = (u^{-1})^* h$$

Write the Ricci curvature equation in harmonic coordinates

$$Ric(\tilde{h}_j) = -\frac{1}{2} \tilde{h}_j^{\alpha\beta} \frac{\partial^2 \tilde{h}_j}{\partial x^\alpha \partial x^\beta} + Q\left(\frac{\partial \tilde{h}_j}{\partial x}\right)$$

where Q is quadratic.

We have $|Ric(\tilde{h}_j)|_{L^\infty} \leq \rho(x_j, g_j)^2 \xrightarrow[j \to +\infty]{} 0$

Fix p large.
So $Ric(\tilde{h}_j)$ tends to zero in L^p. From elliptic ← page 19 in Kazdan's notes
regularity and $\frac{\partial \tilde{h}_j}{\partial x}$ bounded, $\tilde{h}_j^{-1}$ bounded, we
conclude $\tilde{h}_j$ is bounded in L^p_2, for any $p < +\infty$.

From the Sobolev embedding, for p large enough, $\alpha < \beta < 1$

$$C^{1+\beta} \supset L^p_2$$

thus $\tilde{h}_j$ is bounded in $C^{1+\beta}$, and we can assume it
converges in $C^{1+\alpha}$. $\square$

zero in L^2, $\frac{\partial \tilde{h}_j}{\partial x}$ converges to $\frac{\partial \tilde{h}}{\partial x}$ in C^α, $\tilde{h}_j^{-1}$ converges to $\tilde{h}^{-1}$ in $C^{1+\alpha}$

so, by the equation and elliptic regularity,

$\tilde{h}_j$ converges to $\tilde{h}$ in L^2_2. In particular, $\tilde{h}$ is a weak solution in L^2_2 of

$$-\frac{1}{2}\tilde{h}^{\alpha\beta}\frac{\partial^2 \tilde{h}}{\partial x^\alpha \partial x^\beta} + Q\left(\frac{\partial \tilde{h}}{\partial x}\right) = 0$$

We show it is real analytic:

Since $\frac{\partial \tilde{h}}{\partial x} \in C^\alpha$, $Q\left(\frac{\partial \tilde{h}}{\partial x}\right) \in C^\alpha$, $\tilde{h}^{-1} \in C^{1+\alpha}$, we have $\tilde{h} \in C^{2+\alpha}$ by elliptic regularity (page 19 in Kazdan's notes). By induction, we get $\tilde{h} \in C^\infty$, then apply real analytic regularity.

Of course, equation means $\tilde{h}$ is Ricci flat.

Proof of 3. the injectivity radius is upper semi continuous in the C^0 topology on metrics (P. Ehrlich). In fact in the Gromov Hausdorff topology. I mean: if metrics h_j converge to h, then

$$-\lim_{j\to+\infty}\sup \ \mathrm{inject}(x, h_j) \le \mathrm{inject}(x, h).$$

Indeed, if $R = \mathrm{inject}(x,h)$, there exists an y with $d(x,y)$ arbitrarily close to R, which is attained at least twice by $\exp_x$ on $B(0, R+\varepsilon)$ and which is a regular value of $\exp_x$. Thus there are finitely many geodesics γ_k from x to y of length $\le R+\varepsilon$.

Let T_k be a very thin geodesic tube around γ_k. If thin enough, the geodesic γ_k will be the only curve of minimum length joining x to y within T_k.

For each j, there exists a length minimizing curve γ_k^j from x to y in (T_k, h_j). As j tends to infinity, a subsequence converges to a length minimizing curve from x to y in (T_k, h), but this is γ_k. This shows that γ_k^j converges uniformly to γ_k, so γ_k^j is contained in the interior of T_k, that is, γ_k^j is an h_j-geodesic.

For j large enough, $\mathrm{length}_{h_j}(\gamma_k^j) \leq R+\varepsilon$

We conclude that $\exp_x^{h_j}$ is not injective on $B(0, R+\varepsilon)$, that is $\mathrm{inject}(h_j, x) \leq R+\varepsilon$. $\square$

Proof of the Cheeger-Gromoll splitting theorem

It relies on the

Comparison Lemma: Let (M^n, g) have $Ricci(g) \geq 0$

Let f denote the distance function to some point. Then

$$\Delta f \geq \frac{n-1}{f} .$$

Remark: In general f is not smooth, but such an inequality still makes sense

Let γ be a line in M, i.e., $d(\gamma(t), \gamma(s)) = |s-t|$ for all s, t. Construct the Busemann functions of γ (see the course by W. Ballmann)

$$b_{\pm}(x) = \lim_{t \to \pm\infty} t - d(x, \gamma(t))$$

It is an increasing limit so we obtain

$$\Delta b_{\pm} \leq 0$$

in particular $\Delta b_+ + b_- \leq 0$.

From the triangle inequality, we have

$$b_+ + b_- \leq 0$$

everywhere.

Since γ is a line,

$$b_+ + b_- \equiv 0 \quad \text{along } \gamma$$

The maximum principle says that $b_+ + b_-$ vanishes identically.

Since $\Delta b_+ = -\Delta b_- \geq 0$, we have $\Delta b_+ \equiv 0$. We are confronted with the equality case in the comparison lemma above. this equality case says that the full Hessian DDb_+ vanishes. this means that $b_+ : M \to \mathbb{R}$ is a Riemannian submersion with totally geodesic fibres, i.e., M is a riemannian product of some fibre N with $\mathbb{R}$. $\square$

Many details in above proof need be completed & fixed. See Cheeger-Ebin's book or

Arthur Besse's book Einstein Manifolds, § 6. G.

The inflation argument used in Anderson's proof is the prototype of a style of application of compactness theorems. The power of the method depends on the existence of sharp theorems about Ricci flat complete manifolds.

Here is an example

Theorem (M Anderson) Consider 4-dimensional manifolds satisfying the following bounds

$|\text{Ricci}| \leq 1$

Volume $\geq \varepsilon$

diameter $\leq D$

Second Betti number $b_2 \leq B$.

Any Hausdorff limit of 4 manifolds in this class is a $C^{1+\alpha}$ Riemannian 4-manifold with (eventually) orbifold singularities. This means: at finitely many points p, the manifold is not smooth, but diffeomorphic to a quotient $\mathbb{R}^4/\Gamma$. $\Gamma \subset O(4)$ a finite group with the origin as only fixed point. Also, the metric, pulled back to $\mathbb{R}^4$, is of class $C^{1+\alpha}$.

Rough outline of the proof. Analyze a sequence of such metrics (M_j, g_j). If injectivity radius is bounded from below, apply previous convergence theorem. If not, inflate so that injectivity radius is 1. then there is some subsequence converging to a complete manifold (N, h) such that

① (N, h) is Ricci flat

② the volume of balls growths like

$$\text{volume } B(r) \geq \varepsilon r^4$$

③ the curvature tensor is in L^2,

$$\int_N |R|^2 < +\infty$$

Indeed, ② is a consequence of the uniform lower volume bound volume $\geq \varepsilon$ together with Bishop's inequality that gives a lower bound for the volume of small balls in a manifold in terms of Ricci curvature and total volume.

③ is a consequence of the Gauß-Bonnet formula (See the course on Einstein manifolds) for the Euler characteristic

$$2 + b_2 = \chi = \frac{1}{8\pi^2} \int |U|^2 - |Z|^2 + |W|^2 = \frac{1}{8\pi^2} \int |U|^2 + |Z|^2 + |W|^2$$

since $Z = 0$ on N. there some semi continuity of $g \to \int_M |R_g|^2$ is needed.

The sharp theorem on complete Ricci flat manifolds we need is

Theorem: A complete 4-manifold satisfying ① ② and ③ is Asymptotically Locally Euclidean (ALE), i.e., Outside a compact set, N is diffeomorphic to $\mathbb{R}^2 - B/\Gamma$. $\Gamma \subset O(4)$ a finite group, and the metric h converges to Euclidean metric pretty fast.

This theorem is due to many people (Anderson, Bando-Kobayashi, Nakajima). Its proof again involves a scaling argument. Indeed, it amounts to show that the "deflated" manifolds

$$(N, \varepsilon^2 h)$$

converge to $(\mathbb{R}^4/\Gamma, \delta)$ as $\varepsilon \to 0$.

In spherical shells $B^N(p, 2r) - B^N(p, r)$, one needs a lower bound on injectivity radius. Bishop's inequality already gives a lower bound on volumes of balls in the shell. According to Cheeger's Butterfly Lemma, a sectional curvature bound will then imply the injectivity radius bound. To get it, one uses the fact that the curvature tensor of a Ricci flat manifold is harmonic, ie, satisfies an elliptic system of equations of second order. Also, the manifolds $(N, \varepsilon^2 h)$ satisfy a uniform isoperimetric inequality (see lectures by G. Besson). This is a result by P. Li and S. Gallot.

Elliptic equation, in fact, inequation of the form

$$\Delta |R| \leq \text{const}\, |R|^3$$

together with isoperimetric inequality allowsan estimate

$$|R|_{L^\infty} \leq c(|R|_{L^2})$$

Thus, in spherical shells, injectivity radius is bounded below and curvature tends uniformly to zero. the limit is a flat incomplete manifold. Easy to show it is $\mathbb{R}^4/\Gamma$. $\square$

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III Lower bounds on the injectivity radius and applications

Compactness theorems require an injectivity radius bound. This is a very strong assumption, unlikely to be satisfied in any application. Thus the main thrust of the corollaries below stems from the injectivity radius bounds, each of which requiring a deep insight in Riemannian geometry.

1. Previously quoted theorem of Klingenberg
2. Easier variant in <u>even</u> dimensions

<u>Theorem</u> (Synge-Klingenberg): If M^{2n} is orientable, with sectional curvature $0 < K \leq 1$, then the injectivity radius is $\geq \pi$. (see the book by Cheeger-Ebin).

<u>Proof</u>: Show that the space of closed curves with length $< 2\pi$ is disconnected, then apply Synge's proof. □

<u>Corollary</u> (Berger 1983). There exists an $\varepsilon(n) > 0$ such that, if M^{2n} is orientable and admits a metric with $\frac{1}{4} - \varepsilon \leq K \leq 1$, then either M is homeomorphic to S^{2n}, or M is diffeomorphic to a projective space $\mathbb{C}P^n$, $\mathbb{H}P^{n/2}$, $\mathbb{C}aP^2$.

[illegible]

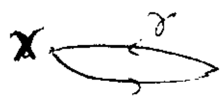
Assume sectional curvature bounds $|K| \leq 1$. Then the volume of a unit ball is controlled by the injectivity radius at the center:

$$\mathrm{vol}\, B(x,1) \leq \mathrm{const}(n)\ \mathrm{inject}(x)$$

<u>Proof</u>: Equip the unit ball $B(0,1)$ in the tangent space T_xM with the pull back metric $\tilde{g} = \exp_x^* g$, so that $\exp_x$ is an isometry.
$K \leq 1$ implies that $\exp_x$ behaves like a covering map: curves in M of length $< \pi$ have a lift: if $\gamma \subset M$ has length $\leq \frac{1}{2}$ and $z \in T_xM$, $|z| < \frac{1}{2}$, $\exp_x z = \gamma(0)$, then there is a unique curve $\tilde{\gamma} \subset T_xM$, with $\tilde{\gamma}(0) = z$ and $\exp_x \circ \tilde{\gamma} = \gamma$.

One can thus define "deck transformations" τ^k, $|k| \leq \frac{1}{2\,\mathrm{inj}(x)}$ on $B(0,\frac{1}{2}) \subset T_xM$: ~~test~~

Assume $\mathrm{inject}(x) < \frac{1}{2}$: there exists a loop γ_0 at x formed with 2 geodesic arcs of equal length



If $|z| < \frac{1}{2}$, define a curve σ_k in M as follows: follow the loop γ_0 k times, then follow $t \to \exp_x(tz)$. Define $\tau^k z$ as the end point of the unique lift of σ_k starting at 0.

Since $\exp_x \circ \tau^k = \tau^k$, τ^k is an isometry. One has $\tau^k \circ \tau^\ell = \tau^{k+\ell}$

One can define "fundamental domains" for the group τ^k:

$$\Delta^k = \{ z : d(z, \tau^k(0)) \leq d(z, \tau^\ell(0)) \text{ for all } \ell, \text{ and } d(z,\tau^k(0)) \leq \tfrac{1}{2} \}$$

then
- $\exp_x : \Delta^k \to B^M(x, \frac{1}{2})$ is surjective for all k
- volume $\Delta^k \cap \Delta^\ell = 0$ for $k \neq \ell$
- volume $\Delta^k =$ volume $B^M(x,\frac{1}{2})$ for all k

thus
$$\text{volume } B^M(x,\tfrac{1}{2}) \leq \frac{\text{volume } \overset{T_xM}{B}(0,\frac{1}{2})}{\text{number of } k} \leq \mathrm{Const}(n)\ \mathrm{inject}(x) \quad \text{if } K \geq -1$$

it is small everywhere there exists a function $f(n, R, i)$

$$\text{inject}(y) \leq f(n, \text{dist}(x,y), \text{inject}(x))$$

with $f(n, R, i) \to 0$ as $i \to 0$

Proof: clear if $d(x,y) = \frac{1}{2}$ since volume $B^n(y, \text{inject}(y)) \geq$ Const(n) inject$(y)^n$. then propagate, using a chain of points z_i, $z_0 = x$, $z_k = y$, $d(z_i, z_{i+1}) = \frac{1}{2}$. $\square$

Corollary: Assume $|K| \leq 1$. For each fixed R, a lower bound on volume $B(x, R)$ implies a lower bound on inject(y) $\forall y \in B(x, R)$.

In particular, in the convergence theorem, the assumptions

$$|K| \leq 1, \quad \text{inject} \geq \varepsilon, \quad \text{diameter} \leq D$$

can be replaced by

$$|K| \leq 1, \quad \text{volume} \geq \varepsilon, \quad \text{diameter} \leq D.$$

Cheeger's Butterfly proof is simpler (See Cheeger-Ebin's Comparison theorem in Riemannian geometry North-Holland).

The above proof is taken from Gromov's Structures Métriques pour les variétés riemanniennes and has the advantage of being local: holds for a ball in a complete manifold.

4. Heintze–Margulis lemma: $\exists\, \varepsilon(n)$ such that,
if (M^n, g) is compact with sectional curvature $-1 \leq K < 0$, then there is a point where the injectivity radius is $\geq \varepsilon(n)$.

Proof: A very rough proof: geodesic loops at x with length $< \varepsilon(n)$ generate an almost nilpotent – and thus cyclic – subgroup of $\pi_1(M)$. this means roughly that only one element of $\pi_1(M)$ is responsible for the short loop (assume $\mathrm{inj} < \varepsilon(n)$ everywhere). Transporting the short loop α along any loop β shows that α and β commute in $\pi_1(M)$, thus $\pi_1(M)$ is cyclic, a contradiction. $\square$

Corollary Gromov 1978: $\exists\, \varepsilon(n, D)$ such that, if a compact manifold M admits a metric with diameter $\leq D$ and curvature

$$-1 - \varepsilon(n, D) \leq K \leq -1,$$

then M admit a constant curvature metric.

Diameter bound is necessary, as shown by Gromov–Thurston 1985. Inventiones Math.

5. Gromov's simplicial volume It is a topological invariant $\|M\|$ of manifolds, with the following property

Theorem (Gromov 1983). $\exists\, \varepsilon(n)$ such that, if M admits a metric such that $\mathrm{Ricci} \geq -1$ and injectivity radius everywhere $\leq \varepsilon(n)$, then $\|M\| = 0$.

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IV Collapsing Riemannian manifolds

One of the possible ways of analysing a sequence of Riemannian manifolds is the following: rescale ("inflate") the metrics in order that the maximum of the sectional curvature (in absolute value) be equal to 1. If, after rescaling, the injectivity radius is bounded from below, the convergence theorem applies, a limit exists and we can say we have some understanding of the sequence.

Can we understand a sequence of metrics with bounded sectional curvature, whose injectivity radius tends to zero?

The answer is positive, but not simple. The theory is mainly due to M. Gromov.

Flat cylinders show that, when injectivity radius tends to zero, a limit may exist, but it has smaller dimension.

Describing this phenomenon requires a new notion of convergence, which we now explain.

Definition - Let $\varepsilon > 0$. An ε-net on a metric space X is a subset $N \subset X$ such that the ε-balls centered at points of N cover X.

Say that two compact metric spaces are "Hausdorff-close" if, for some small $\varepsilon > 0$, there exist ε-nets $N \subset X$ and $N' \subset Y$ such that the Lipschitz distance

$$\mathrm{dist}_L (N, N') < \varepsilon .$$

For non compact spaces, it is preferable to use marked points: Say that X_j, marked at $x_j \in X_j$, converge to X, marked at x, if, for any $R > 0$, the ball $B^{X_j}(x_j, R)$ converge in the Hausdorff sense to $B^X(x, R)$.

1. Example: $M \times \varepsilon N$ converges to M as $\varepsilon \to 0$ if N is compact. Here, εN means N equipped with a metric $\varepsilon^2 g$.

Note that, in this example, the curvature stays bounded only if
- either N is flat
- or $\dim N = 1$.

2. Example: Let $\tilde{M}$ be a fixed riemannian manifold with isometry group G, let $\Gamma_j \subset G$ be discrete subgroups acting freely on $\tilde{M}$, let $M_j = \tilde{M}/\Gamma_j$, $j = 1, 2$.
If Γ_1 and Γ_2 are Hausdorff close as pointed subsets of G, then M_1 and M_2 are Hausdorff close.
For example, let $\tilde{M} = \mathbb{R}^3$ and Γ_ε be generated by a screw motion, translation ε^2, rotation ε. Then Γ_ε converges in $\mathrm{Isom}(\mathbb{R}^3)$ to the Lie group Γ of screw motions with a given axis, and $M_\varepsilon = \mathbb{R}^3/\Gamma_\varepsilon$ converges to $\mathbb{R}^3/\Gamma$ = a half line.

3. Example (Berger 1962) of a sequence of non negatively curved manifolds, with bounded curvature and diameter, whose injectivity radius tends to zero

Consider the 3-sphere $S^3 = \{(z_1, z_2) \in \mathbb{C}^2, |z_1|^2 + |z_2|^2 = 1\}$ and the group of isometries that fixes the great circle $\{z_2 = 0\}$.

It is isomorphic to T^2.

$$\theta, \varphi \mod \mathbb{Z}^2 \to (e^{i2\pi\theta} z_1, e^{i2\pi\varphi} z_2)$$

The orbits of this action of T are exactly the level sets of the function distance to $\{z_2 = 0\}$, $S^3 \to [0, \frac{\pi}{2}]$, i.e., the great circles $\{z_2 = 0\}$ and $\{z_1 = 0\}$ and a family of tori.

For each $\alpha \in \mathbb{R}$, $\alpha \neq 0$, the subgroup $\{\theta = \alpha\varphi\}$ of T^2 acts locally freely on S^3, i.e., gives rise to a non vanishing Killing vector field on S^3, X_α.

Let g_0 denote the round metric on S^3; split it as

$$g_0 = g|_{X_\alpha} \oplus g|_{X_\alpha^\perp}$$

and modify it

$$g_{\alpha,\varepsilon} = \varepsilon^2 g|_{X_\alpha} \oplus g|_{X_\alpha^\perp}$$

Claim: the metrics $g_{\alpha,\varepsilon}$, $\varepsilon \in]0,1]$, have uniformly bounded curvature.

(see O'Neill's formula's in Arthur Besse's Einstein Mfds, chap. 9)

If $\alpha = 1$, then the orbits of X_1 are the fibres of the

Hopf fibration : $S^3 \to S^2$. the family $(S^3, g_{1,\varepsilon})$ converges to S^2 equipped with the round metric of curvature 4 (easy).

If α is rational, the orbits of X_α are the fibers of a Seifert fibration $S^3 \to S^2$, with two singular fibers at the north and south pole : if $\alpha = \frac{p}{q}$,

$$(z_1, z_2) \longmapsto [z_1^p ; z_2^q] \in \mathbb{C}P^1 = S^2$$

The limit is S^2 with a metric that has two conical singularities of angle $\frac{2\pi}{p}$ and $\frac{2\pi}{q}$

If α is irrational, the orbits of X_α are dense in the orbits of T^2, and $(S^3, g_{\alpha,\varepsilon})$ converges to the segment $[0, \frac{\pi}{2}]$.

3 <u>Example</u> : the method above (shortening the metric in the direction of a nonvanishing Killing vector field) applies any time a manifold admits a locally free circle action (Lie groups and compact homogeneous spaces, circle bundles). This applies in particular to nilmanifolds $_\Gamma\backslash^N$, N a nilpotent Lie group, Γ a discrete cocompact subgroup. For instance, when $\dim N = 3$, $_\Gamma\backslash^{N^3}$ is a circle bundle over a torus T^2, so it has metrics with curvature $|K| \leq 1$ arbitrarily close to a flat torus. Now one can show that this flat torus can be taken arbitrarily small.

Indeed, N^3 is the group of triangular matrices $\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ & & 1 \end{pmatrix}$ with $a, b, c \in \mathbb{R}$. The subgroup Γ of matrices with integer entries is discrete and cocompact. The map

$$\delta_\varepsilon : \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ & & 1 \end{pmatrix} \to \begin{pmatrix} 1 & \varepsilon a & \varepsilon^2 c \\ 0 & 1 & \varepsilon b \\ & & 1 \end{pmatrix}$$

is a group automorphism

Let g be some left invariant metric on N^3. Then $\delta_\varepsilon(\Gamma)$ converges to N, so $({}_{\delta_\varepsilon \Gamma}\backslash N, g)$ collapses to a point.

4. Graphmanifolds

Let M_1^3, M_2^3 be circle bundles over surfaces with boundary Σ_1^2, Σ_2^2. If T_1, T_2 are components of the boundary of M_1^3, M_2^3, choose a diffeo $\varphi : T_1 \to T_2$ and glue M_1^3 to M_2^3 along this diffeomorphism. Since φ need not map a circle action to the other, the resulting manifold need not admit a circle action. Manifolds obtained by iterating this surgery are called graphmanifolds (Waldhausen 1967). According to Thurston, hyperbolic manifolds and graphmanifolds are building blocks for all 3-manifolds.

In a special case, we describe metrics with small injectivity radius.

Assume $M_1 = S^1 \times \Sigma^1$, $M_2 = S^1 \times \Sigma^1$ and the diffeomorphism φ exchanges base and fiber

$$\varphi(S^1 \times \{*\}) = \{*\} \times \partial\Sigma^2, \quad \varphi(\{*\} \times \partial\Sigma^1) = S^1 \times \{*\}.$$

We can equip Σ^1 and Σ^2 (assume they have $\chi < 0$) with constant curvature -1 metrics of finite area so that the boundary circles have length ε. Decide that the fiber circles also have length ε. Then φ can be taken to be an isometry, and a Riemannian surgery is possible.

As ε tends to zero, the injectivity radius is $\sim \varepsilon$, the volume stays bounded but the diameter cannot stay bounded.

Cheeger and Gromov have attempted to characterize differentiable manifolds on which "collapses" exist. In fact, collapse can be taken in several different meanings. The strongest sense is with bounded diameter; up to now a characterization does not exist, which led Cheeger and Gromov to a weaker notion of collapse.

Definition: A smooth manifold M is collapsable if it admits a metrics with $|K| \leq 1$ and arbitrarily (uniformly) small injectivity radius.

Theorem: (Cheeger-Gromov 1957) there exist a topological characterization of collapsable manifolds interms of "F-structures".

In dimension 3, this boils down to:

a 3-manifold is collapsable iff it is a graph manifold.

A slightly different problem is to give a metric description of collapsing. The model is Gromov's almost flat manifolds theorem

Gromov's Almost Flat Manifolds Theorem.

There exists a constant $\varepsilon(n)$ such that, if a compact manifold M admits a metric g with

$$|K| \leq 1, \quad \text{diameter} \leq \varepsilon(n)$$

then M is diffeomorphic to a quotient $\Gamma \backslash N$ when N is a nilpotent Lie group and Γ a discrete fixed point free group of affine transformations of N.

Furthermore, there exists left invariant metrics g_0 on N (and so on $\Gamma\backslash N$) such that

$$\mathrm{dist}_{L}(g, g_0)$$

tends to zero uniformly with the diameter of g.

Elaborating on Gromov's work, K. Fukaya gives a description of certain collapses with bounded diameter. the description is complete in the case where there exist. Hausdorff limit is a Riemannian manifold M': we give a rough statement

Theorem (K. Fukaya) There exists an $\varepsilon(n, \mathrm{inject}(M'))$ such that, if

$$\mathrm{dist}_{\mathrm{Hausdorff}}(M, M') < \varepsilon$$

and $|K| \leq 1$ on M, then there is a map

$$M \to M'$$

which is nearly a Riemannian submersion

the fibers are almost flat manifolds

the structure group of the fibration is specified.

When M' has dimension 3, one recovers

- if M' is a point = nilmanifolds in nearly invariant metrics
- if M' is a circle = affine torus bundles over the circle, i.e., Solvmanifolds
- if M' is a circle = circle bundle over surfaces.

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