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Basic Riemannian Geometry (I)

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BASIC RIEMANNIAN GEOMETRY I

by
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These notes correspond to a series of 5 lectures given at the "College on Differential Geometry", ICTP Trieste, October 30 – December 1, 1989.

The emphasis is on metric properties of Riemannian spaces. This point of view leads naturally to basic concepts such as, geodesics, connections, curvature, and even manifolds.

Although this way of organizing the basic material in Riemannian geometry is not traditional, the reader should have no difficulty filling in proofs or finding them in the litterature.* For this reason our presentation will only indicate proofs.

1. GEODESICS

The euclidean distance between points $p=(p_1,\dots,p_n)$ and $q=(q_1,\dots,q_n)$ in $\mathbb{R}^n$ is given by

$$(1.1)\qquad \delta(p,q) = \|p-q\| = \langle p-q, p-q\rangle^{\frac{1}{2}} = \Big(\sum_{i=1}^{n}(p_i-q_i)^2\Big)^{\frac{1}{2}}.$$

* cf. e.g. [CE], [GKM], [K], [KN], [M]...

In terms of this, the <u>euclidean length</u> of a continuous curve $c : [a,b] \to \mathbb{R}^n$ is defined as

$$(1.2) \qquad L_g(c) = \sup \sum_{i=1}^{k} g(t_{i-1}, t_i) \,,$$

where the supremum is taken over all partitions $a = t_0 < t_1 < \dots < t_k = b$ of $[a,b]$. A curve c is called <u>rectifiable</u> if $L_g(c) < \infty$. Any C^1-curve c is rectifiable, in fact

$$(1.3) \qquad L_g(c) = \int_a^b \| c'(t) \| \, dt \,,$$

where $c' : [a,b] \to \mathbb{R}^n$ is the derivative of c. Clearly $g(c(a), c(b)) \leq L_g(c)$ and equality holds if and only if c is a line segment from $c(a)$ to $c(b)$.

Now let $X \subset \mathbb{R}^n$ be a path connected subset of $\mathbb{R}^n$. If confined to X, g is not a reasonable measure for the distance between points in X. Instead define

$$(1.4) \qquad d(p,q) = \inf L_g(c) \,,$$

where the infimum is taken over all continuous curves from p to q in X. This d is a metric on X provided any pair of points in X can be joined by a rectifiable curve in X.

<u>1.5 EXAMPLE</u> Let $U \subset \mathbb{R}^n$ be an open subset and $f : U \to \mathbb{R}$ a C^r-function, $r \geq 1$. Equip $X = \operatorname{graph}(f) = \{(p, f(p)) \in \mathbb{R}^{n+1} \mid p \in U\}$ with the metric d from

(1.4) and identify X with U via $p \leftrightarrow (p, f(p))$. Then U is a metric space in which distances are given as follows

$$d_g(p,q) = \inf_c \int_a^b g_{c(t)}(c'(t), c'(t))^{\frac{1}{2}} dt .$$

Here $g_p(\cdot,\cdot)$ is the inner product in $\mathbb{R}^n$ defined by $g_p(u,v) = (D(id,f)_p u, D(id,f)_p v)$ and the infimum is taken of all piecewise C^1 curves from p to q.

This example suggests the following notion

<u>1.6 DEFINITION</u> A <u>Riemannian C^r-structure</u> on an open set $U \subset \mathbb{R}^n$ is a C^r-map g, which assigns to each $p \in U$ an inner product in $\mathbb{R}^n$. The pair (U,g) is called an n-dimensional <u>Riemannian domain</u>.

Let $e_1, \dots, e_n$ be the standard basis for $\mathbb{R}^n$. For each $p \in U$ the matrix $\{g_{ij}(p)\} = \{g_p(e_i, e_j)\}$ determines a symmetric linear isomorphism $G_p : \mathbb{R}^n \to \mathbb{R}^n$ and

$$(1.7) \qquad g_p(u,v) = (u, G_p v)$$

for all $p \in U$, $u,v \in \mathbb{R}^n$. The change of g from point to point can thus be thought of as fixing the euclidean inner product $(\cdot,\cdot)$ and varying G.

The <u>(Riemannian) length</u> of a piecewise C^1 curve $c : [a,b] \to U$ is now defined (cf. 1.5) as

(1.8) $\quad L(c) = L_g(c) = \int_a^b g_{c(t)}(c'(t), c'(t))^{\frac{1}{2}} dt$.

For connected U, the <u>(Riemannian) distance</u> between $p, q \in U$ is defined by (cf. 1.5)

(1.9) $\quad d(p,q) = d_g(p,q) = \inf L_g(c)$,

where the infimum is taken over all piecewise regular curves c (i.e. $c' \neq 0$ where defined). Local comparison of g with constant inner products shows that indeed (U, d_g) is a metric space whose topology coincides with the induced topology from $\mathbb{R}^n$.

Using the metric d_g from (1.9) it is now possible to define the length of any continuous curve as in (1.2). As in (1.4) this leads to a new metric on U. However, this metric coincides with d_g. Thus (U, d_g) is an example of an <u>inner metric space</u>, which in general is a metric space in which distances are measured by the infimum of lengths of curves.

A <u>normal geodesic</u> in (U, d_g) is a curve c which is locally distance preserving, i.e. for any t in the domain of c

(1.10) $\quad d_g(c(t_1), c(t_2)) = |t_1 - t_2|$

whenever t_1, t_2 are close to t. A <u>geodesic</u> is a curve which up to a linear change of parameter is a normal geodesic. A <u>minimal geodesic</u> is a geodesic $c:[a,b] \to U$ with $L_g(c) = d_g(c(a), c(b))$. Local existence of minimal geodesics is a consequence of local compactness of U and Ascoli's theorem about equicontinuous families of maps.

2 CONNECTIONS

Local minimality of geodesics suggest an approach via the calculus of variations.

A one parameter variation of a curve $c:[a,b] \to U$ is a map

$$V : [a,b] \times (-\varepsilon, \varepsilon) \longrightarrow U$$

such that $V(t,0) = c(t)$. We assume that V is of class C^r, $r \geq 1$ (in general it is useful to consider piecewise C^r variations). With the notation $c_s = V(\cdot, s)$ we have $c = c_0$. The curve

$$X(t) = X_0(t) = \frac{\partial}{\partial s} V(t,s)_{|s=0}$$

is called the <u>variation field</u> of V along $c = c_0$.

We are interested in the variation of $L(c_s)$, i.e. of $\frac{d}{ds} L(c_s)_{|s=0}$. Here we may assume that c is parametrized proportional to arclength, i.e. $g_{c(t)}(c'(t), c'(t)) = \ell^2$ is constant. In this case a straight forward calculation based on (1.7) yields

2.1 FIRST VARIATION FORMULA

$$\frac{d}{ds} L(c_s)|_{s=0} = \ell^{-1} \int_a^b g_{c(t)} ({}^1\nabla_t X(t), c'(t))\, dt$$

where

$${}^1\nabla_t X(t) = \frac{d}{dt} X(t) + {}^1\Gamma_{c(t)} (X(t), c'(t))$$

and

$${}^1\Gamma_p (u,v) = \frac{1}{2} G_p^{-1} DG_p(u)(v)$$

for all $p \in U$, $u, v \in \mathbb{R}^n$.

Note that ${}^1\Gamma_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ defined above is bilinear and

(2.2) $\quad {}^1\nabla_t (X + Y) = {}^1\nabla_t (X) + {}^1\nabla_t (Y)$

(2.3) $\quad {}^1\nabla_t (fX) = f' \cdot X + f\, {}^1\nabla_t (X)$

for all fields X, Y along c and any function $f: [a,b] \to \mathbb{R}$. Moreover, for any X, Y

(2.4) $$\frac{d}{dt} g_{c(t)} (X(t), Y(t)) = g_{c(t)} ({}^2\nabla_t X(t), Y(t)) + g_{c(t)} (X(t), {}^2\nabla_t Y(t))$$

where ${}^2\nabla_t X(t) = \frac{d}{dt} X(t) + {}^2\Gamma_{c(t)} (X(t), c'(t))$ and ${}^2\Gamma_p(u,v) = {}^1\Gamma_p(v,u)$ for all $p \in U$, $u, v \in \mathbb{R}^n$.

Given any smooth map Γ that assigns to any $p \in U$ a bilinear map $\Gamma_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ the expression

(2.5) $$\nabla_t X(t) = \frac{d}{dt} X(t) + \Gamma_{c(t)}(X(t), c'(t))$$

is called the covariant derivative of X along c with Christoffel symbol Γ. Clearly (2.2) and (2.3) hold for any covariant derivative. If moreover Γ is symmetric, i.e. Γ_p is symmetric for every $p \in U$, then

(2.6) $$\nabla_t \frac{\partial}{\partial s} V(t,s) = \nabla_s \frac{\partial}{\partial t} V(t,s)$$

for any C^r variation V. We say that ∇ is metric (cf. 2.4) if

(2.7) $$g_c(X, Y)' = g_c(\nabla_t X, Y) + g_c(X, \nabla_t Y) .$$

Assuming ∇ is a symmetric, metric covariant derivative or connection on (L, g) a direct calculation gives

2.8 FIRST VARIATION FORMULA

$$\frac{d}{ds} L(c_s)_{|s=0} = \ell^{-1} \int_a^b g_c(\nabla_t X, c') = \ell^{-1} \left\{ g_c(X, c')\Big|_a^b - \int_a^b g_c(X, \nabla_t c') \right\}$$

In particular, if $\frac{d}{ds} L(c_s)_{|s=0} = 0$ for all variations with $V(a,s) = c(a)$ and $V(b,s) = c(b)$ we conclude

(2.9) $$\nabla_t c' = c''(t) + \Gamma_{c(t)}(c'(t), c'(t)) = 0 .$$

For any Γ, this is called the geodesic equation for the connection ∇. This second order equation is equivalent to the system

$$(2.10) \qquad c' = d \quad ; \quad d' = -\Gamma_c(d,d)$$

of first order equations. From the existence and uniqueness theorem for first order equations, we get immediately

2.11 THEOREM For every $(p_0, v_0) \in U \times \mathbb{R}^n$ there is an interval $J(p_0, v_0) \ni 0$ and a unique maximal solution to (2.9), $c_{(p_0,v_0)} : J(p_0,v_0) \to U$, such that $c_{(p_0,v_0)}(0) = p_0$ and $c'_{(p_0,v_0)}(0) = v_0$. Moreover, the set $W = \{(q,v,t) \mid (q,v) \in U \times \mathbb{R}^n, t \in J(q,v)\}$ is open in $U \times \mathbb{R}^n \times \mathbb{R}$, and the map $W \to U$, $(q,v,t) \to c_{(q,v)}(t)$ is smooth.

In view of this (cf. (2.8) and (2.9)), the desire to find a connection ∇ satisfying (2.6) and (2.7) is obvious. The answer is contained in

2.12 FUNDAMENTAL LEMMA OF RIEMANNIAN GEOMETRY For every Riemannian structure g on $U \subset \mathbb{R}^n$ there is one and only metric connection ∇ which is symmetric.

The connection in 2.12 is called the Levi-Civita connection. In the proof of 2.12 it is useful to "expand" the notion of covariant derivatives and formalize the setting.

A pair $(p,v) \in U \times \mathbb{R}^n$ as considered in 2.11 is called a <u>tangent vector</u> at $p \in U$. The set of all tangent vectors at p, $\{p\} \times \mathbb{R}^n$ is a vector space isomorphic to $\mathbb{R}^n$ called the <u>tangent space</u> T_pU to U at p. We will also use the notation v_p for the tangent vector (p,v) and TU for the union of all tangent spaces $\bigcup_p T_pU = U \times \mathbb{R}^n$.

If $c : [a,b] \to U$ is a differentiable curve, the tangent vector $(c(t), c'(t))$ is called the <u>velocity vector</u> of c at t, and it is denoted by $\dot{c}(t)$. Note that any tangent vector v_p is $\dot{c}(0)$ for some curve c (e.g. $c(t) = p + t \cdot v$). Any tangent vector can therefor be viewed as an equivalence class of curves with common velocity vector. This view point is extremly useful.

With the formal introduction of tangent spaces we can view a riemannian structure g on $U \subset \mathbb{R}^n$ as a map that assigns to any $p \in U$ an inner product, $g_p = \langle \cdot , \cdot \rangle_p$ on the tangent space T_pU. Moreover, any map $Y : U \to \mathbb{R}^n$ may be viewed also as a map that assigns to any $p \in U$ a tangent vector $(p, Y(p))$ in T_pU. With this interpretation Y is called a <u>vector field</u> on U. Similarly, if $c : [a,b] \to U$ the proper way to view $X : [a,b] \to \mathbb{R}^n$ as a <u>vector field along c</u> is to consider the map $X_c : [a,b] \to TU$ that assigns to any $t \in [a,b]$ the tangent vector $X(t)_{c(t)} \in T_{c(t)} U$.

Now if $Y : U \to \mathbb{R}^n$ is a vector field and $c : [a,b] \to U$ is a curve we can consider the vector field $Y \circ c$ along c. In this case (2.5) reads

$$\nabla_t (Y \circ c)(t) = DY_{c(t)}(c'(t)) + \Gamma_{c(t)}(Y(c(t)), c'(t)),$$

which clearly only depends on $\dot{c}(t)$ and on Y near $c(t)$. Thus if $v_p \in T_pU$ we define

$$(2.13) \qquad \nabla_{v_p} Y = v_p[Y] + \Gamma_p(Y(p), v),$$

where $v_p[Y] = DY_p(v)$ is the <u>directional derivative</u> of Y in direction v_p. Again we can view the covariant derivative $\nabla_{v_p} Y$ of Y in direction v_p as a tangent vector to U at p if we please. Replacing v_p by a vector field $X : U \to \mathbb{R}^n$ we can now introduce the <u>covariant derivative</u> $\nabla_X Y$ of Y in direction X as

$$(2.14) \qquad (\nabla_X Y)(p) = \nabla_{X(p)} Y.$$

Then $\nabla_X Y$ should be viewed a a vector field on U. The following are immediate

$$(2.15) \qquad \nabla_{fX + gZ} Y = f \nabla_X Y + g \nabla_Z Y$$

$$(2.16) \qquad \nabla_X (aY + bZ) = a \nabla_X Y + b \nabla_X Z$$

$$(2.17) \qquad \nabla_X f \cdot Y = X[f] \cdot Y + f \cdot \nabla_X Y$$

for all vector fields X, Y, Z, functions f, g and constants a, b. Here $X[f] : U \to \mathbb{R}$ is the directional derivative $X[f](p) = X_p[f] = Df_p(X(p))$ of f in direction X.

A map that assigns to any pair of vector fields X, Y a vector field $\nabla_X Y$ satisfying (2.15)–(2.17) is called a <u>connection</u> or covariant derivative. Such a map is clearly determined by

$$(2.18) \qquad \nabla_{e_i} e_j = \sum_{k=1}^{n} \Gamma_{ij}^{k} e_k ,$$

where $e_1, \dots, e_n : U \to \mathbb{R}^n$ are the <u>standard coordinate vector fields</u> on U, also sometimes denoted by $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ or $D_1, \dots, D_n$. Now $\Gamma_{ij}^k : U \to \mathbb{R}$ $i, j, k = 1, \dots, n$ define a map $\Gamma : U \to L^2(\mathbb{R}^n, \mathbb{R}^n; \mathbb{R}^n)$ by $\Gamma_{(p)}(\Sigma u_i e_i, \Sigma v_j e_j) = \Sigma u_i v_j \Gamma_{ij}^k e_k$, and the connection ∇ defined by this Γ coincides with the connection we started out with. Thus connections on $U \subset \mathbb{R}^n$ are in 1-1 correspondence with Christoffel symbols.

The proper notation for the covariant derivative $\nabla_t Y$ of a vector field $Y : [a,b] \to TU$ along a curve $c : [a,b] \to U$ is $\nabla_{\frac{\partial}{\partial t}} Y$ where $\frac{\partial}{\partial t}$ as above denotes the "coordinate" vector field on $\mathbb{R}^1 \supset [a,b]$.

Observe that for vector fields $X, Y : U \to \mathbb{R}^n$ and symmetric Γ we have

$$\nabla_X Y - \nabla_Y X = X[Y] - Y[X] =: [X,Y] , \tag{2.19}$$

where the vector field $[X,Y](p) = X_p[Y] - Y_p[X]$ is called the <u>Lie bracket</u> of X and Y. In general

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \tag{2.20}$$

at each $p \in U$ only depends on $X_p, Y_p \in T_pU$. Hence for each $p \in U$, $T_p : T_pU \times T_pU \to T_pU$ is a bilinear map called the <u>torsion tensor</u> of ∇. Clearly ∇ is <u>torsion free</u> $(T \equiv 0)$ if and only if Γ is symmetric.

Note that by definition $[X,Y](p)$ depends on X, Y near p, and the directional derivative is computed as

$$[X,Y]_p[f] = X_p[Y[f]] - Y_p[X[f]] . \tag{2.21}$$

We say that X and Y <u>commute</u> if $[X,Y]=0$. Clearly coordinate vector fields commute. The advantage of (2.21) over (2.19) is that it provides an invariant (coordinate free) describtion of the Lie bracket. The point is, that tangent vectors can also be interpreted as "directional derivatives" acting on functions. One has the following immediate identities

$$v_p[af+bh] = a\,v_p[f] + b\,v_p[h] \tag{2.22}$$

$$v_p[f \cdot g] = v_p[f] \cdot g(p) + f(p)\, v_p[g] . \tag{2.23}$$

A operation satisfying (2.22) and (2.23) for functions f, g defined near p is called a <u>derivation</u> at $p \in U$. Since the map that assigns to a tangent vector the corresponding "directional derivative" is a linear isomorphism between the tangent space T_pU and the vector space of derivations at p, we may also view tangent vectors as derivations. A similar statement then also holds for vector fields.

It is now not difficult to show that the Levi Civita connection for (U, g) (cf 2.12) is characterized by the identity

$$(2.24) \quad \langle \nabla_X Y, Z \rangle = \tfrac{1}{2}\{X[\langle Y,Z\rangle] - Z[\langle X,Y\rangle] + Y[\langle Z,X\rangle] \\ - \langle X,[Y,Z]\rangle + \langle Z,[X,Y]\rangle + \langle Y,[Z,X]\rangle\}$$

for all vector fields X, Y and Z on U.

In the next section we will see that solutions to (2.9) corresponding to the Levi Civita connection are precisely the geodesics introduced in section 1.

3. THE EXPONENTIAL MAP

From now on ∇ will denote the Levi-Civita connection on a riemannian domain (U, g). Solutions to (2.9) will at the moment be referred to a ∇-geodesics.

From (2.11) and the homogeneity properties of (2.9) it follows that the set

$$\mathcal{O} = \{(p,v) \in U \times \mathbb{R}^n \mid 1 \in J(p,v)\}$$

is an open neighborhood of $U \times \{0\}$ in $U \times \mathbb{R}^n$. Moreover, $\mathcal{O}_p = \mathcal{O} \cap T_pU$ is starshaped around $0 \in T_pU$. The <u>exponential map</u>, $\exp : \mathcal{O} \to U$ is defined by

$$(3.1) \qquad \exp(v_p) = c_{(p,v)}(1)$$

for all tangent vectors $v_p = (p,v) \in \mathcal{O}_p \subset T_pU$, $p \in U$. By (2.11) $\exp : \mathcal{O} \to U$ is smooth and $\exp(0_p) = p$ for all $p \in U$. Consider the smooth map

$$(\pi, \exp) : \mathcal{O} \to U \times U \quad , \quad (p,v) \to (p, \exp(v_p))$$

and observe that

$$D(\pi,\exp)_{(p,0)} = \begin{array}{c c} & \begin{array}{cc} n & n \end{array} \\ & \left(\begin{array}{c|c} id & 0 \\ \hline id & id \end{array}\right) & \begin{array}{c} n \\ n \end{array} \end{array}$$

for all $p \in U$. By the inverse function theorem and $(\pi, \exp)|U \times \{0\} = id_{U \times \{0\}}$, it follows that

<u>3.2 THEOREM</u> There is an open neighborhood $\mathcal{D} \subset \mathcal{O}$ of $U \times \{0\}$ and an open neighborhood $\mathcal{V} = (\pi, \exp)(\mathcal{D})$ of the diagonal Δ in $U \times U$ such that $(\pi, \exp) : \mathcal{D} \to \mathcal{V}$ is a <u>diffeomorphism</u>, i.e. it is a smooth bijective map with smooth inverse.

In particular, if $\exp_p : \mathcal{O}_p \to U$ is the restriction of $\exp$ to $\mathcal{O}_p$, then $\exp_p$ is a diffeomorphism of an open neighborhood of $0_p \in T_pU$ onto an open neighborhood of $p \in U$.

The comparison of euclidean geometry of T_pU with riemannian geometry of U via $\exp_p$ is crucial for the understanding of (local) riemannian geometry.

In the first step of this comparison, it is convenient to view the differential of a map as a map between tangent spaces. To be precise, if $f : U \to \mathbb{R}^m$ is differentiable at $p \in U$ with differential $Df_p : \mathbb{R}^n \to \mathbb{R}^m$, the map

$$(3.3) \qquad f_{*p} : T_pU \to T_{f(p)}\mathbb{R}^m \quad , \quad (p, v) \longmapsto (f(p), Df_p(v))$$

is called the <u>tangent map</u>, or the <u>induced map</u> of f at $p \in U$. If there is no danger of confusion we may simply refer to it also as the differential of the map.

The next lemma is often referred to as the "Gauss lemma":

<u>3.4 LEMMA</u> For all $p \in U$, $\exp_p : \mathcal{O}_p \to U$ is a "radial" isometry, i.e.

$$\langle (\exp_p)_* (v_v), (\exp_p)_* (u_v) \rangle_{\exp_p(v)} = \langle v_v, u_v \rangle_p$$

for all $v \in \mathcal{O}_p$ and tangent vectors $u_v \in T_v \mathcal{O}_p$.

The proof of 3.4 is a simple application of the first variation formula (2.8) applied to the variation $V(t,s) = \exp_p(p, t(v+su))$ for

$t \in [0,1]$ and s near zero.

The most important and fairly direct application of the "Gauss-lemma" is contained in

<u>3.5 LEMMA</u> For $p \in U$ choose $\varepsilon > 0$ so that $\exp_p$ restricted to the euclidean ball $B(o_p, \varepsilon) \subset \mathcal{O}_p \subset T_pM$ is a diffeomorphism. Then for any $q = \exp_p(v)$, $v \in B(o_p, \varepsilon)$ and any curve $c : [0,1] \to U$ from p to q

$$L_g(c) \geq L_g(c_{(p,v)} | [0,1]) .$$

In particular, the ∇-geodesic $c_{(p,v)} : [0,1] \to U$ from p to q is a geodesic, and $B(p, \varepsilon) = \exp_p(B(o_p, \varepsilon))$.

It follows from 3.5 that the geodesics introduced in section 1 are smooth curves which satisfy the ∇-geodesic equation (2.9), where ∇ is the Levi Civita connection (2.24). Moreover, they are locally unique!

As another important application of the first variation formula, we shall now see that isometries between Riemannian domains are smooth. Here $F : (U_1, d_{g_1}) \to (U_2, d_{g_2})$ is called an <u>isometry</u> if $F(U_1) = U_2$ and F is distance preserving. In this case $F^{-1} : U_2 \to U_1$ is also an isometry. In particular F is a homeomorphism and $\dim U_1 = \dim U_2$.

3.6 THEOREM An isometry F between riemannian domains (U_1, g_1), (U_2, g_2) is a diffeomorphism. In fact $F_{*p} : T_pU_1 \to T_{F(p)}U_2$ is a linear isometry for all $p \in U_1$, and $F \circ \exp_p = \exp_{F(p)} \circ F_{*p}$.

Proof Fix $p \in U_1$ and choose $\delta > 0$ such that $\exp_q : B(0_q, 2\delta) \to B(q, 2\delta)$ and $\exp_{F(q)} : B(0_{F(q)}, 2\delta) \to B(F(q), 2\delta)$ are diffeomorphisms for all $q \in B(p, 2\delta)$ (cf. 3.2 and 3.5). Since F is an isometry, it maps for each $q \in B(p, \delta)$ the unique minimal geodesic from p to q, to the unique minimal geodesic from $F(p)$ to $F(q)$. Since $\exp_p : B(0_p, \delta) \to B(p, \delta)$ and $\exp_{F(p)} : B(0_{F(p)}, \delta) \to B(F(p), \delta)$ are diffeomorphisms, we only need to show that $\exp_{F(p)}^{-1} \circ F \circ \exp_p : B(0_p, \delta) \to B(0_{F(p)}, \delta)$ is smooth. In fact, we claim that it is the restriction of a linear isometry $T_pU_1 \to T_{F(p)}U_2$. We have already seen that it preserves the norm of vectors. To show that it preserves angles consider vectors $u_p, v_p \in B(0_p, \delta)$, and let $V : [0,1] \times [0,1] \to B(p, 2\delta)$ be the variation

$$V(t,s) = \exp_{\exp(v_p)}(t \cdot \exp_{\exp(v_p)}^{-1}(\exp_p(s \cdot u_p))) .$$

For each s, this is the unique minimal geodesic from $\exp(v_p)$ to $\exp(su_p)$. The first variation formula then yields (cf. 2.8)

$$\frac{d}{ds} L(c_s)_{|s=0} = \|v_p\|_p^{-1} \langle u_p, -v_p \rangle_p = -\|u_p\| \cos \theta$$

where θ is the angle between u_p and v_p. The same argument applies to $F \circ V$ and hence angles are preserved. The argument also shows that $F_{*p} = \exp_{F(p)}^{-1} \circ F \circ \exp_p$.