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COLLEGE ON DIFFERENTIAL GEOMETRY

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Conformal Maps

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Lecture 1. Conformal maps

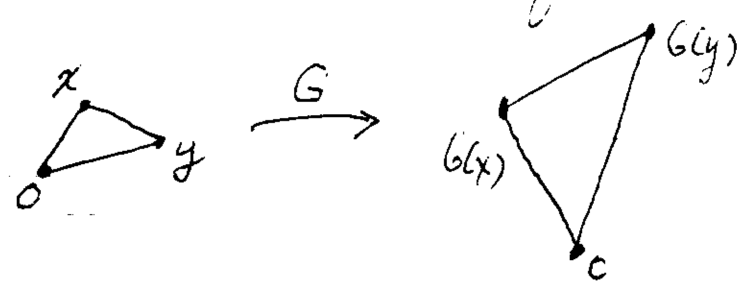
Basic notion: <u>similarity</u>

<u>Def.</u> A map $F: \mathbb{R}^n \to \mathbb{R}^n$ is a <u>similarity</u> if it is a one-to-one correspondence that maps each triangle onto a similar triangle.

If F is a similarity, then so is G, where

$$G(x) = F(x) - F(0), \quad \text{and } G(0) = 0.$$

If $0, x, y$ are the vertices of a triangle, then $0 = G(0), G(x), G(y)$ are the vertices of a similar triangle,



so that $$\frac{|G(x)|}{|x|} = \frac{|G(y)|}{|y|}.$$

Also, if $0, x, y$ lie on a line, so must $0, G(x), G(y)$, and it follows easily that

$\exists\, \lambda > 0$ such that $|G(x)| = \lambda |x|$ for all $x \in \mathbb{R}^n$.
Hence $\dfrac{G(x)}{\lambda}$ is a congruence, which maps every triangle into a congruent triangle, and preserves all lengths. Since it is also a linear map, $G(x)/\lambda$ is given by an orthogonal matrix A, and therefore our original F is of the form

$$F(x) = \lambda A x + c, \quad A \in O(n), \ c \text{ constant}.$$

We can also consider more generally, maps of $\mathbb{R}^m \to \mathbb{R}^n$, $n \geq m$ which take each triangle into a similar triangle. How can one recognize such a map? The answer is:

Lemma 1. Let $y = Bx$ map $\mathbb{R}^m \to \mathbb{R}^n$, $n \geq m$. Then $|Bx| = \lambda |x|$ for all $x \in \mathbb{R}^m$, where $\lambda > 0$ is a constant $\Leftrightarrow$ the matrix $G = B^T B$ satisfies

$$g_{ij} = \lambda^2 \delta_{ij}, \quad i,j = 1, \dots, m, \qquad G = (g_{ij}).$$

Proof. G is a symmetric matrix: $G^T = G$, so that it can be diagonalized using an orthogonal matrix O: $\quad O^T G O = D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_m \end{pmatrix}$.

But $\quad |Ox|^2 = |x|^2$ for all x, since $O \in O(m)$

and $\quad |BOx|^2 = x^T O^T B^T B O x = x^T D x = \sum_{j=1}^{m} d_j x_j^2$.

Hence $|Bx|^2 = \lambda^2 |x|^2$ for all x

$\iff |BOx|^2 = \lambda^2 |Ox|^2$ for all x

$\iff \sum_{j=1}^{m} d_j x_j^2 = \lambda^2 \sum_{j=1}^{n} x_j^2$ for all x

$\iff d_j = \lambda^2, \quad j = 1, \ldots, n$

$\iff D = \lambda^2 I$

$\iff G = \lambda^2 I$, since $D = O^T G O$

and $O \in O(n) \iff O^T = O^{-1}$

so that $O^T(\lambda^2 I) O = \lambda^2 O^{-1} O = \lambda^2 I$.

qed

For $m = n = 2$, there is an even quicker test for similarity

Lemma 2. Let $y = Bx$ be a linear transformation T with $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a similarity $\iff$ either 1. $a = d$, $b = -c$

or 2. $a = -d$, $b = c$.

Note: case 1 occurs $\Leftrightarrow$ $\det B = a^2 + b^2 > 0$;

case 2 — — — — — — — — < 0.

Thus case 1 $\Leftrightarrow$ T preserves orientation,

case 2 — — reverses — .

Proof. ~~Since~~ B is a similarity $\Leftrightarrow$ $B = \lambda A$, where A is an orthogonal matrix: $A \in O(2)$. But then

either case 1. $A = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}$, a rotation ($\det A = 1$)

or case 2. $A = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ \sin\vartheta & -\cos\vartheta \end{pmatrix}$, a reflection ($\det A = -1$)

For $B = \lambda A$, this gives the conditions stated on a, b, c, d.

qed

Differentiable maps $y = F(x)$; $y_i = F_i(x_1, \ldots, x_m)$, $i = 1, \ldots, n$.

$F: D \to \mathbb{R}^n$, D a domain in $\mathbb{R}^m$.

The differential of F at a point p, denoted dF_p is a linear transformation

$$\eta = A\xi, \quad a_{ij} = \frac{\partial F_i}{\partial x_j}(p), \quad i = 1, \ldots, n; \; j = 1, \ldots, m.$$

(If $m = n$, then $\det(a_{ij}) = \left.\dfrac{\partial(y_1, \ldots, y_n)}{\partial(x_1, \ldots, x_n)}\right|_p$, the Jacobian.

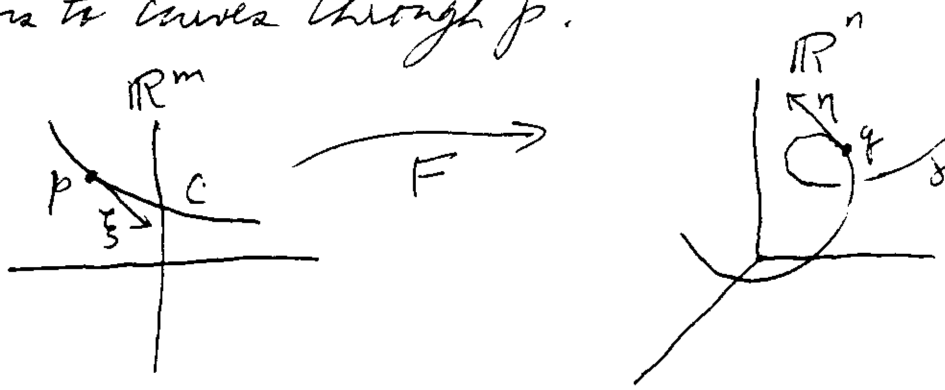
There are two standard interpretations of the differential:

1. Let C be a smooth curve through p, given by $x(t)$, $x(0) = p$.
Let γ be the image of C under F: $y(t) = F(x(t))$.
The tangent vector ξ to C at p is $\xi = x'(0)$
" " " " " η " γ " $q = F(p)$ " $\eta = y'(0)$,

where, by the chain rule,

$$\eta_i = y_i'(0) = \sum_j \frac{\partial y_i}{\partial x_j}(p)\, x_j'(0), \quad \text{or} \quad \eta = dF_p\, \xi.$$

So the differential describes the induced map on tangent vectors to curves through p.



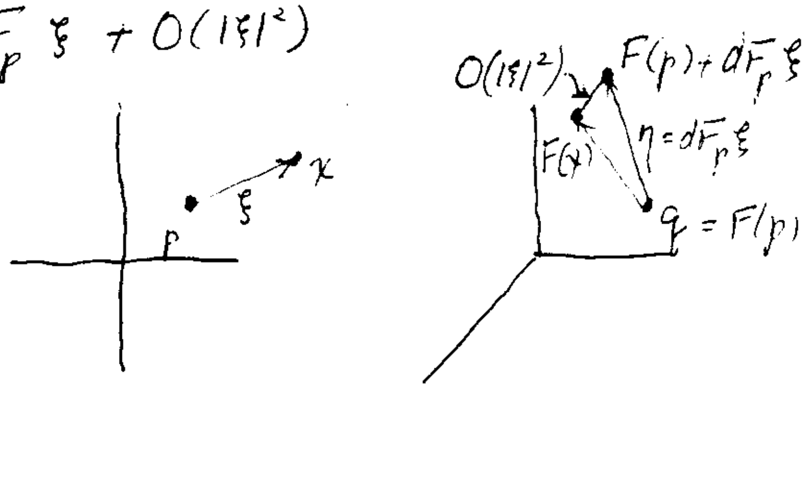
$q = F(p)$

2. If $p = (a_1, \ldots, a_m)$, $x = (x_1, \ldots, x_m)$, $\xi = x - p$,

$$F_i(x) = F_i(p) + \sum \frac{\partial F_i}{\partial x_j}(p)(x_j - a_j) + O(|x-p|^2)$$

$$F(x) - F(p) = dF_p\, \xi + O(|\xi|^2)$$

Thus the differential gives the <u>first order approximation</u> to F near p.



Def. F is conformal if dF_p is a similarity for all p.

Applying Lemma 1, with $a_{ij} = \frac{\partial F_i}{\partial x_j}$, $G = A^T A$,

$$g_{ij} = \sum_k a_{ki} a_{kj} = \sum_k \frac{\partial F_k}{\partial x_i} \frac{\partial F_k}{\partial x_j} = \frac{\partial F}{\partial x_i} \cdot \frac{\partial F}{\partial x_j},$$

and F is conformal $\iff g_{ij} = \lambda^2 \delta_{ij}$, $\lambda = \lambda(p) > 0$.

Note: Sometimes "conformal" is used only for one-to-one maps, in which case the condition we have given is called locally conformal.

By the above interpretations, conformality means that angles between tangent vectors are preserved at each point:

and that arc lengths at p are all multiplied by the same factor $\lambda > 0$:

$$|y'(0)| = \lambda |x'(0)|.$$

(In fact, either of these implies the other.)

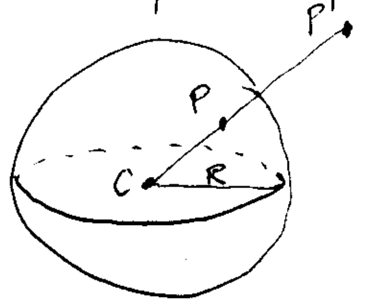
Also, that F, to first approximation, defines a similarity near p.

Examples of conformal maps

1. Any similarity F is a conformal map ($dF_p = F$ for all p).

2. Inversions in a sphere.

An *inversion* maps each ray through the center of a sphere onto itself, by the rule that the image of a point P is the point P' such that the product of the distances of P and P' to the center of the sphere equals the square of the radius.



$$(\overline{CP})(\overline{CP'}) = R^2.$$

Lemma 2. Inversions are conformal maps.

Proof. Since similarities are conformal, it is sufficient to prove the result for a sphere of radius 1 with center at the origin. Then each point $x \neq 0$ maps onto a point y on the same ray through the origin, so that $y = cx$, $c > 0$, and

$$|x|\,|y| = 1 \quad \text{or} \quad |x|\,|cx| = 1, \text{ so that } c = 1/|x|^2.$$

Hence the inversion is the map

$$y = \frac{x}{|x|^2}, \quad \text{or} \quad y_i = \frac{x_i}{r^2}, \quad i = 1, \dots, n, \quad r^2 = \sum x_i^2.$$

Then $2r\dfrac{\partial r}{\partial x_j} = 2x_j$, $\dfrac{\partial r}{\partial x_j} = \dfrac{x_j}{r}$,

and $\dfrac{\partial y_i}{\partial x_j} = \dfrac{\delta_{ij}}{r^2} - 2r^{-3}\dfrac{\partial r}{\partial x_j}x_i = \dfrac{r^2\delta_{ij} - 2x_i x_j}{r^4}$.

$$g_{ij} = \sum_k \frac{\partial y_k}{\partial x_i}\frac{\partial y_k}{\partial x_j} = \frac{1}{r^8}\sum_k (r^2\delta_{ki} - 2x_k x_i)(r^2\delta_{kj} - 2x_k x_i)$$

$$= \frac{1}{r^8}(r^4\delta_{ij} - 2r^2 x_i x_j - 2r^2 x_i x_j + 4r^2 x_i x_j)$$

$$= \frac{1}{r^4}\delta_{ij}$$

qed

<u>Liouville Theorem</u>. The <u>only</u> conformal maps of $\mathbb{R}^n$, $n \geq 3$, are composed of similarities and inversions.

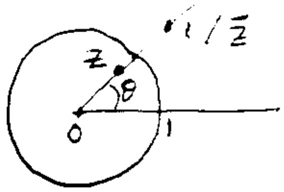
<u>Inversions in the plane</u>. Have complex notation:

$$z = x_1 + ix_2 \qquad w = y_1 + iy_2$$

Inversion in unit circle: $y_1 + iy_2 = \dfrac{x_1 + ix_2}{r^2} = \dfrac{z}{|z|^2}$.

Since $|z|^2 = z\bar{z}$, inversion is $\boxed{w = \dfrac{1}{\bar{z}}}$

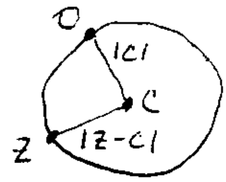
(if $z = re^{i\theta}$, $\bar{z} = re^{-i\theta}$, $\dfrac{1}{\bar{z}} = \dfrac{1}{r}e^{i\theta}$:



Note: Since every inversion in $\mathbb{R}^n$ induces an inversion on each plane through the center of the sphere, one can often use the plane case to get general results. For example, what is the image under inversion in the unit sphere of a sphere through the origin?

By restricting to all z-planes through the origin, it is sufficient to find the answer in the plane:

circle through the origin $|z-c| = |c|$



We can write this equation as $(z-c)(\overline{z-c}) = c\bar{c}$,

or $z\bar{z} - c\bar{z} - \bar{c}z - c\bar{c} = c\bar{c}$, or $z\bar{z} = \bar{c}z + c\bar{z}$.

Under inversion: $w = \frac{1}{\bar{z}}$, or $z = \frac{1}{\bar{w}}$, the image is

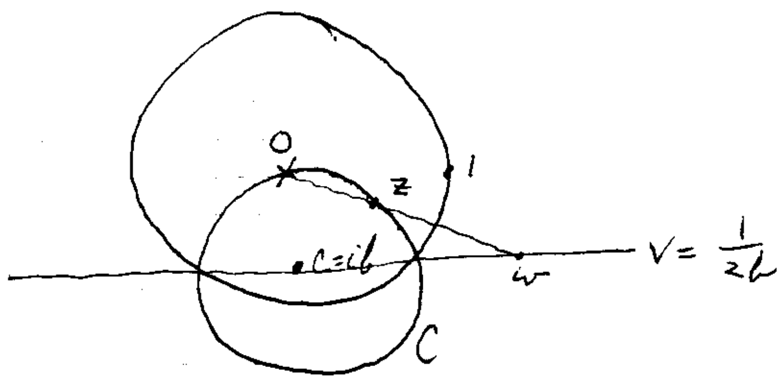
$$\frac{1}{\bar{w}}\,\frac{1}{w} = \frac{\bar{c}}{\bar{w}} + \frac{c}{w}$$

or $$1 = \bar{c}w + c\bar{w} = 2\,\mathrm{Re}\{\bar{c}w\}.$$

But this is a linear equation. If we choose c on the imaginary axis: $c = ib$, then $\mathrm{Re}\{\bar{c}w\} = \mathrm{Re}\{-ibw\}$
$= b\,\mathrm{Im}\{w\}$.

For $w = u + iv$, (and $c = ib$), our equation becomes

$2bv = 1$, or $v = \frac{1}{2b}$, a horizontal line.



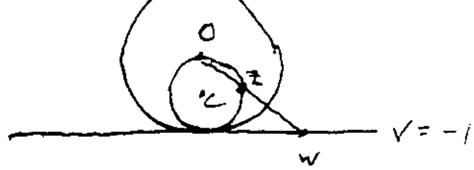
Conclusion: a circle C through the origin with center at $c = ib$, maps onto the line $v = \frac{1}{2b}$ under inversion in the unit circle.

By rotating this picture about the vertical axis, we conclude that:

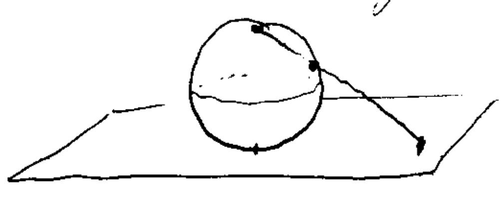
under inversion in the unit sphere, the image of a sphere through the origin is a plane.

Since inversion is conformal, the restriction gives a conformal map of the sphere on the plane, called <u>stereographic projection</u>. In fact, we have, by varying b, a whole one-parameter family of such maps, but one can prove that they are essentially equivalent. The two most popular versions are

1. choosing $b = -\frac{1}{2}$ gives $v = -1$

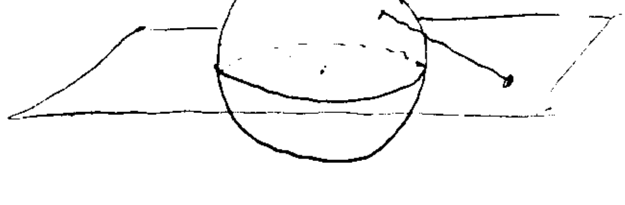


This corresponds to projecting a sphere onto a tangent plane.



2. to get the image line $v = \frac{1}{2b}$ to pass through the center of the circle: $c = ib$, we need $\frac{1}{2b} = b$ or $b^2 = \frac{1}{2}$, $b = \frac{\sqrt{2}}{2}$.

This corresponds to projecting the sphere onto a plane through the center.



Note that the original sphere in which we were inverting has disappeared in these pictures, and we are left with the conformal map between the secondary sphere and a plane.

References

For conformal maps and Liouville's Theorem, you can consult §15 of Modern Geometry - Methods and Applications, Part I, by B.A. Dubrovin, A.T. Fomenko, and S.P. Novikov, Springer 1984. (Also in French: Géometrie Contemporaine)

Many introductory texts in differential geometry have a proof of Liouville's Theorem.

For a detailed treatment of linear and differentiable maps $F: \mathbb{R}^2 \to \mathbb{R}^2$ in the spirit of this lecture, see R. Osserman, Two-Dimensional Calculus, Harcourt-Brace 1968 (reprinted by Krieger).