INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
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SMR.404/5

COLLEGE ON DIFFERENTIAL GEOMETRY

(30 October - 1 December 1989)

Complex functions and conformal metrics

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Lecture 2. Complex functions and conformal metrics.

We now restrict attention to conformal maps $F: \mathbb{R}^2 \to \mathbb{R}^n$, and start with $n = 2$.

If $F: \mathbb{R}^2 \to \mathbb{R}^2$, we can use complex notation:

$$z = x_1 + i x_2, \quad w = y_1 + i y_2, \quad w = f(z).$$

Fundamental fact: $f'(z_0)$ exists $\Leftrightarrow$ the Cauchy-Riemann equations hold. $\dfrac{\partial y_1}{\partial x_1} = \dfrac{\partial y_2}{\partial x_2}$, $\dfrac{\partial y_2}{\partial x_1} = -\dfrac{\partial y_1}{\partial x_2}$.

This is just a condition on dF_{z_0}. In fact

Case 1. $f'(z_0) = 0 \iff dF_{z_0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Case 2. $f'(z_0) \neq 0 \iff dF_{z_0} = \lambda \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad \lambda > 0$

$$\iff f'(z_0) = \lambda e^{i\theta}$$

Thus f (or F) is conformal, with $\det dF_{z} = \lambda^2 = |f'(z)|^2$ and near z_0 f behaves like a rotation through angle $\theta = \arg f'(z_0)$ followed by a dilation by $\lambda = |f'(z_0)|$.

<u>Def</u> f is (complex) <u>analytic</u> in a domain D, or <u>holomorphic</u> in D if $f'(z_0)$ exists for all $z_0 \in D$.

Basic facts on complex analytic functions $f(z)$ in D.

1. $f(z)$ is represented by a power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ near each point z_0 in D
2. $f(z_0) = w_0 \Rightarrow \exists \delta > 0 : f(z) \neq w_0$ for $0 < |z - z_0| < \delta$
3. the zeros of $f'(z)$ are isolated.

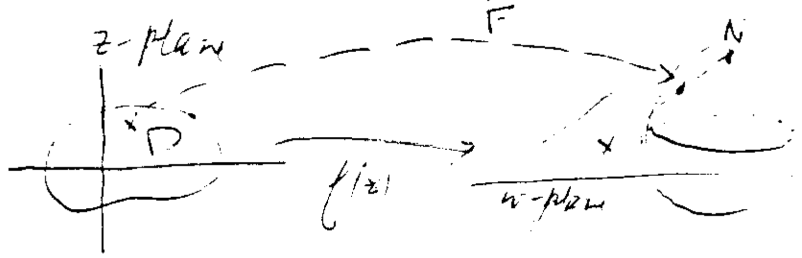
Basic theorem: <u>Riemann mapping theorem</u>. If D and D' are any two simply-connected plane domains, other than the whole plane, then there exists a conformal map of D onto D'; that is, $\exists f(z)$, holomorphic in D, $f'(z) \neq 0 \ \forall z \in D$, and $f(z)$ is a one-to-one map of D onto D'.

Thus, in contrast to $\mathbb{R}^n$, $n \geq 3$ where by Liouville's Theorem, hardly any domains are conformally equivalent, in $\mathbb{R}^2$ there are no constraints at all.

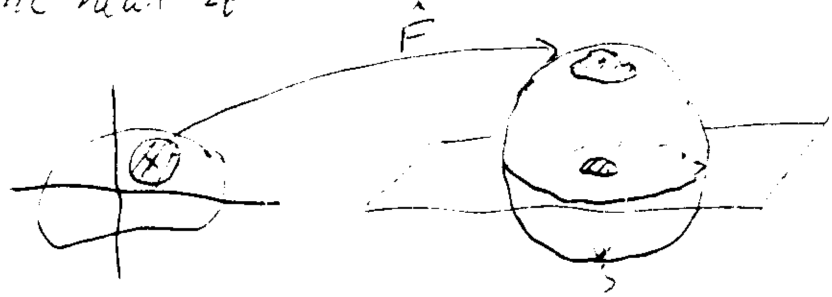
<u>Meromorphic functions</u>. We often want to allow functions to be holomorphic in a domain D except at isolated points where they tend to infinity. Such functions are called <u>meromorphic</u>. In this case

that near a singular point z_0, such a function $f(z)$ must have the form $f(z) = \dfrac{g(z)}{(z-z_0)^\nu}$ for some $\nu \geq 1$, where $g(z)$ is analytic near z_0 and $g(z_0) \neq 0$. The function $f(z)$ is then said to have a <u>pole of order ν</u> at z_0.

A good way to think of meromorphic functions is by composing the map $f(z)$ with stereographic projection into the unit sphere S^2, to get a map $\hat{F} : D \to S^2$.



At a pole z_0 of f, $\hat{F}$ simply maps onto the "north pole" N of S^2. The condition f be meromorphic near z_0 is equivalent to the condition that $\hat{F}$ composed with stereographic projection from the <u>South</u> pole be holomorphic near z_0

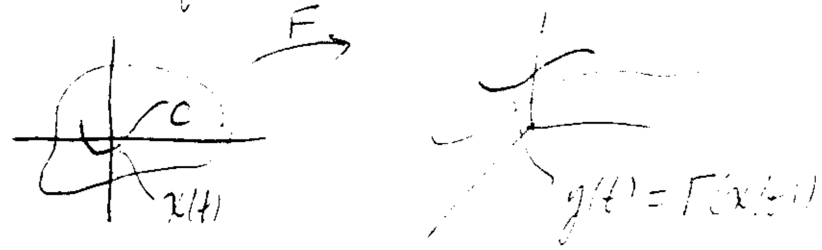


The sphere S^2 thought of as $\mathbb{C}\cup\{\infty\}$ under stereographic projection is called the <u>Riemann sphere</u>. A meromorphic function is simply a complex analytic map into the Riemann sphere.

We next look at differentiable maps

$$F : D \to \mathbb{R}^n, \qquad D \subset \mathbb{R}^2.$$

For any curve C in D, given by $x(t)$, we can compute the length of the image curve $y(t) = F(x(t))$.



$$\left(\frac{ds}{dt}\right)^2 = \left|\frac{dy}{dt}\right|^2 = \sum_{i=1}^{n}\left(\sum_{j=1}^{2} \frac{\partial y_i}{\partial x_j}\frac{dx_j}{dt}\right)^2 = \sum_{j,k=1}^{2} g_{jk}\frac{dx_j}{dt}\frac{dx_k}{dt},$$

where $\quad G = (g_{ij}) = A^T A, \quad dF_{x(t)}\,\xi = A\xi.$

$$F \text{ is conformal} \iff g_{ij} = \lambda^2 \delta_{ij}, \quad \lambda > 0$$

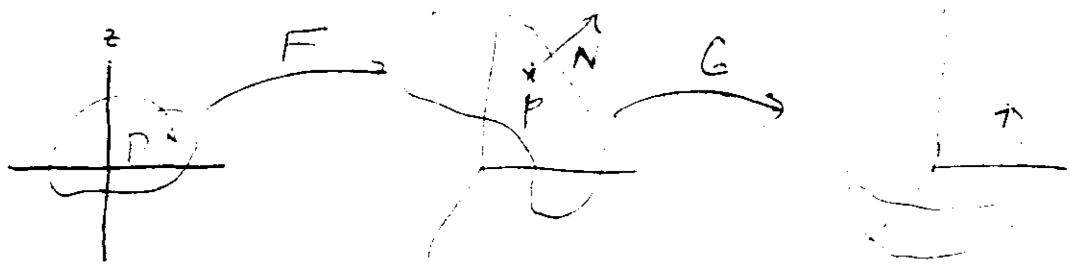
$$\iff \left(\frac{ds}{dt}\right)^2 = \lambda^2 \left|\frac{dz}{dt}\right|^2, \qquad z = x_1 + i x_2.$$

A smooth map $F: D \to \mathbb{R}^n$ is said to define a *regular surface* if $dF_p \xi \neq 0$ $\forall \xi \neq 0$ at each $p \in D$. Equivalently

$$\sum_{i,j=1}^{2} g_{ij}\,\xi_i \xi_j = |dF_p \xi|^2$$

is a positive definite quadratic form. This form is then called the *first fundamental form* of the surface.

In the special case $n = 3$ Gauss introduced the map into the unit sphere defined by the unit normal N at each point:



That map G is called the *Gauss map* or *spherical map* of the surface. Gauss defined the *curvature* K by

$$K(p) = \det dG_p$$

where dG_p takes tangent vectors to curves in the surface at p into tangent vectors at the image point on S^2.

But the normal to S^2 at the point n is just n, and so dG_p may be thought of as a linear map of the tangent plane to the surface at p into itself. Thus the determinant of dG_p is well-defined, independent of choice of a basis.

The curvature K is now called the *Gauss curvature* of the surface. The *theorema egregium* of Gauss states that even though the definition of K depends on the way the surface lies in space, the value of K is an *intrinsic quantity*, depending only on measurements *along the surface*; that is, depending only on the coefficients g_{ij} of the first fundamental form. In fact, Gauss gave an explicit formula for K in terms of the g_{ij}. For a conformal map, where $g_{ij} = \lambda^2 \delta_{ij}$, the formula is particularly simple:

$$K = -\frac{\Delta \log \lambda}{\lambda^2}, \qquad \left(\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right).$$

Def. A function $\lambda(z)$ on a domain D is said to define a <u>conformal metric</u> on D by the formula

$$\frac{ds}{dt} = \lambda(z)\left|\frac{dz}{dt}\right| , \qquad \lambda(z) > 0 \tag{1}$$

for the arclength of any smooth curve in D.

If $F: D \to \mathbb{R}^n$ is conformal, then equation (1) is precisely the formula for the arclength of the <u>image</u> of the curve under F, when

$$|dF_z(\xi)| = \lambda(z)|\xi|$$

or equivalently, $g_{ij} = \lambda^2 \delta_{ij}$.

<u>Question.</u> Does <u>every</u> positive function $\lambda(z)$ on a domain $D \subset \mathbb{R}^2$ arise in this way through a conformal map $F: D \to \mathbb{R}^n$?

<u>Answers</u> 1. <u>No,</u> for F smooth, $n=3$ (Hilbert) i.e. - there exists a (smooth) $\lambda(z)$ on $D = \{|z|<1\}$ for which no smooth $F: D \to \mathbb{R}^3$ induces the metric $\lambda(z)\left|\frac{dz}{dt}\right|$.

2. Yes, for F smooth, n sufficiently large (Nash). i.e. every conformal metric in every domain D can be realized by a smooth surface in some $\mathbb{R}^n$. (Nash embedding theorem.)

3. Yes, for $n=3$ and $F \in C^1$ (Kuiper). Every conformal metric can be realized on a surface ~~that~~ in $\mathbb{R}^3$ that is C^1, but need not be C^2. Note that such a surface has a tangent plane at each point that varies continuously with the point, but may not vary differentially. In other words, the normal N is continuous, but may not be differentiable. So the Gauss map G is well-defined and continuous, but dG may not exist, and hence also not K in Gauss' original definition. On the other hand the intrinsically defined K exists as $-\Delta \log \lambda / \lambda^2$.

4. Who cares? (Riemann); i.e. conformal metrics are interesting to study in their own right,

whether or not they arise as the metrics induced on actual surfaces in space. Such metrics are called <u>Riemannian metrics</u>. We shall give a number of examples.

<u>References</u>. The standard (and probably best) reference for analytic and meromorphic functions is L.V. Ahlfors, <u>Complex Analysis</u>. It contains, in particular, a proof of the Riemann mapping theorem.

The elementary theory of surfaces is in many introductory texts on differential geometry, including the classic Struik and the more recent classic by do Carmo. An emphasis on surfaces in $\mathbb{R}^n$ is in the first chapter of R. Osserman, <u>A Survey of Minimal Surfaces</u>.

Gauss' basic paper (in Latin) on differential geometry together with an English translation and many useful comments are in a recent commemorative volume edited by Dombrowski.

A good source for Hilbert's Theorem is the beautiful set of lecture notes, <u>Differential Geometry in the Large</u>, published recently (1983) as volume 1000 of the Springer Lecture Notes in Mathematics.

Nash's embedding theorem appeared in Annals of Mathematics 63 (1956), pp. 20-63. Kuiper's theorem followed soon after, by a closer look at Nash's proof.

Riemann's collected works were printed in paperback by Dover in 1953. (All in German.) A great place to read both Gauss and Riemann (in English) as well as many clarifying comments and further developments is Volume 2 of Spivak's <u>Comprehensive Introduction to Differential Geometry</u>.