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Basic Riemannian Geometry (II)

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4. CURVATURE

From the last section we know that for each $p \in U$ the exponential map $\exp_p$ is a radial isometry (3.4). The deviation of $\exp_p$ to being an isometry is measured by curvature. This deviation is in turn best understood (cf 3.6) in terms of the induced map $(\exp_p)_*$.

Fix $p \in U$ and $v \in \mathcal{E}_p \subset T_pU$. To describe $(\exp_p)_{*v} : T_v\mathcal{E}_p \to T_{\exp_p(v)}U$ recall that any tangent vector $u_v \in T_v\mathcal{E}_p$ is represented by a curve $\sigma(s) = v + s\cdot u$ for some $u \in T_xU$. Hence $(\exp_p)_{*v}(u_v)$ is represented by the curve $s \to \exp_p(v + s\cdot u)$. As in the proof of (3.4) consider the variation

$$V(t,s) = \exp_p(t(v+su))$$

for $t \in [0,1]$ and s near zero. Using the notation from section 2 the variation field $X = X_0$ along $c = c_0$ is given by $X(t) = (\exp_p)_*(tu_{tv})$. Since each c_s is a geodesic and hence satisfies a second order equation we expect each X_s to do the same. Now $\nabla_{\frac{\partial}{\partial t}} \dot{c}_s = 0$ for all s and t, so

$$(4.1) \qquad 0 = \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} \dot{c}_s = \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} \dot{c}_s - \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \dot{c}_s + \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \dot{c}_s$$

$$= \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} \dot{c}_s - \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \dot{c}_s + \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial t}} X_s$$

in view of (2.6). To understand the first two terms replace $\dot{c}_s$ by a general vector field Z along V. A direct computation using the definition (2.5) gives

$$\nabla_s \nabla_t Z(t,s) = \frac{\partial^2}{\partial s \partial t} Z + D\Gamma_{V(t,s)}(\tfrac{\partial}{\partial s} V)(Z, \tfrac{\partial}{\partial t} V)$$

$$+ \Gamma_{V(t,s)}(\tfrac{\partial}{\partial s} Z, \tfrac{\partial}{\partial t} V) + \Gamma_{V(t,s)}(Z, \tfrac{\partial^2}{\partial s \partial t} V)$$

$$+ \Gamma_{V(t,s)}(\tfrac{\partial}{\partial t} Z, \tfrac{\partial}{\partial s} V) + \Gamma_{V(t,s)}(\Gamma_{V(t,s)}(Z, \tfrac{\partial}{\partial t} V), \tfrac{\partial}{\partial s} V))$$

and similarly for $\nabla_t \nabla_s Z(t,s)$. Consequently

(4.2) $$\nabla_s \nabla_t Z - \nabla_t \nabla_s Z = D\Gamma_{V(t,s)}(\tfrac{\partial}{\partial s} V)(Z, \tfrac{\partial}{\partial t} V) - D\Gamma_{V(t,s)}(\tfrac{\partial}{\partial t} V)(Z, \tfrac{\partial}{\partial s} V)$$

$$+ \Gamma_{V(t,s)}(\Gamma(Z, \tfrac{\partial}{\partial t} V), \tfrac{\partial}{\partial s} V) - \Gamma_{V(t,s)}(\Gamma(Z, \tfrac{\partial}{\partial s} V), \tfrac{\partial}{\partial t} V)$$

which clearly depends only on Z, $\frac{\partial}{\partial s} V$ and $\frac{\partial}{\partial t} V$ at (t,s). In view of this define for each $p \in U$ the 3-linear map $R_p : T_pU \times T_pU \times T_pU \to T_pU$ by

(4.3) $$R_p(u_p, v_p) z_p = [D\Gamma_p(u)(z,v) - D\Gamma_p(v)(z,u)$$

$$+ \Gamma_p(\Gamma_p(z,v),u) - \Gamma_p(\Gamma_p(z,u),v)]_p$$

for all $u, v, z \in \mathbb{R}^n$. This is called the <u>curvature tensor</u> for ∇ at p. Inserting this into (4.1) yields (take $s=0$)

$$(4.4) \qquad \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial t}} X + R(X,\dot c)\dot c = 0 .$$

This equation is called the <u>Jacobi equation</u>, and any vectorfield X along a geodesic c, which satisfies (4.4) is called a <u>Jacobi field</u> along c. The Jacobi equation is clearly a second order linear differential equation. In particular

<u>4.5 THEOREM</u> For any (maximal) geodesic $c : J \to U$ and tangent vectors $u, v \in T_{c(0)}U$ there is a unique Jacobi field $X : J \to TU$ along c with $X(0) = u$, and $\nabla_{\frac{\partial}{\partial t}} X(0) = v$.

Returning to (4.3) observe that for any $u_p, v_p, z_p \in T_pU$ there is a variation V and a vector field Z such that $V(0,0) = p$, $\frac{\partial V}{\partial s} V(0,0) = u$, $\frac{\partial}{\partial t} V(0,0) = v$ and $Z(0,0) = z$ (take e.g. $V(t,s) = p + t\cdot v + s\cdot u$ and $Z(t,s) = z$ for all t,s near zero). For general vector fields X, Y, and Z on U one finds

$$(4.6) \qquad R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z .$$

The easiest way to see this is to observe that the right hand side is linear with respect to functions in all three variables. It then follows that for each $p \in U$ it only depends on the values of X, Y and Z at p (i.e. the expression is "<u>tensorial</u>"). Choosing X, Y and Z as above then establishes (4.6).

From the above discussion it follows in particular that for $v \in \mathcal{E}_p \subset T_pU$ and $u \in T_pU$, the vector field $X(t) = \exp_*(tu_{tv})$ is the unique Jacobi field along $c_v : J_v \to U$ with initial conditions

$$(4.7) \qquad X(0) = 0_p \,, \quad \nabla_{\frac{\partial}{\partial t}} X(0) = u \,.$$

Since we already know that $\exp_p$ is a radial isometry, we are only interested in $\|X(t)\|_{\exp_p(tv)}$ compared to $\|tu\|_p$ when u is perpendicular to v. Assume therefore that $\|u\|_p = \|v\|_p = 1$ and $u \perp v$. Rather than estimating $\|X(t)\|$ directly we consider the smooth function $t \to \|X(t)\|^2 = \langle X(t), X(t)\rangle_{\exp_p(tv)} = f(t)$ and its Taylor series. We abreaviate from now on $\nabla_{\frac{\partial}{\partial t}} X$ to simply X'. Then

$$
\begin{aligned}
f(0) &= \langle X(t), X(t)\rangle (0) = 0 \\
f'(0) &= 2\langle X'(t), X(t)\rangle (0) = 0 \qquad\qquad (\text{cf. } 2.7) \\
f''(0) &= 2\{\langle X''(t), X(t)\rangle + \langle X'(t), X'(t)\rangle\}(0) = 2\|u\|^2 = 2 \\
f'''(0) &= 2\{\langle X'''(t), X(t)\rangle + 3\langle X''(t), X'(t)\rangle\}(0) = \\
&= 2\{\langle X'''(0), X(0)\rangle + 3\langle R(X(0), v)v, u\rangle\} = 0 \qquad (4.4) \\
f''''(0) &= 2\{\langle X''''(t), X(t)\rangle + 4\langle X'''(t), X'(t)\rangle + 3\langle X''(t), X''(t)\rangle\}(0) \\
&= 8\langle X'''(0), u\rangle
\end{aligned}
$$

By (4.4) $X''' = -(R(X,\dot c)\dot c)' = -R'(X,\dot c)\dot c - R(X',\dot c)\dot c -$

$R(X, \ddot{c}')\dot{c} - R(X, \dot{c})\ddot{c}'$. For $t=0$ we get $X'''(0) = -R(u,v)v$, and hence

$$(4.8) \qquad \| \exp_*(tu)_{tv} \|^2 = t^2 - \tfrac{1}{3}\langle R(u,v)v, v\rangle t^4 + O(t^5)$$

Here $\langle R(u,v)v, u\rangle$ only depends on span(u,v) and not the particular orthonormal basis u, v. For general independent u, v this number is

$$(4.9) \qquad K = \frac{\langle R(u,v)v, u\rangle}{\|u\|^2\|v\|^2 - \langle u, v\rangle^2}$$

and it is called the <u>sectional curvature</u> of the 2-plane, span(u,v). In view of (4.8), the sectional curvature determines whether $\exp_p$ is expanding ($K<0$) or contracting ($K>0$).

More or less straight forward purely algebraic manipulations yields

<u>4.10 THEOREM</u> The curvature tensor R of the Levi-Civita connection ∇ of a riemannian structure $\langle , \rangle$ on $U \subset \mathbb{R}^n$ satisfies the identities

(i) $\quad R(X,Y)Z = -R(Y,X)Z$

(ii) $\quad R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$

(iii) $\quad \langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle$

(iv) $\quad \langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$

for all vector fields X, Y, Z and W on U.

If we k denote the "biquadratic" form

$$(4.11) \qquad k(X,Y) = \langle R(X,Y)Y, X\rangle$$

then a lengthy computation gives

$$(4.12) \qquad \begin{aligned} \langle R(X,Y)Z, W\rangle = \tfrac{1}{6}\{ & k(X+W, Y+Z) - k(Y+W, X+Z) \\ & - k(X+W, Y) - k(X+W, Z) + k(Y+W, X) \\ & + k(Y+W, Z) - k(X, Y+Z) + k(Y, X+Z) \\ & + k(W, X+Z) - k(W, Y+Z) + k(X, Z) \\ & - k(Y, Z) - k(W, X) + k(W, Y)\} . \end{aligned}$$

A particular consequence of this formula is (cf 4.9 and 4.11) that the curvature tensor R is completely determined by the sectional curvature (and of course vice versa)! Another is that the <u>curvature tensor of a space of constant (sectional) curvature K</u> is given by

$$(4.13) \qquad R(X,Y)Z = K\{\langle Y,Z\rangle X - \langle X,Z\rangle Y\}$$

The <u>Ricci-tensor</u>, C_1R is defined for each $p \in L$ as the bilinear map

$$(4.14) \qquad C_1R(u_p, v_p) = \operatorname{trace}(w_p \to R(w_p, u_p)v_p)$$

for all $u_p, v_p \in T_pL$. Clearly $\text{Ricc.} = \dfrac{C_1R(u_p,u_p)}{\|u_p\|^2}$,

$u_p \neq 0$ only depends on the line spanned by u_p. This is called the <u>Ricci curvature</u>. Clearly

$$(4.15) \qquad \mathrm{Ricci}(\mathcal{L}) = \sum_{i=1}^{n-1} K(\sigma_i)$$

for $\mathcal{L} = \mathrm{span}\{X\}$ and $\sigma_i = \mathrm{span}\{X, u_i\}$, where $X, u_1, \dots, u_{n-1}$ is an orthonormal basis for T_pU. Averaging once more defines the <u>scalar curvature</u> $S: U \to \mathbb{R}$, i.e.

$$(4.16) \qquad S(p) = \sum_{i=1}^{n} c_1 R(u_i, u_i) = 2 \sum_{1 \le i < j \le n} K(\sigma_{ij}),$$

where $u_1, \dots, u_n$ is an orthonormal basis for T_pU and $\sigma_{ij} = \mathrm{span}(u_i, u_j)$. Note, that it is only for $n > 2$ that these curvatures are essentially different.

Using the concepts developed in this section it is not so difficult to show the existence of <u>convex balls</u> (metric balls in which any two points are joined by a unique minimal geodesic which is entirely contained in the ball)

<u>4.17 THEOREM</u> For every $p \in U$ there is an $\varepsilon > 0$ (depending on curvature bounds near p) such that $B(p, \varepsilon)$ is convex. In particular the inner metric on $B(p, \varepsilon)$ coincides with the induced metric from (U, d_g).

5 RIEMANNIAN SPACES

On the basis of (4.17) and (3.6) we are now ready to globalize the idea of a riemannian domain

5.1 DEFINITION An n-dimensional Riemannian space is an inner metric space (M^n, d) in which every point has an open neighborhood isometric to an n-dimensional Riemannian domain.

A cover of M^n by open sets U_α and corresponding isometries $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subset \mathbb{R}^n$ to Riemannian domains defines in view of (4.17) and (3.7) a smooth atlas on M. In particular, a Riemannian n-space is a smooth n-manifold. The description of tangent vectors in terms of curves clearly agrees no matter which local descriptions φ_α are used. Thus for ever $p \in M$ we have a well defined tangent space T_pM isomorphic to $\mathbb{R}^n$. Moreover the various descriptions of inner product $g_\alpha(p)$ via φ_α in T_pM also coincide. We therefor have an inner product g_p in each tangent space T_pM of M i.e. a Riemannian manifold in the usual sense. The converse is immediate from the approach in section 1.

Since (clearly) isometries of riemannian domains preserve Levi-Civita connections we get a connection ∇ defined on global vector fields of M and satisfying (2.15), (2.16), (2.17), (2.19) and (2.7) or equivalently $X\,g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$. This connection (the Levi Civita connection on M) is also given by the formula (2.24).

It is therfor clear that all the concepts introduced in the first 4 sections carry over to Riemannian spaces (manifolds) immediately. In particular, there is an exponential map, $\exp : \mathcal{O} \longrightarrow M$ defined on the open set $\mathcal{O} = \{ v \in TM \mid 1 \in J(v) \}$. Moreover, $\exp$ is smooth, $\exp(0_p) = p$ and $\exp_p : \mathcal{O}_p = \mathcal{O} \cap T_pM \rightarrow M$ is a diffeomorphism near $0_p \in T_pM$.

M is said to be <u>geodesically complete</u> provided $J(v) = \mathbb{R}$ for all tangent vectors $v \in TM$, or equivalently all maximal geodesics on M are defined on the whole real line. The following result is of fundamental importance

<u>5.2 HOPF-RINOW</u> A Riemannian space is a complete metric space if and only if it is geodesically complete. In this case any two points can be joined by a minimal geodesic.

The simplest Riemannian manifolds are those of constant curvature. These are also called space forms, and they play an important role in comparison theory.

Obviously the euclidean space $\mathbb{R}^n$ (with constant riemannian structure) is flat, i.e. has constant curvature zero. Moreover any flat, complete and simply connected Riemannian manifold is isometric to euclidean space (the exponential map provides an isometry).

To find examples with non zero curvature we can appeal to conformal changes of metrics. Two riemannian metric $g, \tilde{g}$ on a smooth manifold M are said to be conformally equivalent if there is a function $f: M \to \mathbb{R}$ such that $\tilde{g} = e^f \cdot g$. Similarly, a diffeomorphism $F: M \to \tilde{M}$ between riemannian manifolds is said to be conformal if $\tilde{g}(F_* u, F_* v) = e^f \cdot g(u,v)$. This is the case if and only if F_* preserves angles.

Now suppose $\tilde{g} = e^f g$, and let $\tilde{\nabla}$, ∇ be the Levi Civita connections of $\tilde{g}$, and g respectively. In order to express $\tilde{R}$ in terms of R and f we introduce the gradient vector field of f as

(5.3) $$g(\operatorname{grad} f, v) = df(v) = v[f]$$

for all tangent vectors v. Moreover, the Hessian tensor of f is given by

$$(5.4) \qquad H_f(X) = \nabla_X \operatorname{grad} f$$

and the correspondin Hessian form by

$$(5.5) \qquad h_f(X,Y) = g(H_f(X), Y).$$

A straight forward but lengthy computation based on (2.24) and (4.6) gives

5.6 THEOREM For vector fields X, Y, an Z on M,

(i) $\tilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}\{X[f]Y - Y[f]X - g(X,Y)\operatorname{grad} f\}$

(ii) $\tilde{R}(X,Y)Z = R(X,Y)Z +$
$$\tfrac{1}{2}\{h_f(X,Z)Y - h_f(Y,Z)X + g(X,Z)H_f(Y) - g(Y,Z)H_f(X)\}$$
$$+\tfrac{1}{4}\{(Y[f]Z[f] - g(Y,Z)\|\operatorname{grad} f\|^2)X$$
$$-(X[f]Z[f] - g(X,Z)\|\operatorname{grad} f\|^2)Y$$
$$+(X[f]g(Y,Z) - Y[f]g(X,Z))\operatorname{grad} f\}$$

and in particular

(iii) $e^{f(p)}\tilde{K}(\sigma) = K(\sigma) - \frac{1}{2}\{h_f(v,v) + h_f(u,u) + \frac{1}{2}(\|\operatorname{grad} f\|^2 - v[f]^2 - u[f]^2)\}$

for any two plane $\sigma \subset T_pM$ spanned by an orthonormal basis u, v relative to g_p. All right hand sides are expressed intirely in terms of g.

We will now use 5.6 when g is the usual flat riemannian metric on $\mathbb{R}^n$. For $k \in \mathbb{R}$, let

$$U_k = \begin{cases} \mathbb{R}^n & , \quad k \geq 0 \\ D^n(|k|^{-\frac{1}{2}}) = \{ p \in \mathbb{R}^n \mid \|p\|^2 < |k|^{-1} \} & , \quad k < 0 \end{cases}$$

be equipped with the riemannian structure $g_k = \varphi_k \cdot g$, where

$$\varphi_k = \frac{4}{(1 + k\|p\|^2)^2} \quad , \quad p \in U_k .$$

Thus g_k is conformally equivalent to the flat metric g and $f_k = \log \varphi_k = \log 4 - 2\log(1 + k\|p\|^2)$.

<u>5.7 THEOREM</u> The riemannian manifold (U_k, g_k) has constant curvature $h \equiv k$. Moreover, for $k \leq 0$ it is complete.

The proof is not difficult, and is left to the reader. The last part appeals to the part of Hopf-Rinow's theorem which assert that if for one $p \in M$, $\exp_p$ is defined on all of T_pM, then M is complete.

For $k > 0$ in (5.7), a picture shows that (U_k, g_k) is isometric via the stereographic projection to the sphere $S^n(\frac{1}{\sqrt{k}})$ minus one point. Thus $S^n(\frac{1}{\sqrt{k}})$ is a complet 1-connected manifold with curvature k.
We now claim that any complete simply connected space form with curvature k is isometric to one of the above spaces. Con-

proof is not via the Cartan-Ambrose-Hicks theorem but directly via the exponential map. For this purpose we need to understand the Jacobi fields X in a space of constant curvature (cf. section 4). These can be described in terms of <u>parallel fields.</u> Here a vector field Z along a curve $c : [a,b] \to M$ is called parallel if $\nabla_{\frac{\partial}{\partial t}} Z = Z' = 0$. This is a linear first order equation. Thus for every $u \in T_{c(a)}M$ there is a unique parallel field Z along c with $Z(a) = u$.

Now let $c : \mathbb{R} \to M$ be a geodesic in a manifold with constant curvature k. Then for $u, v \in \dot{c}(0)^{\perp}$ the unique Jacobi field X along c with $X(0) = u$ and $X'(0) = v$ is given by

$$(5.8) \qquad X(t) = cs_k(t)\, U + sn_k\, V ,$$

where U, V are the parallel fields along c determined by $u, v \in T_{c(0)}M$ and

$$sn_k(t) = \begin{cases} \frac{1}{\sqrt{k}} \sin \sqrt{k}\, t & , k > 0 \\ t & , k = 0 \\ \frac{1}{\sqrt{|k|}} \sinh \sqrt{|k|}\, t & , k < 0 \end{cases} \qquad cs_k(t) = \begin{cases} \cos \sqrt{k}\, t & , k > 0 \\ 1 & , k = 0 \\ \cosh \sqrt{|k|}\, t & , k < 0 \end{cases}$$

This follows directly from (4.4) and (4.13).

Let now M^n be a complete riemannian manifold with constant curvature k and let $\overline{M}^n_k$ be the corresponding complete simply connected model described above. Choose $p \in M$, $\bar{p} \in \overline{M}_k$ and identify T_pM and $T_{\bar{p}}\overline{M}_k$ by a linear isometry. According to (5.8) (with $U=0$) and the characterization of $\exp_{p*}$, $\exp_{\bar{p}*}$ in terms of Jacobi fields (cf section 4) it is obvious that $F = \exp_p \circ \exp_{\bar{p}}^{-1} : \overline{M}_k \to M$ is a local isometry when $k \leq 0$ and hence a covering map since $\overline{M}_k$ is complete. For $k > 0$ one first has to omit $-\bar{p}$ from $\overline{M}_k = S^n(\frac{1}{\sqrt{k}})$. However, without introducing new concepts it follows that $F : S^n(\frac{1}{\sqrt{k}}) - \bar{p} \to M$ extends uniquely to a local isometry $\overline{F} : S^n(\frac{1}{\sqrt{k}}) \to M$.

In particular any 1-connected complete manifold of constant curvature is isometric to one of the previously described model spaces.

We conclude this section with a brief discussion (riemannian) submanifolds. A subset $L \subset M^n$ is said to be a k-dimensional (embedded) <u>submanifold</u> of M if for every $p \in L$ there is a diffeomorphism $\phi : V \to U \subset \mathbb{R}^n$ defined near p such that $\phi(p) = 0$ and $\phi(V \cap L) = U \cap \mathbb{R}^k \times \{0\}$. Such diffeomorphisms are called submanifold charts. The collection $\{\varphi_{|L}\}$ form an atlas for L, making it a k-dimensional smooth manifold.

If M is a Riemannian manifold any submanifold $L \subset M$ will inherit a Riemannian structure. If the Riemannian distance on L coincides with that from M, we say that L is <u>totally geodesic</u>. In this case curvatures on L can simply be read off from curvatures of M. In general the curvature of L is desribed in terms of the curvature of M and the "relative curvature" of L in M, called the second fundamental form. If X and η are vector field along L in M with X every where tangent to L and η every where normal to L we define the <u>second fundamental form</u> by

$$S_\eta X = (\nabla_X \eta)^T \quad (5.9)$$

where ∇ is the Levi Civita connection on M and ^{T} denotes the tangential projection to L. S is bilinear with respect to functions and for every $p \in L$, $S_p : T_pL^\perp \times T_pL \to T_pL$ is a bilinear map. S also described the change in riemannian metrics of a family L_s of submanifolds obtained by varying L in direction of η. L is totally geodesic if and only if $S \equiv 0$.

The tensor S is also called the shape operator of L. It determines uniquely a bilinear map

$$(5.10)\qquad \alpha_p : T_pL \times T_pL \to T_pL^\perp$$

by the condition $\langle \alpha(X,Y), \eta\rangle = -\langle Y, S_\eta X\rangle$. This α is symmetric and (also) called the second fundamental form of $L \subset M$. For fixed normal vector $\eta_p \in T_pL^\perp$ we call $\ell_{\eta_p} = \langle \alpha_p(\cdot,\cdot), \eta_p\rangle$ the second fundamental form in direction η_p. Letting $\overline{R}$, $\overline{k}$ denote the curvature tensor of L, and its corresponding biquadratic form (cf. 4.11) we have

<u>5.11 THEOREM</u> Let M^{n+k} be a riemannian manifold and $L^n \subset M$ a submanifold with induced riemannian structure, and $n \geq 2$. For $p \in L$, $\eta_1, \dots, \eta_k \in T_pL^\perp$ an orthonormal basis and any $u, v, w, z \in T_pL$, we have

(i) $\overline{R}(u,v)w = (R(u,v)w)^T + \sum_{i=1}^{k} \{ \ell_i(v,w)\, S_i(u) - \ell_i(u,w)\, S_i(v)\}$

(ii) $\langle \overline{R}(u,v)w, z\rangle = \langle R(u,v)w, z\rangle + \sum_{i=1}^{k} \{\ell_i(v,w)\,\ell_i(u,z) - \ell_i(u,w)\,\ell_i(v,z)\}$

(iii) $\overline{k}(u,v) = k(u,v) + \sum_{i=1}^{k} \det \begin{pmatrix} \ell_i(u,u) & \ell_i(u,v) \\ \ell_i(u,v) & \ell_i(v,v)\end{pmatrix}$,

where $\ell_i = \ell_{\eta_i}$ and $S_i = S_{\eta_i}$.

This is the famous "Theorema egregium" of Gauss. The proof is straight forward use of the definitions. Using 5.11 is it also easy to see that $S^n(\frac{1}{\sqrt{k}})$ has constant curvature k.