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The hyperbolic plane and the Schwarz-Pick-Ahlfors Lemma

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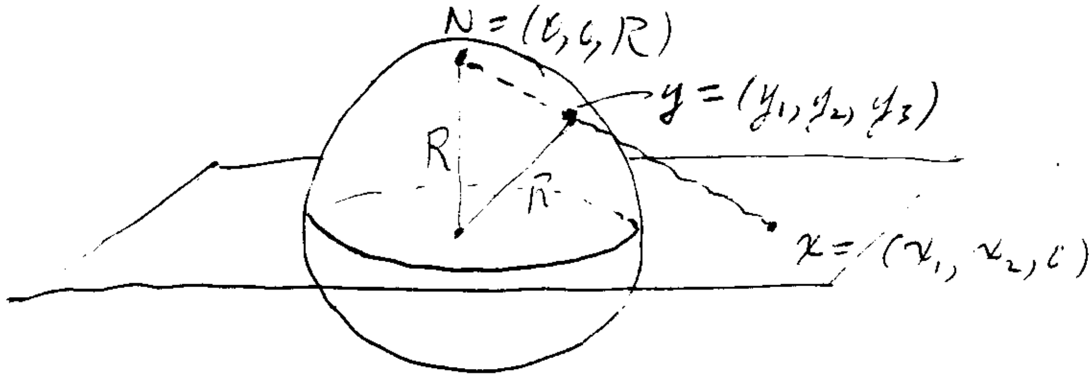
Lecture 3. The hyperbolic plane and the Schwarz-Pick-Ahlfors Lemma.

To construct examples of interesting conformal metrics, start with some defined via conformal maps

$$F: \mathbb{R}^2 \longrightarrow \mathbb{R}^3.$$

For example, let F be stereographic projection of the plane into the sphere of radius R:



We may consider the plane $\mathbb{R}^2$ to be the horizontal plane through the origin so that $(x_1, x_2) \longleftrightarrow (x_1, x_2, 0)$. Then $y = F(x)$ is the intersection with the sphere $y_1^2 + y_2^2 + y_3^2 = R^2$ of the line through x and the North pole $N = (0, 0, R)$. Then

$$y = tx + (1-t)N \quad \text{for some } t \in \mathbb{R}$$

(since the line through x and N consists of all points of the given form as t runs through the reals.) To determine t, use the fact y lies on the sphere:

$$R^2 = |y|^2 = y\cdot y = t^2|x|^2 + (1-t)^2|N|^2 \quad (\text{since } x \perp N)$$
$$= t^2|x|^2 + (1-2t+t^2)R^2$$
$$= R^2 - 2R^2 t + (|x|^2 + R^2)t^2$$

or $\quad 2R^2 t = (|x|^2 + R^2)t^2$

Hence either $t=0$, which gives the point N, or $t \neq 0$, in which case $\quad t = \dfrac{2R^2}{|x|^2 + R^2}$.

Then $\quad 1 - t = \dfrac{|x|^2 - R^2}{|x|^2 + R^2}$.

Putting these values in the equation for y, and writing out that equation in components gives for $F(x)$:

$$(y_1, y_2, y_3) = \left(\frac{2R^2}{|x|^2+R^2}x_1,\ \frac{2R^2}{|x|^2+R^2}x_2,\ \frac{|x|^2-R^2}{|x|^2+R^2}R\right).$$

This is the explicit equation for stereographic projection. (Exactly the same derivation, without change, gives stereographic projection $\mathbb{R}^n \longrightarrow S^n(R) \subset \mathbb{R}^{n+1}$.)

One can then compute $g_{ij} = \dfrac{\partial y}{\partial x_i}\cdot\dfrac{\partial y}{\partial x_j} = \sum\limits_k \dfrac{\partial y_k}{\partial x_i}\dfrac{\partial y_k}{\partial x_j}$.

The answer turns out to be

$$g_{ij} = t^2 \delta_{ij}, \quad \text{where } t = \frac{2R^2}{|x|^2 + R^2}.$$

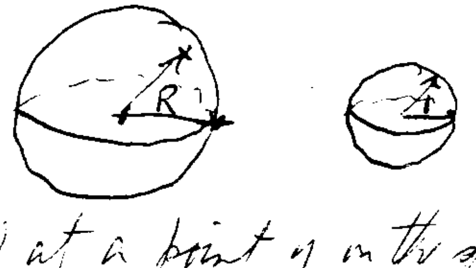
(Thus the conformal stretching factor happens to be precisely equal to the coefficient of x in the original expression for y.) We therefore get the conformal metric

$$\lambda^2 \left|\frac{dz}{dt}\right|^2 = \left(\frac{ds}{dt}\right)^2$$

where we have gone back to complex notation: $z = x_1 + i x_2$, and

$$\lambda = \frac{2R^2}{R^2 + |z|^2} = \frac{2}{1 + \frac{1}{R^2}|z|^2}.$$

The interesting thing is the coefficient $\frac{1}{R^2}$ of $|z|^2$ in this expression. That is in fact exactly the Gauss curve of the image sphere, as follows directly from Gauss' definition:



the normal N at a point y on the sphere of radius R is just $N = \frac{y}{R}$. Hence the Gauss map G is the map

$$G: y \longmapsto \frac{y}{R}$$

and the matrix of dG is $\begin{pmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{R} \end{pmatrix}$.

Hence $\quad K = \det dG = \frac{1}{R^2}$.

But the *Theorema Egregium* tells us that

$$K = -\frac{\Delta \log \lambda}{\lambda^2}.$$

We conclude that

(1) $$\lambda(z) = \frac{2}{1+\alpha|z|^2} \implies -\frac{\Delta \log \lambda}{\lambda^2} = \alpha \quad \text{for all } \alpha > 0.$$

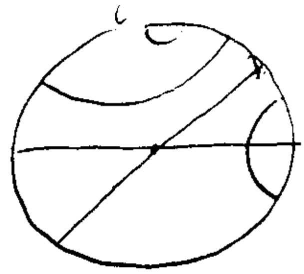
Of course that could be checked by a direct calculation. But where in the calculation does one use $\alpha > 0$? Nowhere! In fact, (1) holds for any constant α, positive, negative, or zero. That led Riemann to suggest studying the conformal metrics

(2) $$\frac{ds}{dt} = \frac{2}{1+\alpha|z|^2}\left|\frac{dz}{dt}\right|$$

for *all* α. In particular, if $\alpha = -1$, then the metric defined by (2), with $\lambda = \frac{2}{1-|z|^2}$

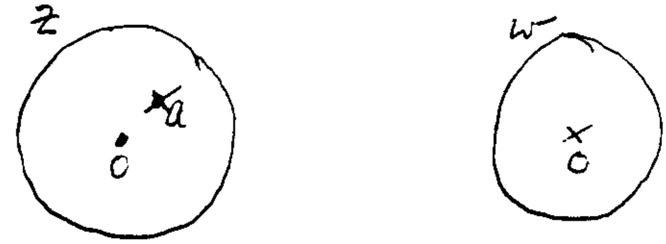
is defined in the unit disk $|z|<1$, and by (1), it has constant Gauss curvature $K \equiv -1$. The unit disk with that metric is called the _hyperbolic plane_.

Although Riemann described all this (and more) in the paper he submitted in 1854 for his "_Habilitation_", it was not published until 1867, a year before Beltrami's famous paper on non-euclidean geometry. The two papers together played a profound role in the acceptance of non-euclidean geometry, which until that point roused little interest. However, it does not appear that it was realized until even later that the metric (2), with $\alpha = -1$, provides an _exact_ model for Lobachevsky's geometry, where the set of "lines" in the non-euclidean plane consists of the set of lines through the origin together with circular arcs orthogonal to the boundary circle.



To see why that is the case, consider a _linear fractional_ transformation of the unit disk:

(3) $$w = \frac{z-a}{1-\bar{a}z}, \quad |a|<1$$



This maps $|z|<1$ onto $|w|<1$ and $a \mapsto 0$. One finds

$$\frac{dw}{dz} = \frac{1-|a|^2}{(1-\bar{a}z)^2}$$

and

$$1-|w|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2}.$$

Using the hyperbolic metric in the two disks:

$$\lambda(z)\left|\frac{dz}{dt}\right| = \frac{2}{1-|z|^2}\left|\frac{dz}{dt}\right| \quad \text{and} \quad \lambda(w)\left|\frac{dw}{dt}\right| = \frac{2}{1-|w|^2}\left|\frac{dw}{dt}\right|,$$

we find that

$$\lambda(w)\left|\frac{dw}{dt}\right| = \frac{2}{1-|w|^2}\left|\frac{dw}{dz}\right|\left|\frac{dz}{dt}\right| = \frac{2|1-\bar{a}z|^2}{(1-|a|^2)(1-|z|^2)} \cdot \frac{1-|a|^2}{|1-\bar{a}z|^2}\left|\frac{dz}{dt}\right|$$

$$= \frac{2}{1-|z|^2}\left|\frac{dz}{dt}\right| = \lambda(z)\left|\frac{dz}{dt}\right|.$$

That means that if C is any smooth curve defined by $z(t)$, $a \le t \le b$, in $|z|<1$, and C' is the image curve,

$$\text{length of } C' = \int_a^b \lambda(w(t))\left|\frac{dw}{dt}\right| = \int_a^b \lambda(z(t))\left|\frac{dz}{dt}\right| = \text{length of } C$$

where "length" means "in the hyperbolic metric".

⑦

Thus, the transformation (3) preserves the hyperbolic lengths of all curves, or is an <u>isometry</u>.

For any two points z_1, z_2 in $|z|<1$, the <u>hyperbolic</u> or <u>non-euclidean distance</u> $d(z_1, z_2)$ is defined as

$$d(z_1, z_2) = \inf \text{ length } C$$

over all curves C in $|z|<1$ joining z_1 to z_2. It is easy to see that z_1 and z_2 lie on the same ray through the origin, then the line segment joining them is the shortest curve or <u>geodesic</u> from z_1 to z_2. In that case

$$d(z_1, z_2) = \int_{|z_1|}^{|z_2|} \frac{2}{1-r^2}\, dr \quad , \quad \text{for } |z_2| > |z_1|.$$

But rays through the origin are carried by the map (3) into circles orthogonal to the boundary circle, and since (3) defines an isometry, these orthogonal circles are geodesics between any pair of points on them. Thus, the "lines" of non-euclidean geometry are still geodesics, or shortest curves joining points on them.

Finally, we give an example that is very important, but less explicit.

Let $w_1, \ldots, w_n$ be any finite set of points in $\mathbb{C}$, and let D be the whole plane $\mathbb{C}$ with the points w_k deleted.

Theorem. If $n \geq 2$, then there exists a conformal metric $\lambda(w) \left|\frac{dw}{dt}\right|$ on D satisfying

1. $K = -\dfrac{\Delta \log \lambda}{\lambda^2} \equiv -1$,

2. near each w_j, $\rho(w) \sim \dfrac{c_j}{|w-w_j| \log|w-w_j|}$, $c_j \neq 0$,

3. near ∞, $\rho(w) \sim \dfrac{c_0}{|w| \log|w|}$, $c_0 \neq 0$.

The existence of this metric follows from the fact that the universal covering surface of D can be mapped conformally onto the unit disk $|z| < 1$, and that the hyperbolic metric on $|z|<1$ can be "pulled back" to a metric on D. That pulled-back metric has the properties listed.

(9)

Schwarz' Lemma. Let $f(z)$ be analytic and satisfy $|f(z)| < 1$ in $|z|<1$. If $f(0)=0$, then $|f'(0)| \le 1$.

The proof is elementary, but this lemma will also follow from a much more general one given below.

First note that geometrically, the Schwarz lemma states that under the hypotheses, the map f is a contraction at the origin; for any curve $z(t)$ with $z(0)=0$, the image curve $w(z(t))$ satisfies

$$\left|\frac{dw}{dt}\right|_{t=0} = \left|\frac{dw}{dz}\right|_{z=0} \left|\frac{dz}{dt}\right|_{t=0} \le \left|\frac{dz}{dt}\right|_{t=0}.$$

(Schwarz' lemma goes on to say that it is a strict contraction, $|f'(0)| < 1$, unless f is a rotation. But we don't need that.)

It follows that $f(z)$ also defines a contraction in the hyperbolic metric at the origin, since

$$\lambda(z) = \frac{2}{1-|z|^2}, \quad \lambda(0) = 2,$$

and

$$\lambda(0)\left|\frac{dw}{dt}\right|_{t=0} = 2\left|\frac{dw}{dt}\right|_{t=0} \le 2\left|\frac{dz}{dt}\right|_{t=0} = \lambda(0)\left|\frac{dz}{dt}\right|_{t=0}.$$

What about at any other point, a? If $f(a)=b$, let

$$Z=\frac{z-a}{1-\bar a z}, \quad W=\frac{w-b}{1-\bar b w}.$$

Then $w=f(z)$ induces a map $W=F(Z)$, with $F(0)=0$ (since $Z=0 \Leftrightarrow z=a$ + $W=0 \Leftrightarrow w=b$). By Schwarz' Lemma, F is a hyperbolic contraction at the origin; but the correspondences $z \Leftrightarrow Z$ and $w \Leftrightarrow W$ are hyperbolic isometries. Hence

$$(4) \qquad \underset{(\text{at } b)}{\frac{2}{1-|w|^2}\left|\frac{dw}{dt}\right|} = \underset{(\text{at } 0)}{2\left|\frac{dW}{dt}\right|} \le \underset{(\text{at } 0)}{2\left|\frac{dZ}{dt}\right|} = \underset{(\text{at } a)}{\frac{2}{1-|z|^2}\left|\frac{dz}{dt}\right|};$$

i.e. $f(z)$ is a hyperbolic contraction everywhere. (This last inequality can also be proved by a direct computation.)

Inequality (4) is the Schwarz-Pick lemma: If $|f(z)|<1$ for $|z|<1$, then hyperbolic lengths are (weakly) decreasing.

$$\text{hyp. length}(f(C)) \le \text{hyp. length}(C)$$

for every curve C in $|z|<1$. That is,

$$\int_{f(C)} \frac{2}{1-|w|^2}\,|dw| \le \int_C \frac{2}{1-|z|^2}\,|dz| \quad . \tag{5}$$

The next step was taken by Ahlfors who thought of (5) in somewhat different terms. Rather than comparing the hyperbolic lengths of the curve C and its image under the map f, Ahlfors thought of comparing the lengths of C in two different metrics, the original one, with $\lambda(z) = \frac{2}{1-|z|^2}$ and the "pulled-back metric" from the w-plane under the map f. In fact, the left side of (5) can be written as an integral over C

$$\int_{f(C)} \frac{2}{1-|w|^2}\,|dw| = \int_C \frac{2}{1-|f(z)|^2}\,|f'(z)|\,|dz|$$

so that one is really comparing the metrics

$$\lambda(z) = \frac{2}{1-|z|^2} \quad \text{and} \quad \hat{\lambda}(z) = \frac{2\,|f'(z)|}{1-|f(z)|^2}$$

in the unit disk. Inequalities (4) and (5) simply say

$$\hat{\lambda}(z) \le \lambda(z) \quad \text{for all } z \in \{|z|<1\}. \tag{6}$$

Somewhat in the spirit of Riemann, Ahlfors asked, what if one compares the hyperbolic metric $\lambda(z)$ with any other metric $\hat{\lambda}(z)$ in the unit disk, regardless of whether $\hat{\lambda}(z)$ arises as the pull-back metric under a mapping. The answer he found was beautifully simple.

Lemma (Ahlfors). Let $\lambda(z)$ be the standard hyperbolic metric: $\lambda(z) = \frac{2}{1-|z|^2}$, and let $\hat{\lambda}(z)$ define another conformal metric in $D: |z| < 1$. If the curvature $\hat{K}$ of $\hat{\lambda}$ satisfies

$$\hat{K} \leq -1 \quad \text{everywhere in } D$$

then

$$\hat{\lambda}(z) \leq \lambda(z) \quad \text{in } D.$$

Proof. Assume first that $\hat{\lambda}(z)$ is continuous in the closed disk $|z| \leq 1$. We set

(7) $$u = \log \lambda, \quad v = \log \hat{\lambda}$$

and

(8) $$h = v - u.$$

Since $\lambda(z) = \frac{2}{1-|z|^2} \to \infty$ as $|z| \to 1$, and we are assuming that $\hat\lambda$ and therefore v is continuous, hence bounded, in $|z| \le 1$, it follows that

$$h(z) = \log \hat\lambda(z) - \log \lambda(z) \to -\infty \quad \text{as } |z| \to 1.$$

It follows that $h(z)$ must assume a maximum value M at some interior point.

(9) $$\max_{|z|\le 1} h(z) = h(z_0) = M.$$

We wish to show that $M \le 0$, which implies by (7) and (8) that $v \le u$, or that $\hat\lambda \le \lambda$ in D. We argue by contradiction. If $M > 0$, then by (9), $h(z) > 0$ in some neighborhood, say $|z - z_0| < \delta$. That means

(10) $$v(z) > u(z) \quad \text{for } |z - z_0| < \delta.$$

But then

(11) $$\Delta h = \Delta \log \hat\lambda - \Delta \log \lambda = -\hat K \hat\lambda^2 - \lambda^2 \quad \left(\text{since } K = -\frac{\Delta \log \lambda}{\lambda^2} = -1\right)$$
$$\ge \hat\lambda^2 - \lambda^2 \quad (\text{since } \hat K \le -1 \Rightarrow -\hat K \ge 1)$$
$$= e^{2v} - e^{2u} \quad (\text{by } (7))$$
$$> 0 \quad \text{in } |z - z_0| < \delta \quad (\text{by } (10)).$$

But $h(x)$ has a maximum at z_0 implies

$h_x = h_y = 0$ at z_0, and

$h_{xx} \leq 0$, $h_{yy} \leq 0$ at z_0, so that

$\Delta h = h_{xx} + h_{yy} \leq 0$ at z_0.

This contradicts (11), hence the assumption that $M > 0$, and therefore proves the lemma, under the hypothesis that $\hat{\lambda}$ was continuous in the closed disk. To eliminate that hypothesis, choose any $R < 1$ and set

$$\lambda_R(z) = \frac{2R}{R^2 - |z|^2} \quad \text{in } |z| < R.$$

Then since $\hat{\lambda}(z)$ is continuous in $|z| \leq R$, and since the Gauss curvature of $\lambda_R(z)$ is again -1, the above argument applies without change in the disk $|z| < R$, to show that

$$\hat{\lambda}(z) \leq \lambda_R(z) \quad \text{for } |z| < R.$$

But this holds for all $R < 1$, and as $R \to 1$, $\lambda_R(z) \to \lambda(z)$, so that $\hat{\lambda}(z) \leq \lambda(z)$ in $|z| < 1$.

qed

Ahlfors goes on to show that the lemma holds in much more generality, where $\hat{\lambda}$ is allowed to have certain kinds of singularities, as long as one can assure that the maximum of the function h in the proof occurs away from the singularities. The simplest example is to allow $\hat{\lambda}(z)$ to degenerate on a set E; that is, $\hat{\lambda}(z) = 0$ for $z \in E$. Then the function $v(z) = \log \hat{\lambda}(z) \to -\infty$ as $z \to a \in E$, and the same is true of $h(z) = v - u$. Thus h must take on its maximum at a point $z_0 \in D \setminus E$, and argument in the proof goes through without change.

In particular, the original Schwarz-Pick lemma was the statement $\hat{\lambda}(z) \le \lambda(z)$, with

$$\hat{\lambda}(z) = \frac{2|f'(z)|}{1-|f(z)|^2}, \qquad \lambda(z) = \frac{2}{1-|z|^2},$$

and this $\hat{\lambda}(z)$ has zeros wherever $f'(z) = 0$. But the conclusion still holds, as indicated, since away from the zeros of f', $\dfrac{\Delta \log \hat{\lambda}}{\hat{\lambda}^2} = -1$, as one can verify either by a computation, or by observing that by the very definition of a pulled-back metric, f is an <u>isometry</u> between the metric

$$\frac{2|f'(z)|}{1-|f(z)|^2}|dz| \quad \text{in } |z|<1$$

and $\frac{2}{1-|w|^2}|dw|$ ~~in the~~ in the image.

But Gauss curvature is preserved under isometries, and the image has $K \equiv -1$, hence so does the pull-back.

References. The metric (2), with α either positive or negative is given (in a very slightly different form) by Riemann in his <u>Habilitationsschrift</u>; see Riemann's Collected Works, Dover 1953, p. 282.

The Schwarz Lemma is in most introductory books on complex variables, including Ahlfors' <u>Complex Analysis</u>, where the Schwarz-Pick Lemma is given as an exercise.

Ahlfors' Lemma is in his paper "An extension of Schwarz's Lemma," Trans. Amer. Math. Soc. 43 (1938), 359-364.

Pick proved his version of Schwarz' Lemma in 1916: Mathematische Annalen 77.

Nevanlinna's book, <u>Analytic Functions</u>, Springer 1970, makes extensive use of conformal metrics. Schwarz-Pick is on p. 48, the hyperbolic metric on the plane with a finite set of points removed is on pp. 248-250, and many related results are given in Chapter IV.