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THIRD AUTUMN WORKSHOP ON ATMOSPHERIC
RADIATION AND CLOUD PHYSICS
27 November - 15 December 1989

"Principles and General Equations of Radiative Transfer"

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Please note: These are preliminary notes intended for internal distribution only.

If the radiant flux is falling at a target one speaks about irradiance and uses the symbol E instead of M.

The integration of the radiant ~~energy~~ flux density over a defined area yields the radiant flux ϕ which is measured in Watt. This is the radiant energy Q transferred in time:

$$\phi = dQ / dt,$$

where the energy is measured in Joule (Ws).

Often it is desirable to express the amount of energy that is received by an object during a certain time. This can be expressed by the radiant exposure

$$H = \int_{\Delta t} E \, dt \quad \text{in } J m^{-2}.$$

The term radiant intensity I may only be used in connection with point radiation sources:

$$I = d\phi / d\Omega \quad W sr^{-1}.$$

Finally we have the energy density in a volume, which can be expressed by

$$u = \frac{4\pi}{c} L \quad J m^{-3}$$

with c = velocity of light.

Terminology and Definitions

Radiant energy transferred per unit time across unit area into a specific direction denoted by a cone with unit solid angle is a conservative quantity of the radiation field that can only be changed by interaction with matter. This quantity is called radiance (L) and measured in $W m^{-2} sr^{-1}$. If α is the (linear) opening angle of the cone, the solid angle Ω is related to α by $\Omega = 2\pi(1 - \cos\alpha)$ and measured in steradian (sr).

The total amount of energy transferred per unit time across unit area is called the radiant flux (area) density M and is measured in $W m^{-2}$. M can be computed by integrating the radiance over the hemisphere taking into account that the crossed unit area changes its cross section with $\cos\vartheta$ if ϑ is the angle between the area normal and the direction of the radiance.

The same symbol M is also used if a radiation source is considered, but is then called radiant exitance.

Table: Radiometric Quantities

Name	Symbol	Unit	Definition
Radiant Energy	Q	J (Ws)	
Radiant Flux	ϕ	W	$\phi = dQ/dt$
Radiant Flux Density	M	$W m^{-2}$	$M = d^2Q/dA\,dt$
Radiant Exitance	M	$W m^{-2}$	$= d\phi/dA$
Irradiance	E	$W m^{-2}$	$E = d\phi/dA$
Radiance	L	$W m^{-2} sr^{-1}$	$L = \frac{d^2\phi}{dA \cos\vartheta \, d\Omega}$
Radiant Exposure	H	$J m^{-2}$	$H = \int_{\Delta t} E\,dt$
Radiant Intensity	I	$W sr^{-1}$	$I = d\phi/d\Omega$
Energy Density of the Radiation Field	u	$J m^{-3}$	$u = \frac{4\pi}{c} L$

A = area
ϑ = angle between direction of radiance and area normal
c = velocity of light
t = time

Extinction Law (Lambert and Bouguer)

A radiance that is transmitted through a medium (e.g. air) is partly extinguished by absorption and scattering processes. If the radiance L enters the medium then after passing the distance ds the radiance $L + dL$ will leave the medium. dL will be proportional to L and the pathlength:

(1) $$dL = -\sigma L \, ds$$

The proportionallity factor σ is called the extinction coefficient. It can be expressed in m^{-1} if s is given in m.

Since the extinction can be due to absorption of gases or aerosol and to scattering at molecules or aerosols it can be splitted up as follows

(2) $$\sigma = a_{gas} + a_{aerosol} + \sigma_R + \sigma_M .$$

a_{gas} and $a_{aerosol}$ are the absorption coefficients for gases and aerosol respectively. σ_R is the Rayleigh scattering coefficient of molecules and σ_M the Mie scattering coefficient for aerosols (solid particles and cloud droplets). The two scattering coefficients can be combined to one:

(3) $$\sigma_{scat} = \sigma_R + \sigma_M$$

It is sometimes usefull to define the extinction coefficients with respect to the interactive mass expressed by the density ρ rather than the pathlength. The mass per unit area is given by $\rho\, ds$. In this case the extinction and absorption coefficients have to be expressed in kg

(4) $$dL = -\left(a_{gas}\,\rho_{gas} + a_{aer.}\,\rho_{aer} + \sigma_R\,\rho_{air} + \sigma_M\,\rho_{aer.}\right) L\, ds.$$

N.b. that ρ_{gas} is the density of the absorbing gas only, not of the air!

From equation (1) follows by integration from pathlength 0 to s:

$$L(s) = L(0)\, e^{-\int \sigma\, ds}$$

The integral in the exponent can only be solved if either $\sigma = \text{const}$ or $\sigma = \sigma(s)$ is known along the path. The exponential function is called transmission τ:

$$\tau = e^{-\int \sigma\, ds} = e^{-u}$$

where $u = \int \sigma\, ds$ is the optical pathlength.

Optical Pathes in the Atmosphere

In a horizontally homogeneous atmosphere it is often appropriate to relate all radiation computations to the vertical direction z. If the direction of a radiance is given by a vector $\vec{s}$ the following relation holds between the increments dz and ds:

$$ds = \frac{dz}{\cos\zeta} = \frac{dz}{\mu}$$

where $\mu = \cos\zeta$ is the cosine of the angle between the vertical (positive in the upward direction) and the direction into which the radiation proceeds.

This relation does not hold for the curved earth but is a good approximation up to $\zeta \approx 75°$ even in that case. More generally μ^{-1} has to be interpreted as the relative airmass which is not infinite for $\zeta = 90°$ but limited to approximately 40.

It is also useful to introduce the (vertical) optical depth δ of the atmosphere which is the negative product of the height and the extinction coefficient:

$$d\delta = -\sigma\, dz$$

All radiation quantities are so far regarded as monochromatic quantities, valid for one single wavelength only. Since the extinction and absorption coefficients are all wavelength dependent, the integration over a certain wavelength intervall constitutes a major problem. As an example: the molecular ~~extinction~~ scattering coefficient decreases with increasing wavelength λ proportional to $\lambda^{-4.09}$, the aerosol scattering coefficient approximately to $\lambda^{-1.5}$. As we shall see the gas absorption coefficients can vary over several orders of magnitude within very narrow spectral intervals.

The transmission averaged over a wavelength interval $\Delta\lambda$ around λ must therefore be written

$$\tau(\lambda, \Delta\lambda) = \frac{1}{\Delta\lambda} \int e^{-\int \sigma(\lambda)\, ds} \, d\lambda$$

Often the extinction law is also presented in decadal form for the whole atmosphere:

$$L(z=0) = L(z=\infty)\, 10^{-\sigma'_\lambda \cdot m}$$

where m is the relative airmass, and for the total spectrum

$$\int L(z=0)\, d\lambda = \int L(z=\infty)\, 10^{-\sigma'_\lambda \cdot m}\, d\lambda = \int L(z=\infty)\, d\lambda \cdot 10^{-\sigma'(m) \cdot m}$$

In this notation the extinction coefficient becomes dependent on the airmass

The optical depth is zero at the top of the atmosphere and is related to the optical path length by

$$d\delta = du \cdot \cos\zeta = \mu \, du$$

Atmospheric turbidity caused by aerosols is sometimes taken care of by a turbidity factor T which is defined in the following way. If we write in the optical depth explicitly ~~with average value for the whole spectrum~~,

$$d\delta = -\left(\sigma_R + \sigma_M + a_{gas} + a_{aerosol}\right) dz$$

$$= -\left(1 + \frac{\sigma_M}{\sigma_R} + \frac{a_{gas}}{\sigma_R} + \frac{a_{aeros}}{\sigma_R}\right) \sigma_R \, dz = -T \sigma_R \, dz$$

then T can be interpreted as the number of Rayleigh atmospheres, which have to be put on top of each other to provide the same extinction effect as the real atmosphere. The values of range between 1.5 and 5 (for extremely polluted areas) if the whole solar spectrum is considered.

Radiation Laws

a) M. Planck

The energy density within a closed system can be described by a function that is only dependent on the Temperature. Planck derived this law for a harmonic oscillator. He called the closed system a black body. If the frequency of the oscillator is f the energy density can be written as

$$u_{f,B} = u_f(T) = \frac{8\pi h f^3}{c^3} \frac{1}{e^{\frac{hf}{kT}} - 1}$$

The term $e^{hf/kT}$ is due to the statistical distribution of the energy levels that are excited in the closed system. This leads to the following formula for the radiance of a black body:

$$L_{B,f} = \frac{c}{4\pi} u_f = \frac{2hf^3}{c^2} \frac{1}{e^{hf/kT} - 1}$$

This expression can be rewritten in terms of the wavelength λ or the wavenumber ν by the relation

$$f = c/\lambda = \nu c \quad ; \quad \nu = 1/\lambda \ \mathrm{cm}^{-1}$$

to give

$$L_{B,\nu} = \frac{c_1}{\pi} \nu^3 \frac{1}{e^{c_2 \nu / T} - 1}$$

$$L_{B,\lambda} = \frac{c_1}{\pi} \lambda^{-5} \frac{1}{e^{c_2/\lambda T} - 1}$$

$$c_1 = 2\pi h c^2 = 3.7418 \cdot 10^{-16} \ W m^2$$
$$c_2 = hc/k = 1.4388 \ cm \ K$$

In using these formula it must be recognized that the $L_{B\lambda}$, $L_{B,\nu}$ are spectral quantities from which the energy per time and area follows by multiplication with the spectral interval.

$$L_B(\lambda, \Delta\lambda) = L_{B,\lambda} \Delta\lambda$$
$$L_B(\nu, \Delta\nu) = L_{B,\nu} \Delta\nu$$

and because $\nu = 1/\lambda$, $d\nu = - \frac{d\lambda}{\lambda^2}$ it follows with $\Delta\nu = - \Delta\lambda/\lambda^2 = - \nu^2 \Delta\lambda$; $- \Delta\nu/\Delta\lambda = + \nu^2 = + \lambda^{-2}$

$$L_{B,\lambda} = \nu^2 L_{B,\nu}$$

A Black Body can experimentally be manifested by a double walled box kept at constant temperature with an opening of an area smaller than 1/100 of the surface of the inner wall.

b) Stefan - Boltzmann

If the Planck law is integrated over all wavelengths respectively wavenumbers, a simple formula results for the total emission of a Black Body into the hemisphere

$$M_B = \pi \int_0^\infty L_B \, d\nu = \sigma T^4$$

where $\sigma = \dfrac{2\pi^5 k^4}{15 c^3 h^3} = 5.\cancel{6}\cancel{6}7961 \cdot 10^{-8}\ Wm^{-2}K^{-4}$

For simplification the Black Body radiance is often written $B_\nu(T) = L_{B,\nu}$ or $B_\lambda(T) = L_{B,\lambda}$.

Scattering Processes: Rayleigh scattering

Scattering in the atmosphere takes place at molecules and larger particles, aerosols and cloud droplets. The interaction of molecules with the radiation field can be treated like the antenna problem as long as no absorption phenomena take place. An antenna radiates by periodic changes of their dipole moment, the distance of positive and negative . The radiation is maximum at 90 degrees off the antenna axis and zero in the direction of this axis. Since the axis of the molecules are statistically distributed ~~and~~ the scattered radiation field will have rotational symmetry with respect to the direction of the exiting light wave. The pase function, which results, is given by

$$\phi(\vartheta) = \frac{3}{4}\left(1+\cos^2\vartheta\right)$$

while the scattering extinction coefficient is

$$\sigma_R = \frac{32\,\pi^3}{3N\,\lambda^4}\,(n-1)^2$$

The General Equation of Radiative Transfer

To derive the general equation of radiative transfer we consider a beam of radiation at an arbitrary point P at pressure p and altitude z in the atmosphere. The direction of the radiance at this point is denoted by the vector $\vec{s}$ with the length s. Between the increment of the height dz and the increment of the pathlength ds exist the relation

$$ds = dz / \mu$$

if $\mu = \pm \cos \zeta$ with ζ = angle between the vertical and $\vec{s}$.

Within the small distance ds the radiance L is changed to $L + dL$. The change dL is due to absorption, emission and scattering processes which we can describe as follows:

Absorption: $(dL)_{abs} = - a(P) \cdot L(P)\, ds$

Scattering out of the path:

$$(dL)_{scatt,\,out} = - \sigma_{scat}(P)\, L(P)\, ds$$

Emission:

$$(dL)_{em} = a(P)\, J_{em}\, ds$$

Scattering into the path ds from all other directions $ds'(\mu',\varphi')$:

$$(dL)_{scat,\,in} = \frac{1}{4\pi}\int_{\mu'=1}^{+1}\int_{\varphi'=0}^{2\pi} L(P,\mu',\varphi')\,\phi(P,\mu,\varphi,\mu',\varphi')\,\sigma_{scat}(P)\,d\mu'd\varphi'ds$$

$$= \sigma_{scat}(P)\,J_{scat}\,ds$$

$$J_{scat} = \frac{1}{4\pi}\iint L(P,\mu',\varphi')\,\phi(P,\mu,\varphi,\mu',\varphi')\,d\mu'd\varphi'$$

The total change in the path s is

$$\frac{dL}{ds} = \sum\left(\frac{dL}{ds}\right)_i$$

$$= -L(P)\left[a(P)+\sigma_{scat}(P)\right] + a(P)\,J_{em} + \sigma_{scat}\,J_{scat}$$

If we define

$$a(P) + \sigma_{scat}(P) = \sigma(P) \quad \text{(total extinction)}$$

$$J_{em} = \frac{\sigma(P)}{a(P)}\hat{J} \quad \text{and} \quad J_{scat} = \frac{\sigma(P)}{\sigma_{scat}}\tilde{J}$$

as well as

$$\hat{J} + \tilde{J} = J^*$$

we can write

$$\frac{dL}{ds} = -\sigma(P)\,L(P) + \sigma(P)\,J^*$$

or

$$\frac{1}{\sigma(P)}\frac{dL}{ds} = - L(P) + J^*(P)$$

By introducing the optical depth of the atmosphere with the increment

$$d\delta = -\sigma\, dz = -\sigma \mu\, ds$$

we can write

$$\mu\left(\frac{dL}{d\delta}\right)_P = L(P) - J^*(P)$$

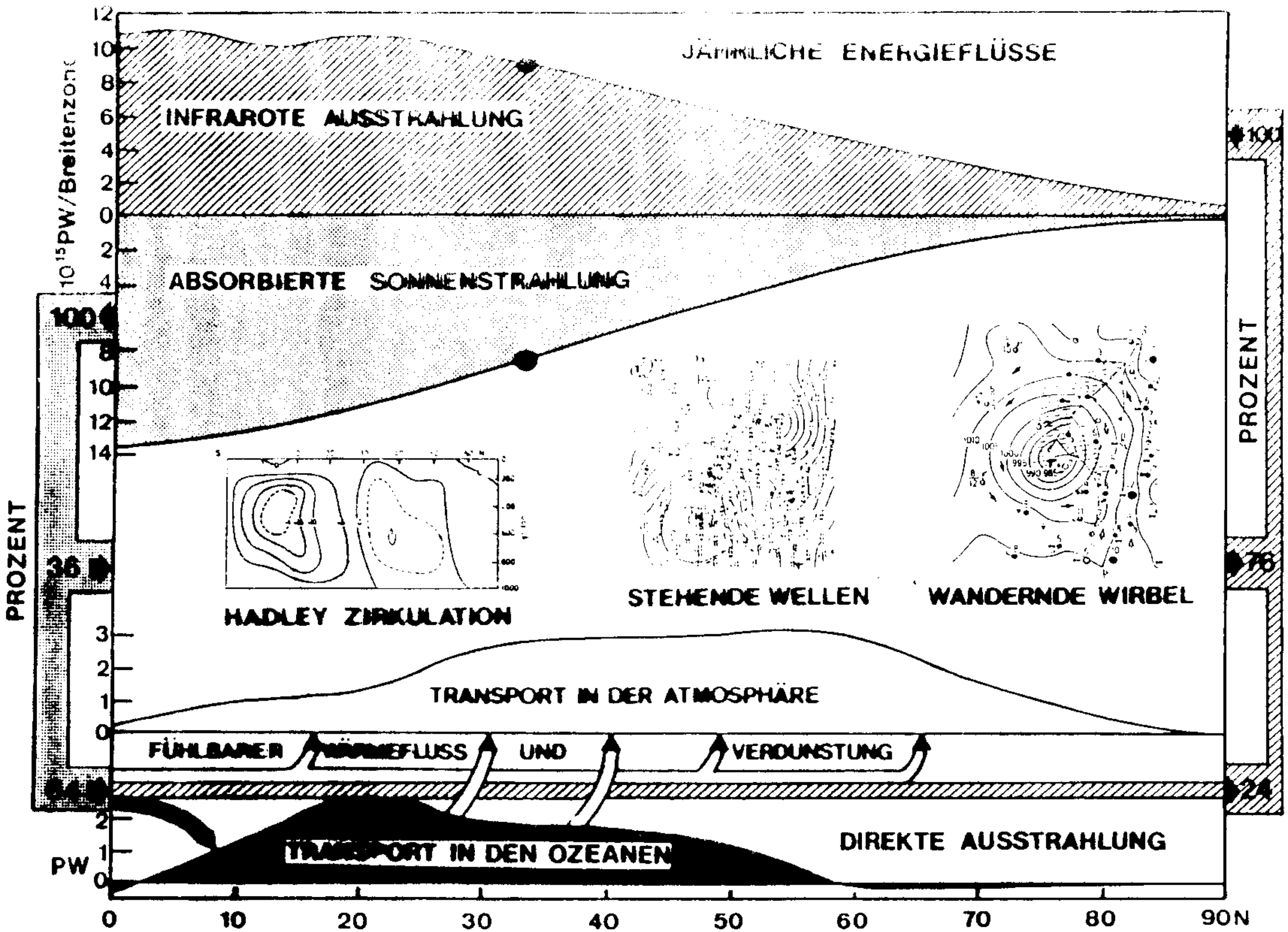
In a horizontally homogenous atmosphere we may write

$$\boxed{\mu\left(\frac{dL}{d\delta}\right)_z = L(z) - J^*(z)}$$

This is the radiative transfer equation (RTE) in its most general form. It can be used for the solution of all radiation problems provided the following parameters are known:

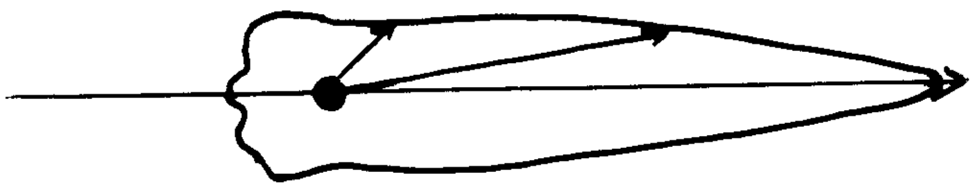
$\sigma_{scat}(z)$, $a(z)$, $\phi(z; \mu, \varphi; \mu', \varphi')$, J_{em},

and the boundary conditions $L(0)$, $L(\infty)$.



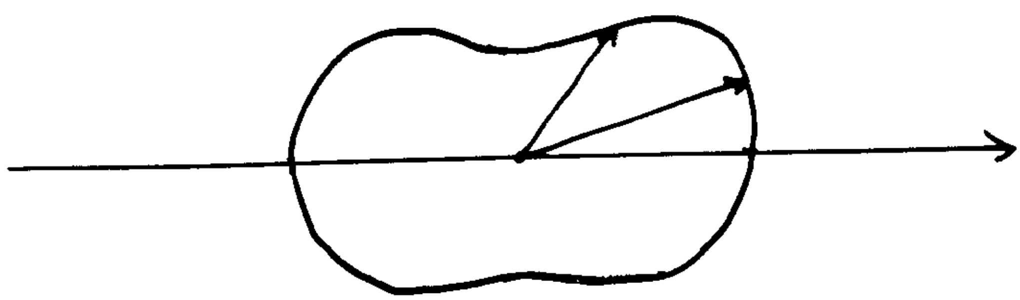
JÄHRLICHE ENERGIEFLÜSSE
10^{15} PW/Breitenzone
INFRAROTE AUSSTRAHLUNG
ABSORBIERTE SONNENSTRAHLUNG
PROZENT
100
36
64
76
24
HADLEY ZIRKULATION
STEHENDE WELLEN
WANDERNDE WIRBEL
TRANSPORT IN DER ATMOSPHÄRE
FÜHLBARER WÄRMEFLUSS UND VERDUNSTUNG
TRANSPORT IN DEN OZEANEN
DIREKTE AUSSTRAHLUNG
PW
0
10
20
30
40
50
60
70
80
90N

Aerosol scattering:

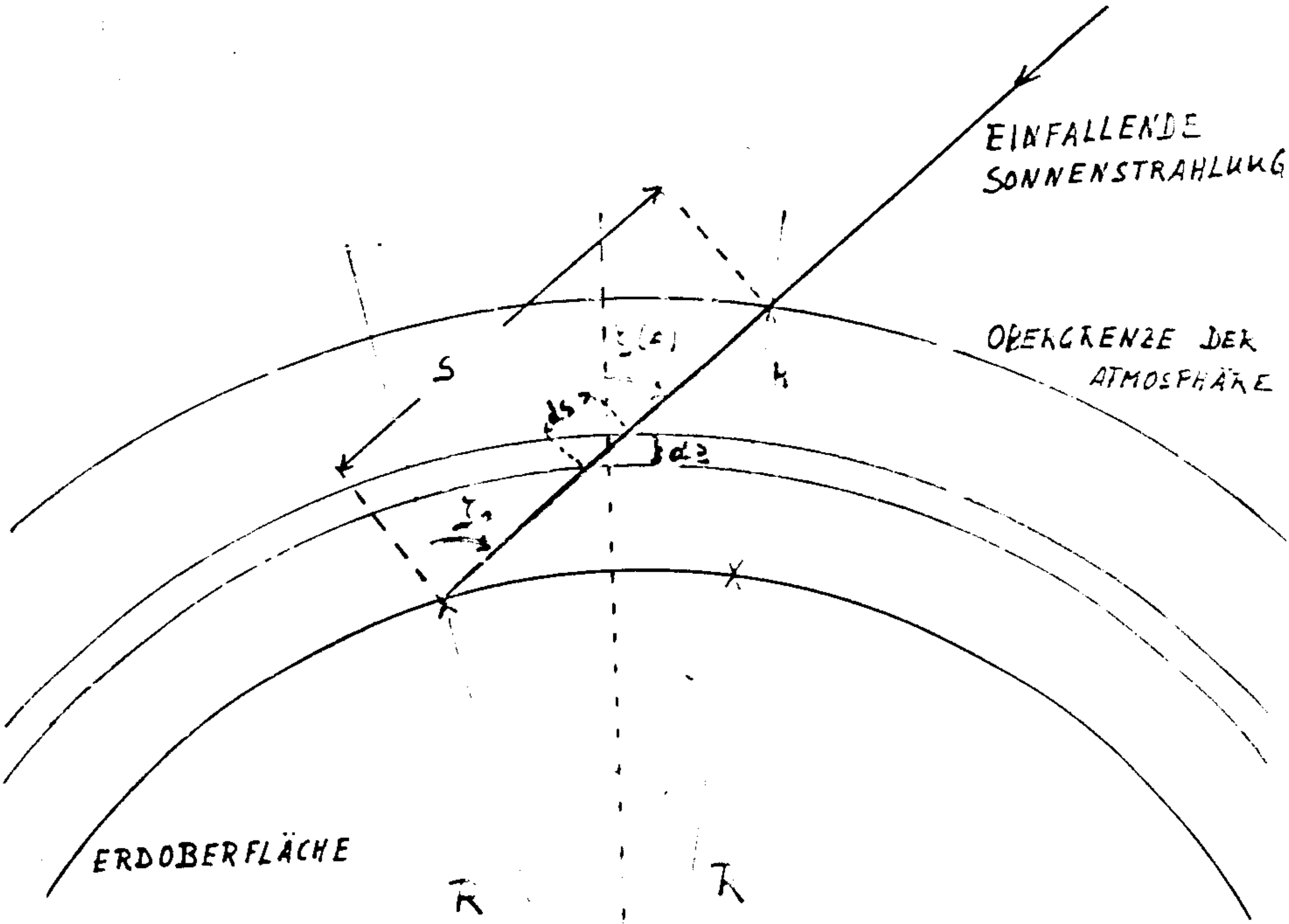


scattering centers equal or larger than λ

Molecular scattering:



scattering centers much smaller than λ



$$(R+H)^2 = s^2 + R^2 - 2sR\cos(180-\zeta_0$$

$$s^2 + 2sR\cos\zeta + (R^2 - [R+H]^2) = 0$$

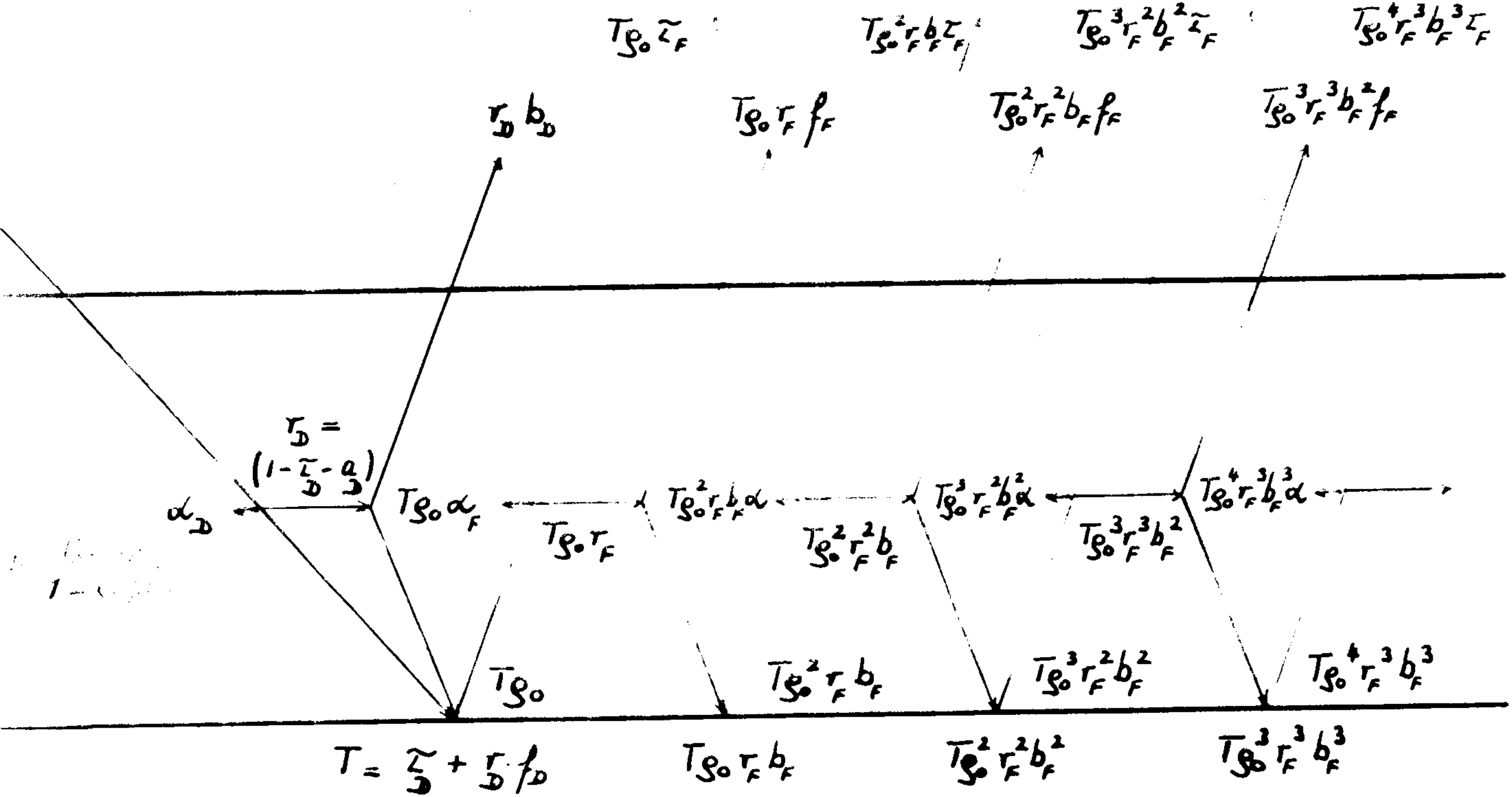
$$s = -R\cos\zeta + \sqrt{(R\cos\zeta)^2 + 2RH + H^2}$$

$$ds = dz / \cos\zeta(z)$$

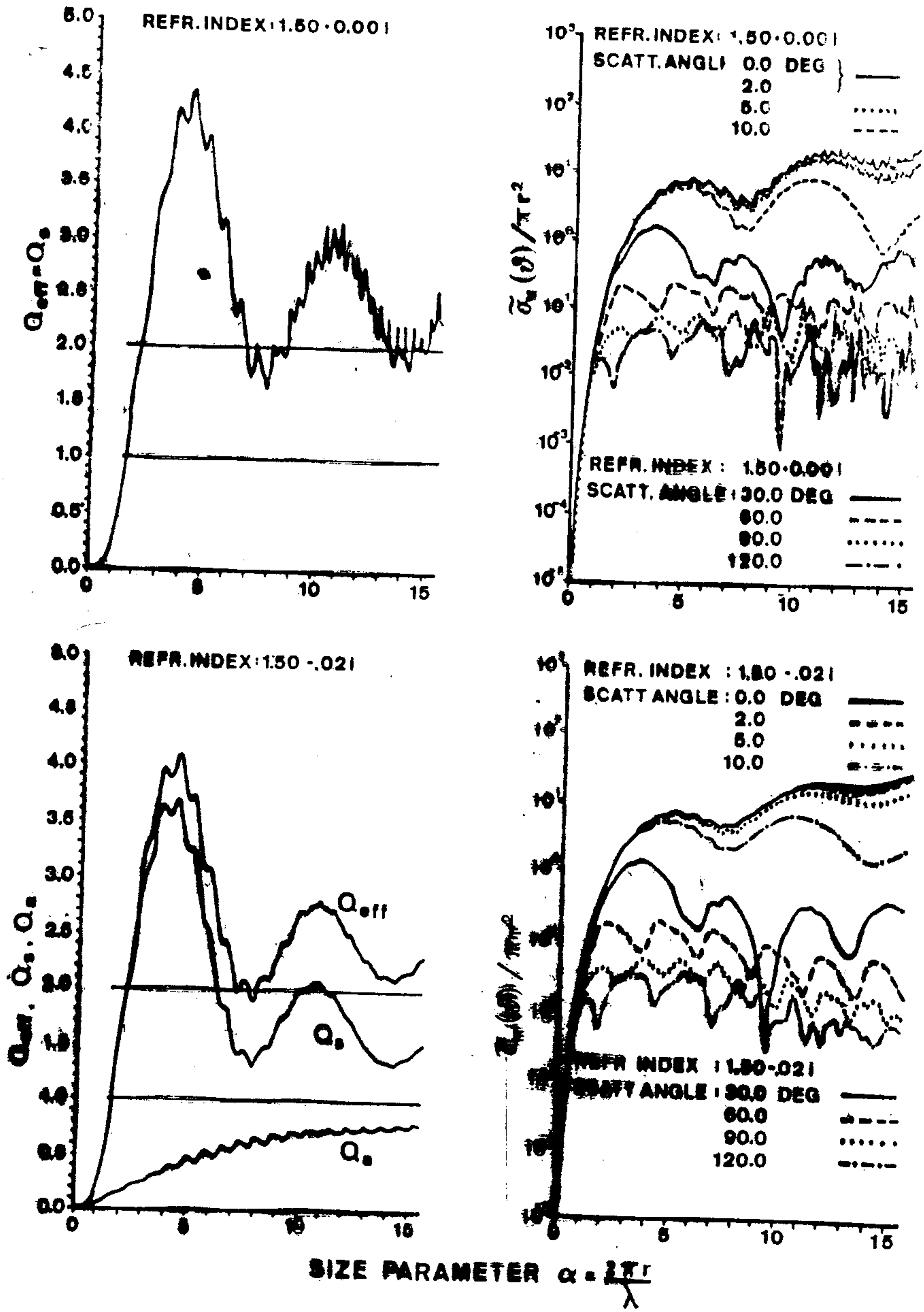
RELATIVE LUFTMASSE:

$$M_{rel} = \frac{s}{H}$$

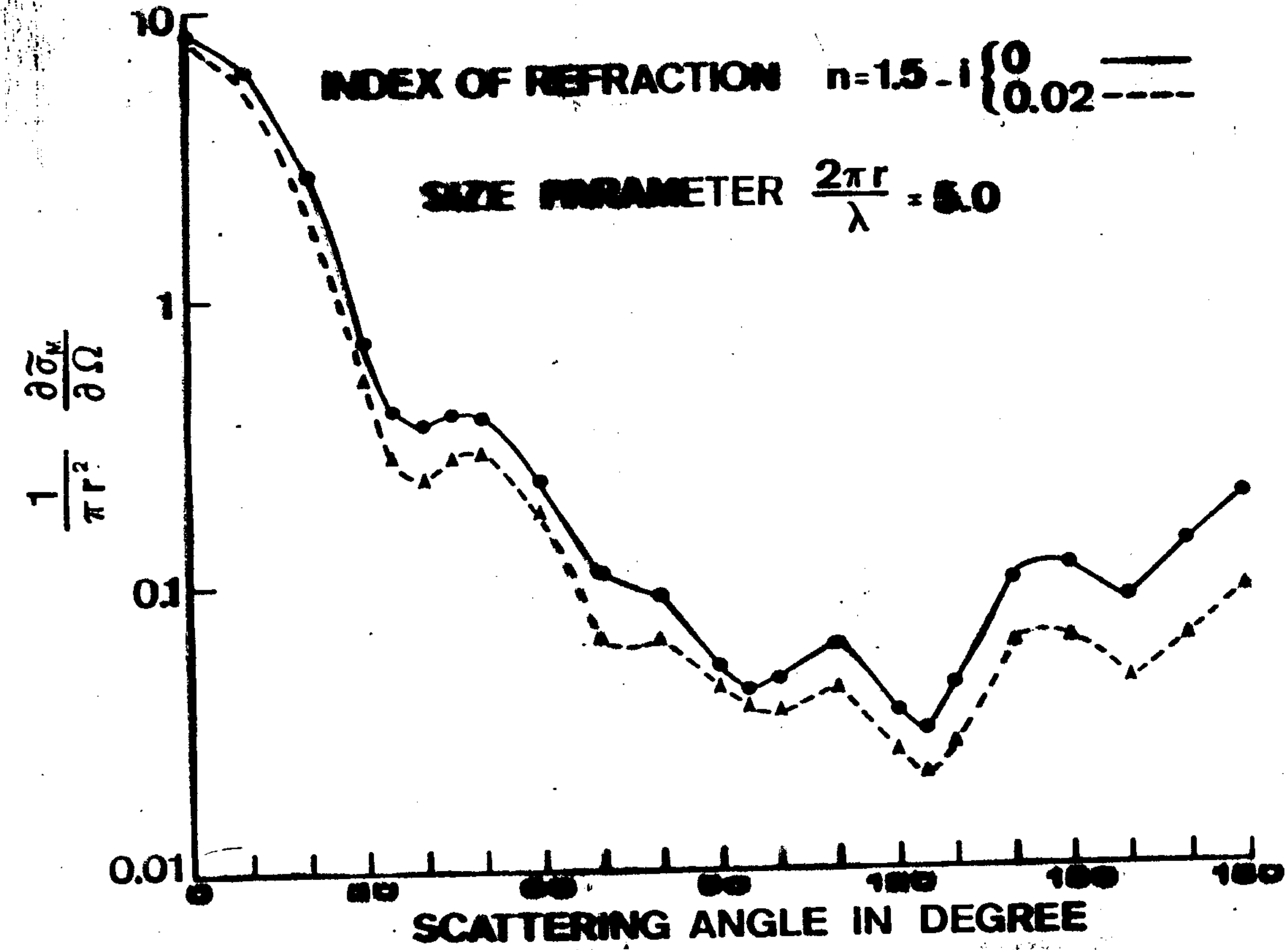
$$\rho_\rho - r_D b_D + T\rho_0 \left[\tau_F + r_F f_F\right] / (1 - \rho_0 r_F b_F)$$



$$\tau = T \sum_{n=0}^{\infty} (\rho_0 r_F b_F) = \frac{\tau_D + r_D f_D}{1 - \rho_0 r_F b_F}$$



REFR. INDEX : 1.50 - 0.001
REFR. INDEX : 1.50 - 0.001
SCATT. ANGLE 0.0 DEG
2.0
5.0
10.0
REFR. INDEX : 1.50 - 0.001
SCATT. ANGLE : 30.0 DEG
60.0
90.0
120.0
REFR. INDEX : 1.50 - .021
Q_{eff}
Q_s
Q_a
REFR. INDEX : 1.50 - .021
SCATT ANGLE : 0.0 DEG
2.0
5.0
10.0
REFR. INDEX : 1.50 - .021
SCATT ANGLE : 30.0 DEG
60.0
90.0
120.0
SIZE PARAMETER $\alpha = \frac{2\pi r}{\lambda}$



INDEX OF REFRACTION n=1.5 - i {0 ; 0.02
SIZE PARAMETER $\frac{2\pi r}{\lambda}$ = 5.0
$\frac{1}{\pi r^2} \frac{\partial \tilde{\sigma}_M}{\partial \Omega}$
10
1
0.1
0.01
SCATTERING ANGLE IN DEGREE

$$\rho_r = r_D b_D + T\rho_0 \frac{\tau_F + r_F f_F}{1 - \rho_0 r_F b_F}$$

$$\alpha = \alpha_D + T\rho_0 \frac{\alpha_F}{1 - \rho_0 r_F b_F}$$

$$\tau = \frac{T}{1 - \rho_0 r_F b_F}$$

$$T = \tau_D + r_D f_D$$

$$\alpha_i + r_i + \tau_i = 1 \ , \qquad b_i + f_i = 1$$

$$i = \begin{cases} D & \text{direct solar radiation} \\ F & \text{diffuse shortwave flux} \end{cases}$$

b = backward , f = forward scattering

Conservation of energy:

$$1 = \rho_r + \alpha + \tau(1 - \rho_0)$$

$$= r_D b_D + \alpha_D + \frac{T}{1 - \rho_0 r_F b_F}(1 - \rho_0 r_F b_F)$$

$$= r_D b_D + \alpha_D + \tau_D + r_D f_D = 1$$