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We consider a homogeneously broadened ring laser with spherical mirrors in which the atomic line is resonant with three degenerate transverse modes of the resonator. Upon increasing the pump parameter, the cylindrically symmetric single-mode configuration becomes unstable and the system approaches a new stationary state characterized by three coexisting modes and an asymmetric field distribution that emerges continuously at the instability threshold. We also show that the injection of a sufficiently strong external signal with the same spatial profile as the cylindrically symmetric mode of the cavity can restore the original symmetry of the output field.

The phenomenon of spatial symmetry breaking in systems far from thermal equilibrium is common to many fields of science and is basic to our understanding of spatial patterns. It is currently the subject of active investigations in Synergetics [1] and in the theory of dissipative structures [2]. Recent studies have predicted the appearance of translational symmetry breaking phenomena in passive [3] and active [4] nonlinear optical systems and traced their origin to the loss of stability of the spatially homogeneous stationary state in the presence of dissipation. These effects are examples of Turing instabilities [5], even if the operating mechanism is diffraction rather than diffusion. Related spatial pattern formation phenomena have been identified also in other optical systems characterized by gaussian intensity profiles [6], although in these instances the emerging patterns are not the consequence of symmetry breaking in the strict sense of the word.

In this paper we analyze the spontaneous breaking of cylindrical symmetry in a nonlinear optical system, an effect that was discovered numerically in ref. [7] and observed experimentally in ref. [8]. The novelty of our investigation is the development of a sufficiently simple dissipative model that allows an analytic characterization of the symmetric stationary state and of the mechanism leading to the loss of stability. We consider a ring laser with spherical mirrors and assume that the length of the active region is much smaller than the Rayleigh length of the cavity field. Actually, this restriction is not essential for the purpose of producing the required results, but it is convenient in order to keep our calculations as simple as possible. In fact, it allows us to neglect the longitudinal variations of the beam width and field phase along the active sample so that the transverse profile of the cavity modes is described by the orthonormal functions [9]

$$A_{p,0}(\rho,\varphi)=(2/\sqrt{2\pi})\,L_p^0(2\rho^2)\exp(-\rho^2)\,, \qquad (1a)$$

$$A_{p,l}(\rho,\varphi)=(2/\sqrt{\pi})\,(2\rho^2)^{1/2}\times\left(\frac{p!}{(p+1)!}\right)^{1/2}L_p^l(2\rho^2)\exp(-\rho^2)\begin{cases}\cos l\varphi\\ \sin l\varphi\end{cases}$$

$$p,l=0,1,2,\ldots, \qquad (1b)$$

where p is the radial and l is the angular index; ρ denotes the radial coordinate, $r=(x^2+y^2)^{1/2}$, normalized to the beam waist w, and L_p^l are Laguerre polynomials of the indicated argument. The cavity resonances depend on the sum of the indices $2p+l$, a situation that produces modal degeneracy.

We assume that the active medium is a homogeneously broadened system of two-level atoms and that the excited region has a gaussian transverse shape of radius r_p, i.e. the equilibrium population difference is described by the function

$$\chi(\rho)=\exp(-2\rho^2/\psi^2), \quad \psi=2r_p/w. \tag{2}$$

We also assume the atomic line to be resonant with three cavity modes characterized by the same longitudinal index and by the transverse indices: (i) $p=l, l=0$, (ii) $p=0, l=2$, cosine angular profile, and (iii) $p=0, l=2$, sine angular profile. We label with A_i ($i=1, 2, 3$) the transverse profiles of these three modes, given by eq. (1). In addition, we stipulate that:

(a) All the other cavity modes either suffer from large losses, or their frequency separation from the atomic line is much larger than the atomic linewidth.

(b) The uniform field condition

$$\alpha L \ll 1, \quad T \ll 1,$$
$$\text{with } C\equiv\alpha L/2T=\text{arbitrary}, \tag{3}$$

is satisfied, where α is the gain coefficient per unit length, L is the length of the active region, T is the transmissivity coefficient of the mirrors and C is the pump parameter.

Under these conditions we can show that the dynamics of the system is governed by the three-mode model

$$\frac{df_i}{dt}=-\kappa\left(f_i-2C\int_0^{2\pi}d\varphi\int_0^{\infty}d\rho\,\rho\,A_i(\rho,\varphi)\,P(\rho,\varphi,t)\right), \tag{4a}$$

$$\partial P/\partial t=\gamma_\perp\,[F(\rho,\varphi,t)\,D(\rho,\varphi,t)-P(\rho,\varphi,t)], \tag{4b}$$

$$\partial D/\partial t=-\gamma_\parallel\,[\mathrm{Re}(F^*(\rho,\varphi,t)\,P(\rho,\varphi,t)) + D(\rho,\varphi,t)-\chi(\rho)], \tag{4c}$$

$$F(\rho,\varphi,t)=\sum_{i=1}^{3}A_i(\rho,\varphi)\,f_i(t), \tag{5}$$

where F is the normalized slowly-varying envelope of the electric field, f_i ($i=1, 2, 3$) are the complex mode amplitudes, P and D are the normalized atomic polarization and population inversion, respectively. The cavity damping constant, or cavity linewidth, κ is given by

$$\kappa=cT/\Lambda, \tag{6}$$

where Λ is the total cavity length. $\gamma_\perp$ and $\gamma_\parallel$ are the relaxation rates of the atomic polarization and population inversion, respectively, $\gamma_\perp$ coincides with the atomic linewidth.

The set of equations (4), in addition to the trivial stationary solution $f_i=0$ ($i=1, 2, 3$), admits a set of single-mode stationary solutions governed by the steady state equations ($i=1, 2, 3$)

$$1=2C\int_0^{2\pi}d\varphi\int_0^{\infty}d\rho\,\rho\,\frac{A_i^2(\rho,\varphi-\varphi_0)\exp(-2\rho^2/\psi^2)}{1+A_i^2(\rho,\varphi-\varphi_0)\,|f_i|^2}, \tag{7}$$

$$|f_j|=0, \quad \text{for } j\neq i,$$

corresponding to the stationary output intensity

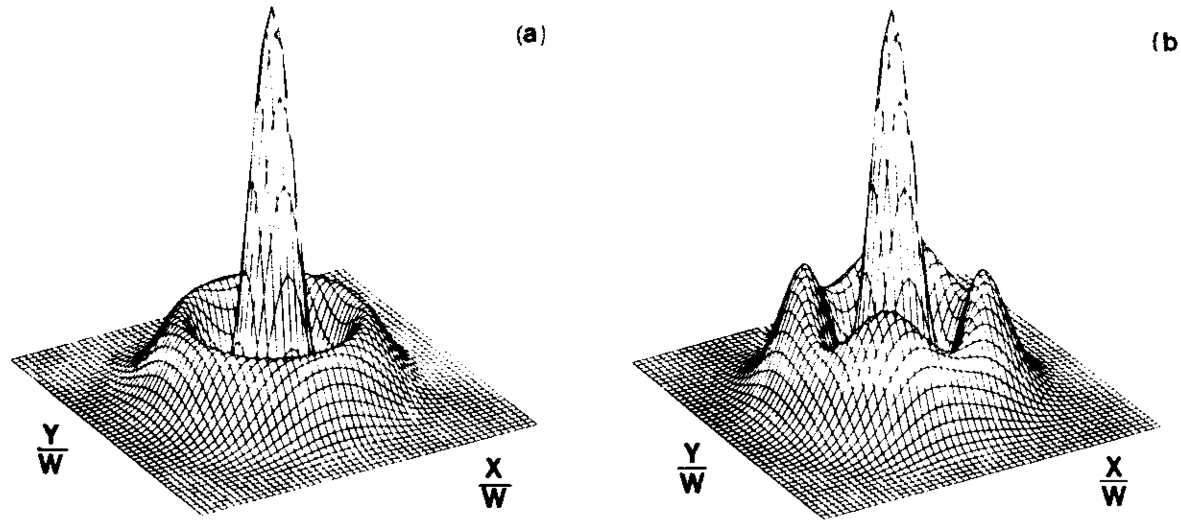
$$|F(\rho,\varphi)|^2=A_i^2(\rho,\varphi-\varphi_0)\,|f_i|^2$$

for each single-mode steady state. The phase of F is arbitrary. The appearance of the parameter φ_0 indicates that, for a fixed value of the index i, eq. (7) describes an infinite class of stationary solutions that can be obtained by a rotation around the longitudinal axis. In particular, the solution for $i=1$ is cylindrically symmetric (see fig. 1a), and the class of solutions defined by eq. (7) for $i=2$, under variation of φ_0, coincides with that obtained for $i=3$. The lasing threshold values for the symmetric and asymmetric single-mode stationary solutions are given by

$$(2C)^{S}_{thr}=\frac{(1+\psi^2)^3}{\psi^2(1+\psi^4)}, \quad (2C)^{A}_{thr}=\frac{(1+\psi^2)^2}{\psi^6}. \tag{8}$$

Hence the symmetric solution has the lowest threshold.

Now we perform an exact linear stability analysis of the single-mode stationary states. In the resonant configuration, the linear stability analysis of each nontrivial solution leads to a pair of eigenvalue equations, one for the amplitude fluctuations, the other for the fluctuations of the phase. An analysis



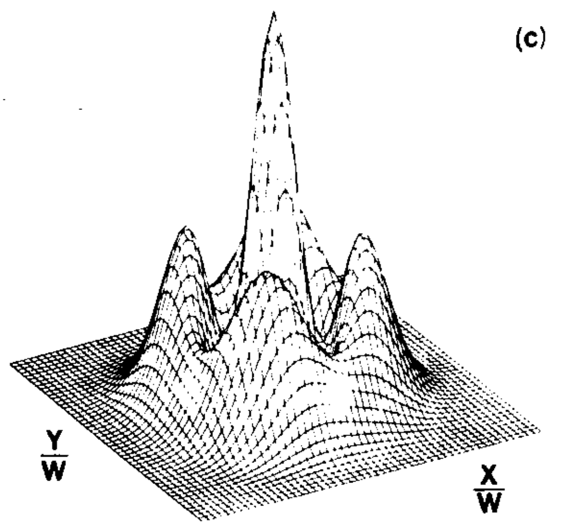


Fig. 1. Transverse profile of the output intensity $|F(\rho, \varphi)|^2$ for $\psi = 1.5$; (a) the unstable nontrivial single-mode stationary solution for $2C/(2C)^{S}_{thr} = 1.5$; (b) the multimode stationary state for the same value of C as in (a); (c) same as (b) but for $2C/(2C)^{S}_{thr} = 1.91$.

of these equations yields the following results:

(a) The asymmetric single-mode stationary solutions are unstable already at threshold [i.e. $2C = (2C)^{A}_{thr}$].

(b) The symmetric stationary solution becomes unstable when the pump parameter exceeds the instability threshold $(2C)^{I}$ defined by the condition

$$1 = (2C)^{I} \int_0^{2\pi} d\varphi \int_0^{\infty} d\rho\, \rho \frac{A_2^2(\rho, \varphi) \exp(-2\rho^2/\psi^2)}{1 + A_1^2(\rho)|f_1|^2} \quad (9)$$

where $|f_1|^2$ is the intensity of mode 1 in the symmetric stationary state, and A_2 can be replaced by A_3. Eq. (9) results from the eigenvalue equation for the phase fluctuations. The physical meaning of condition (9) is that the symmetric steady state becomes unstable when the residual gain experienced by the asymmetric modes exceeds their losses. Fig. 2 illustrates the behavior of the ratio $(2C)^{I}/(2C)^{A}_{thr}$ as a function of ψ; this ratio approaches unity for increasing values of ψ, and increases markedly when ψ decreases, i.e. when the active medium is confined to a narrow region around the axis where the strength of the modes with $l \neq 0$ vanishes.

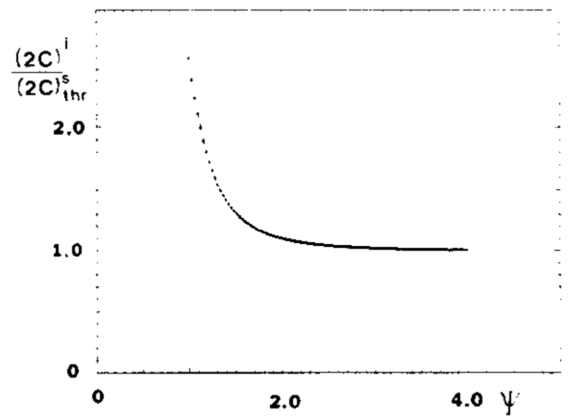


Fig. 2. Variation of the instability threshold with the pump waist. The ratio of the instability threshold to the laser threshold is plotted as a function of ψ which is twice the ratio of the pump waist to the beam waist.

We have solved numerically eqs. (4) and (5) for $2C>(2C)^i$, $\kappa/\gamma_\perp=\gamma_\parallel/\gamma_\perp=1$, starting from an initial condition in which $|f_1|^2$ is given by the symmetric stationary solution, while f_2 and f_3 are very small to simulate the fluctuations that trigger the instability. An example of the type of solutions that results by this procedure is shown in fig. 1a. In the course of time the modes 2 and 3 grow and the system approaches a multimode stationary state which is, of course, asymmetric (fig. 1b, c). In steady state, the

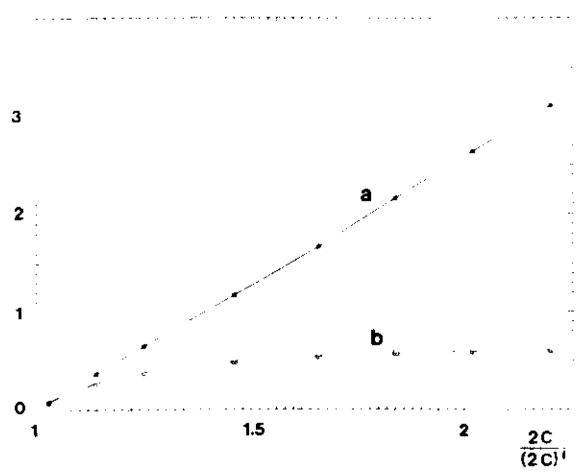


Fig. 3. Variation of the multimode stationary state with the pump parameter, with $\psi=1.5$. Curve (a) shows the behavior of $(2/\pi)(|f_2|^2+|f_3|^2)$, curve (b) shows the ratio $(|f_2|^2+|f_3|^2)/|f_1|^2$.

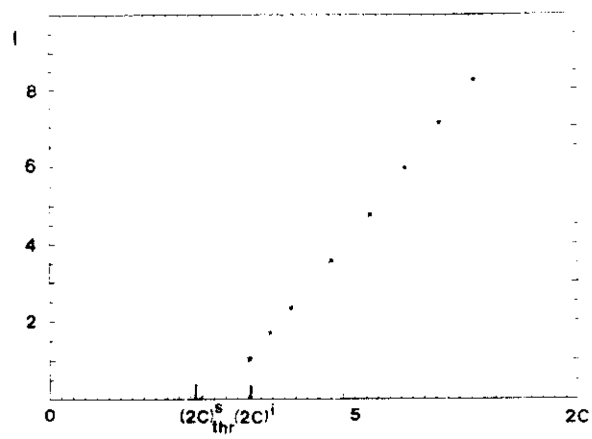


Fig. 4. Variation of the total output intensity I [eq. (13)] with the pump parameter, for $\psi=1.5$. The solid curve refers to the symmetric single-mode stationary solution which becomes unstable for $2C>(2C)^i$; the broken curve refers to the multimode steady state which emerges continuously for $2C=(2C)^i$.

field $F(\rho,\varphi)$ acquires the form

$$F(\rho,\varphi)=(2/\pi)^{1/2}\{(1-v)f_1 + v[f_2\cos 2\varphi+f_3\sin 2\varphi]\}\exp(-v/2)\,, \qquad (10)$$

where we have set $v=2\rho^2$ and we have taken into account the explicit expressions of the Laguerre polynomials. The phase of f_1 and the moduli $|f_2|$, and $|f_3|$ depend on the initial fluctuations with the following features:

(i) The difference between the phases of f_2 and f_3 is equal to 0 or π, while the difference between the phases of f_1 and f_2 is equal to $\pm\pi/2$.

(ii) The combination $|f_2|^2+|f_3|^2$ is independent of the initial fluctuation.

With this in mind, it is easy to obtain from eq. (10) the following expression for the intensity of the multimode steady state

$$|F(\rho,\varphi)|^2 = (2/\pi)\{(1-v)^2|f_1|^2+v^2[(|f_2|^2+|f_3|^2)/2] \times[1+\sin(4\varphi+\theta)]\}\exp(-v)\,, \qquad (11)$$

where we set

$$\sin\theta=\frac{|f_2|^2-|f_3|^2}{|f_2|^2+|f_3|^2}\,,\quad \cos\theta=\pm\frac{2|f_2|\,|f_3|}{|f_2|^2+|f_3|^2}\,, \qquad (12)$$

and where the + (or −) sign is selected when the phase difference between f_2 and f_3 is equal to 0 (or π) and θ represents the global phase of the output laser field. Hence, as expected, we find an infinite class of multimode stationary solutions obtained by rotation around the longitudinal axis; the initial fluctuation selects the angle θ. The total intensity is given by

$$I \equiv \int_0^{2\pi} d\varphi \int_0^{\infty} d\rho\, \rho\, F(\rho, \varphi)\,|^2 = |f_1|^2 + |f_2|^2 + |f_3|^2 . \quad (13)$$

On increasing the pump parameter beyond the instability threshold, the contribution from the asymmetric mode increases, as one can see from fig. 1c and fig. 3. On the other hand, the ratio $(|f_2|^2 + |f_3|^2)/|f_1|^2$ seems to approach a constant value (fig. 3). Finally fig. 4 shows the variation of the total output intensity given by eq. (13) and it compares it with the intensity of the symmetric stationary solution. Clearly, the multimode configuration has a better change to exploit the available gain and gives rise to larger power output. Both figs. 3 and 4 suggest that the asymmetry emerges continuously at the instability threshold $(2C)^1$.

In conclusion we have shown explicitly that beyond the instability threshold the system approaches

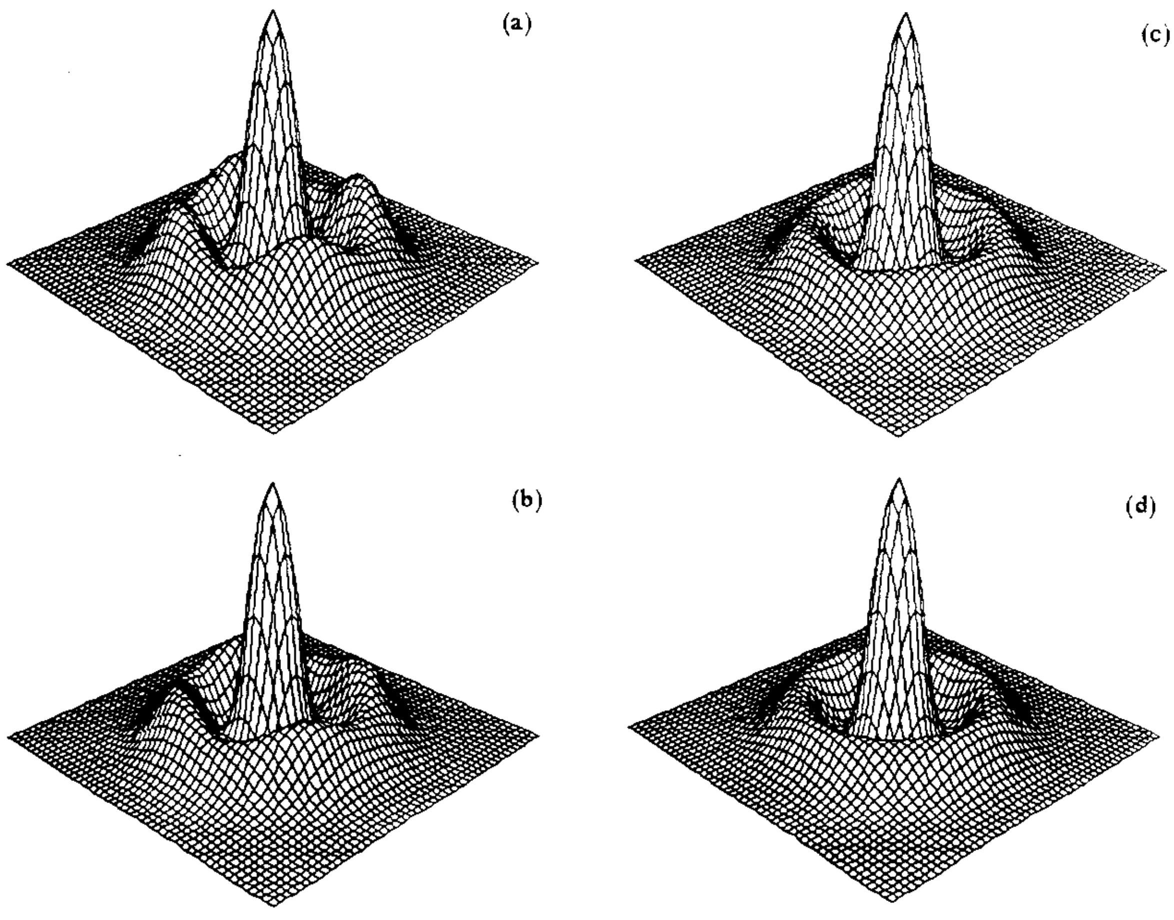


Fig. 5. Transverse intensity profile $|F(\rho, \varphi)|^2$ at the output of the laser, initially in the stable symmetry broken configuration, under the action of an injected signal which is mode matched to the $p=1$, $l=0$ configuration. Figures (a) through (d) show the restoration of the symmetric state for growing strength of the injected field.

a new steady state in which symmetric and asymmetric modes coexist and produce a spontaneous breaking of the cylindrical symmetry. We believe that a detailed experimental observation and comparison with theory is possible and would be of significant value in the context of the subject of Turing instabilities and spatial pattern formation.

It is interesting to observe that the stable asymmetric stationary state can be destabilized, in turn, by the injection into the laser of a coherent external signal of sufficient strength and a transverse shape which is mode-matched to the symmetric $p=1$, $l=0$ configuration. This effect is illustrated in figs. 5a-d for progressively larger amplitudes of the injected signal. We note that the restoration of the original cylindrical symmetry is a threshold effect that emerges for finite values of the injected signal strength.

We conclude this paper with two additional remarks:

(i) The same cylindrical symmetry breaking phenomenon can be observed starting from different symmetric configurations, for example from the mode $p=2$, $l=0$; this initial state, after destabilization, also turns into a stationary multimode configuration in which the symmetric and asymmetric modes coexists.

(ii) We have examined the same problem for absorptive optical bistability with an input field matched to the $p=1$, $l=0$ mode, and found that in this case the instability occurs only in the unphysical negative slope branch of the steady state cycle of transmitted versus incident power.

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