INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

H4.SMR/449-37

**WINTER COLLEGE ON HIGH RESOLUTION SPECTROSCOPY**

**(8 January - 2 February 1990)**

# QUANTUM OPTICS AND SQUEEZED STATES

**A.S. Shumovsky**

**Laboratory of Theoretical Physics**
**Joint Institute for Nuclear Research**
**141980 Dubna, USSR**

WINTER COLLEGE ON HIGH RESOLUTION SPECTROSCOPY

# QUANTUM OPTICS AND SQUEEZED STATES

A.S.Shumovsky
Laboratory of Theoretical Physics
Joint Institute for Nuclear Research
141980 Dubna
USSR

Lecture 1

## Quantum States of the Photon Field and Their Statistical Properties

*"Even though these subjects seem irrelevant, I must touch on all of them; for they have a bearing on what happened latter".*
*Kenneth Roberts - Northwest Passege*

The quantum statistical properties of light represent an important branch of modern physics. This fact is connected with the development of spectroscopy, optical communication, precision quantum measurement, etc. Investigation of the statistical properties of light plays also an important role in the understanding of the collective states of the fields with the different physical nature.

In quantum mechanics, complete information about the statistical and fluctuation properties of a system contains in the density matrix $\rho$ defined by the Liouville equation

$$i \hbar \, \partial\rho/\partial t = [ H , \rho ] , \qquad (1.1)$$

where $H$ is the hamiltonian of the system. In the case of quantum optics $H$ describes an interaction between light and matter. Let $\mathfrak{A}$ be a dynamical variable in the Schrödinger representation. Then its time-dependent mean value is defined in the following manner:

$$\langle \mathfrak{A} \rangle_t = \mathrm{Tr} \, ( \mathfrak{A} \, \rho(t) ) . \qquad (1.2)$$

The Heisenberg representation is constructed from the Schrödinger representation by the relation

$$\mathfrak{A}(t) = U^{-1}(t) \, \mathfrak{A} \, U(t)$$

where $U(t)$ is the unitary operator obeying the equation

$$i \hbar \, \partial U/\partial t = H U , \quad U( t_0 ) = 1 .$$

Hence the mean value (1.1) can be represented in the form

$$\langle \mathfrak{A} \rangle_t = \mathrm{Tr}\,(\,\mathfrak{A}(t)\,\rho(t_0)\,)\,. \qquad (1.3)$$

Operator $\mathfrak{A}(t)$ obeys the Heisenberg equation of motion:

$$i\,\hbar\,d\mathfrak{A}/dt = [\,\mathfrak{A}\,,\,H\,]\,. \qquad (1.4)$$

For the model problems of quantum optics the density matrix of the initial state $\rho\,(t_0)$ is usually written in the form

$$\rho\,(t_0) = \rho_F \otimes \rho_M\,, \qquad (1.5)$$

where $\rho_F$ is the initial density matrix of a radiation field and $\rho_M$ is the same for the matter. This condition (1.5) expresses the hypothesis of the switching on of the radiation-matter interaction at the time moment $t_0$.

Next, to define $\rho_F$, we discuss some important states of the quantized electromagnetic field.

In the quantum electrodynamics the electric and magnetic intensities, $\vec{E}$ and $\vec{B}$, are regarded as operators satisfying Maxwell's equations

$$\vec{\nabla} \times \vec{B} = -\,c^{-1}\,\partial\vec{B}/\partial t\,,$$

$$\vec{\nabla} \times \vec{B} = c^{-1}\,\partial\vec{E}/\partial t\,,$$

$$(\,\vec{\nabla}\vec{E}\,) = 0\,, \quad (\,\vec{\nabla}\vec{B}\,) = 0\,.$$

The operators $\mathbf{E}$ and $\mathbf{B}$ can be defined in terms of the vector potential operator $\mathbf{A}$:

$$\vec{E} = -\,c^{-1}\,\partial\vec{A}/\partial t\,, \quad \vec{B} = \vec{\nabla} \times \vec{A}\,.$$

The operator $\mathbf{A}$ obeys the wave equation

$$\vec{\nabla}^2\vec{A} - c^{-2}\,\partial^2\vec{A}/\partial t^2 = 0\,.$$

For the field in a normalization cubic volume $L^3$ with periodic boundary conditions the vector potential operator $\mathbf{A}$ can be expanded in terms of plane waves in the following form



$$\vec{A}\,(\,\vec{x}\,) = (\,2\pi\hbar c/L^3)^{-1/2} \sum_{k}\sum_{\sigma=1}^{2} k^{-1/2}\,(\,\vec{e}_\sigma(k)\,a_{k\sigma}\,e^{i(\vec{k}\vec{x}\,-\,ckt)} + \vec{e}^{\,*}_\sigma(k)\,a^+_{k\sigma}\,e^{-i(\vec{k}\vec{x}\,-ckt)}\,)\,. \qquad (1.6)$$

Here $a_{k\sigma}$ ( $a^+_{k\sigma}$ ) is the annihilation (creation) operator for a photon of momentum $\hbar\mathbf{k}$ and polarization $\sigma$, $e_\sigma(k)$ is a unite polarization vector satisfying the condition

$$\vec{e}_\sigma(k)\,\vec{e}^{\,*}_{\sigma'}(k) = \delta_{\sigma\sigma'}\,, \quad \vec{k}\,\vec{e}_\sigma(k) = 0\,.$$

The photon operators obey the following commutation rules

$$[\,a_\lambda\,,\,a^+_{\lambda'}\,] = \delta_{\lambda\lambda'}\,, \qquad (1.7)$$

$$[\,a_\lambda\,,\,a_{\lambda'}\,] = [\,a^+_\lambda\,,\,a^+_{\lambda'}\,] = 0\,,$$

where $\lambda = (\,k\,,\,\sigma\,)$ is the mode index.

The energy of a free field in a volume $L^3$ is described by the hamiltonian

$$H = \frac{1}{8\pi}\int_{L^3} (\,\vec{E}^2 + \vec{B}^2\,)\,d^3x\,.$$

So, using the Maxwell's equations together with (1.6) we obtain

$$H = \sum_\lambda \hbar\,\omega_\lambda\,(\,\hat{n}_\lambda + 1/2\,) \qquad (1.8)$$

where $\omega_\lambda = kc$ and so-called number operator $\hat{n}_\lambda$ is defined by the expression

$$\hat{n}_\lambda = a^+_\lambda\,a_\lambda\,.$$

<u>Fock state</u> (number state) is defined as an eigenstate of the <u>number operator</u>

$$\hat{n}\,|\,n\,\rangle = n\,|\,n\,\rangle \qquad (1.9)$$

where $n$ is a real number and we omit the mode index for simplicity.

Vacuum state of the field is defined by

$$a\,|\,0\,\rangle = 0\,, \quad \langle\,0\,|\,0\,\rangle = 1\,.$$

Because

$$a^{+} \mid 0 \rangle = \mid 1 \rangle ,$$

$$a^{+} \mid n \rangle = \sqrt{(n+1)} \mid n+1 \rangle , \quad a \mid n \rangle = \sqrt{n} \mid n-1 \rangle$$

any Fock state can be constructed from the vacuum state

$$\mid n \rangle = ( n! )^{-1/2} (a^{+})^{n} \mid 0 \rangle , \quad \langle n \mid n' \rangle = \delta_{nn'} . \qquad (1.10)$$

The countable set of states $\mid n \rangle$, $n = 0, 1, 2, \ldots$, forms a complete orthogonal system, the so-called Fock basis. Any state of the radiation field can be expanded in terms of Fock state. The appropriate expansion of a density matrix is

$$\rho = \sum_{n,m} \rho( n , m ) \mid n \rangle \langle m \mid , \quad \rho( n , m ) = \langle n \mid \rho \mid m \rangle . \qquad (1.11)$$

For the Fock state $\rho( n , m ) = \delta_{nm}$ and $\rho = \mid n \rangle \langle n \mid$.

The mean number of photons in the Fock state is

$$\langle \hat{n} \rangle = \langle n \mid \hat{n} \mid n \rangle = n ,$$

whereas mean square of the number of photons is

$$\langle ( \hat{n} )^2 \rangle = \langle n \mid ( \hat{n} )^2 \mid n \rangle = n^2 .$$

So, the variance of the number of photons in the Fock state is

$$\langle ( \Delta \hat{n} )^2 \rangle = \langle ( \hat{n} ) \rangle - \langle \hat{n} \rangle = 0 . \qquad (1.12)$$

One can say therefore that it is the state with a fixed number of photons.

Besides the number of photons, the operator of phase $\hat{\varphi}$ can also be considered. The following definition was given by Dirac

$$a = e^{i\hat{\varphi}} \sqrt{(a^{+}a)} . \qquad (1.13)$$

The famous uncertainty relation of the form

$$\langle ( \Delta \hat{n} )^2 \rangle \langle ( \Delta \hat{\varphi} )^2 \rangle \geq 1/4 \qquad (1.14)$$



is usually used. It follows from (1.12) and (1.14) that from the quantum-mechanical point of view the phase is a completely uncertain characteristic of the field in the Fock state.

It should be noted that (1.13) is not a mathematically correct definition. It was shown by Carruthers and Nieto ( Rev. Mod. Phys. 40(1968)411) that the operator $U = \exp ( i \hat{\varphi} )$ is not unitary, since

$$\langle 0 \mid U^{+} U \mid 0 \rangle = 0 ,$$

in contradiction with $U^{+}U = \hat{1}$. Following to Carruthers and Nieto, one can define the phase operator $\hat{\varphi}$ by the formal expressions:

$$e^{i\hat{\varphi}} = ( \hat{n} + 1 )^{-1} a^2 , \quad e^{-i\hat{\varphi}} = ( a^{+} )^2 ( \hat{n} + 1 )^{-1} . \qquad (1.15)$$

Nevertheless, expressions (1.13) and (1.14) can be used to obtain estimations in many cases.

Chaotic state is defined as a state of the field being in the equilibrium with the medium. Perhaps, it is the most widely distributed type of the natural radiation in the Universe.

In statistical mechanics, the equilibrium state of a system is defined by the condition of maximum of its entropy. The quantum definition of the entropy is

$$S = - \operatorname{Tr} ( \rho \ln \rho ) \qquad (1.16)$$

or, in terms of the Fock states,

$$S = \sum_{n,m} \langle n \mid \rho \mid m \rangle \langle m \mid \ln \rho \mid n \rangle .$$

The maximum is defined with the supplementary condition

$$\operatorname{Tr} ( \rho a^{+}a ) = \langle \hat{n} \rangle_c . \qquad (1.16a)$$

Then

$$\rho_c = \sum_{n} \rho_c(n) \mid n \rangle \langle n \mid , \qquad (1.17)$$

$$\rho_c(n) = \frac{\langle \hat{n} \rangle_c^n}{(1 + \langle \hat{n} \rangle_c)^{1+n}}, \quad \langle \hat{n} \rangle_c = ( e^{\hbar\omega/\vartheta} - 1 )^{-1}$$

where $\vartheta$ is a radiation temperature. This is the well-known Bose-Einstein distribution. The expression (1.17) can also be represented in the form

$$\rho_c = \frac{\exp(-H/\vartheta)}{\mathrm{Tr}\,\exp(-H/\vartheta)}$$

where $H = \hbar\,\omega\,\hat{n}$ .

Chaotic state is not a pure quantum state because

$$\mathrm{Tr}\,\rho_c^2 \neq 1 .$$

The number of photon variance in a chaotic state is

$$\langle (\Delta \hat{n})^2 \rangle_c = \langle \hat{n} \rangle_c ( 1 + \langle \hat{n} \rangle_c ) . \qquad (1.18)$$

From the expressions (1.14), (1.17) and (1.18) it follows that the variance of phase $\langle (\Delta \hat{\phi})^2 \rangle$ tends to zero when $\vartheta \to \infty$ and tends to infinity when $\vartheta \to 0$. Hence the phase uncertainty decreases with the increase of the radiation temperature in the chaotic state.

Coherent state is defined as an eigenstate of a photon annihilation operator

$$a \mid \alpha \rangle = \alpha \mid \alpha \rangle , \quad \langle \alpha \mid a^+ = \alpha^* \langle \alpha \mid \qquad (1.19)$$

It was introduced by Glauber ( Phys. Rev. 130 (1963) 2529 ).

Let us expand $\mid \alpha \rangle$ over the Fock basis

$$\mid \alpha \rangle = \sum_{n=0}^{\infty} \lambda_n \mid n \rangle \qquad (1.20)$$

where $\lambda_n$ are some complex numbers. When we substitute expression (1.19) into (1.20), we get the result

$$\sum_{n=1}^{\infty} \lambda_n \sqrt{n} \mid n - 1 \rangle = \alpha \sum_{n=0}^{\infty} \lambda_n \mid n \rangle .$$



So, the following recurrence relation

$$\lambda_n = \frac{\alpha^n}{\sqrt{n!}} \lambda_0$$

takes place, and $\lambda_0 = 1$ from the normalization condition. Hence, the coherent state can be constructed from the vacuum state in the following manner

$$\mid \alpha \rangle = D(\alpha) \mid 0 \rangle \qquad (1.21)$$

where the displacement operator

$$D(\alpha) = \exp ( \alpha a^+ - \alpha^* a )$$

obeys the conditions

$$\begin{cases} D^+(\alpha)\, a\, D(\alpha) = a + \alpha \\ D^+(\alpha)\, a^+ D(\alpha) = a^+ + \alpha^* \\ D^+(\alpha) = D^{-1}(\alpha) = D(-\alpha) . \end{cases}$$

The scalar product of two coherent states is

$$\langle \beta \mid \alpha \rangle = \exp ( - \mid \alpha \mid^2/2 - \mid \beta \mid^2/2 + \alpha \beta^* ) .$$

Hence the coherent states are not orthogonal to each other, but any state of another sort can be expanded over the coherent states.

It follows from the expression (1.21) that the possibility to discover, in a coherent state, exactly n photons is

$$\mid \langle n \mid \alpha \rangle \mid^2 = \frac{\mid \alpha \mid^{2n}}{n!} e^{-\mid \alpha \mid^2} . \qquad (1.22)$$

This is the Poisson distribution with $\langle \hat{n} \rangle = \mid \alpha \mid^2$ and

$$\langle (\Delta \hat{n})^2 \rangle = \langle \hat{n} \rangle . \qquad (1.23)$$

By analogy with the expression (1.11) the density matrix can be expressed in terms of the coherent states with complex $\alpha$ :

$$\rho = \int P(\alpha) \mid \alpha \rangle \langle \alpha \mid d^2\alpha . \qquad (1.24)$$

Here $P(\alpha)$ is interpreted as a quasi-probability distribution because in the coherent state the quantum mean value of the normally ordered form coincides with the classical mean value with the probability distribution $P(\alpha)$. Comparing (1.24) with (1.11) and using (1.20) we obtain

$$\rho(n,m) = \int P(\alpha) \frac{\alpha^n (\alpha^*)^m}{\sqrt{n!\, m!}} e^{-|\alpha|^2} d^2\alpha .$$

**References**


Bogolubov N.N. and Shirkov D.V. - Introduction to the Theory of Quantized Field. Interscience. New York. 1959.

Born M. and Wolf E. - Principles of Optics. Pergamon. Oxford. 1965

Perina J. - Quantum Statistics of Linear and Nonlinear Optical Phenomena. D.Reidel Pub. Co. 1984.




Lecture 2

## SQUEEZED STATE OF FIELD

*"It is not really difficult to construct a series of inferences, each dependent upon its predecessor and each simple in itself."*
*Sir Arthur Conan Doyl - The Dancing Men.*

Now we discuss the quantum fluctuation properties of the radiation field. Instead of the creation and annihilation nonhermitian operators $a^+$ and $a$, let us consider two hermitian quadrature operators $X_1$ and $X_2$ defined by the expressions

$$X_1 = ( a^+ + a )/2 , \quad X_2 = i ( a^+ - a )/2 . \qquad (2.1)$$

and obeying the commutation relation

$$[ X_1 , X_2 ] = i/2 . \qquad (2.2)$$

The known quantum expression for a plane monochromatic electromagnetic wave is

$$E(t) = \lambda ( a e^{-i\omega t} + a^+ e^{i\omega t} ) . \qquad (2.3)$$

In terms of the quadrature operators it reduces to

$$E(t) = 2 \lambda ( X_1 \cos \omega t + X_2 \sin \omega t ) . \qquad (2.4)$$

Since for a quantum harmonic oscillator

$$a = ( \omega q + i p ) ( 2\hbar\omega )^{-1/2}$$

$$a^+ = ( \omega q - i p ) ( 2\hbar\omega )^{-1/2}$$

from the comparison with (2.1) it follows that the quadrature operators can be interpreted as a dimensionless coordinate and a dimensionless momentum, respectively,

$$X_1 = ( \omega / 2\hbar )^{1/2} q, \quad X_2 = ( 2\hbar\omega )^{-1/2} p .$$

The following uncertainty relation for quadratures

$$\langle ( \Delta X_1 )^2 \rangle \langle ( \Delta X_2 )^2 \rangle \geq | \langle [ X_1 , X_2 ] \rangle |^2 / 4 = 1/16 \qquad (2.5)$$

can be obtained. If an exact equality in (2.5) holds, then corresponding quantum state of the field is a state with minimal uncertainty. From the definition of the coherent state it follows that

$$\langle ( \Delta X_1)^2 \rangle = \langle ( \Delta X_2)^2 \rangle = 1/4. \qquad (2.6)$$

So, $| \alpha \rangle$ is the minimal uncertainty state.

Let us consider now the uncertainty of the coherent state on the $X_1$ , $X_2$ plane. It follows from (1.21) and (2.1) that

$$\langle X_1 \rangle_\alpha = \mathrm{Re}\, \alpha , \quad \langle X_2 \rangle_\alpha = \mathrm{Im}\, \alpha .$$

In accordance with (2.6) each component of vector $| \alpha \rangle$ can be measured with the minimal accuracy

$$\delta = ( \langle ( \Delta X_1 )^2 \rangle_\alpha )^{1/2} = 1/2 .$$

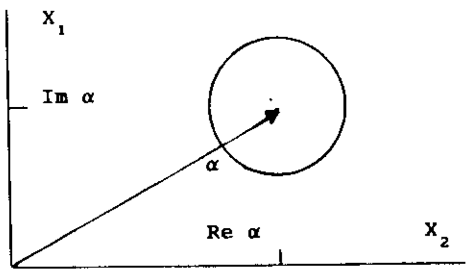


Fig. 2.1. Plot of uncertainty in a coherent state.

So, the uncertainty of $\alpha$ can be pictured as an area of a circle with radius 1/2 (see Fig. 2.1).

In the Fock state (1.9)

$$\langle X_1 \rangle_n = \langle X_2 \rangle_n = 0$$

and

$$\langle ( \Delta X_1 )^2 \rangle_n = \langle ( \Delta X_2 )^2 \rangle = (1 + 2n)/4 > 1/4 .$$

It is not a minimal uncertainty state at $n > 0$ . The quantum accuracy in the measurement of quadrature components of field is

$$\delta = ( 1 + 2n )^{1/2} / 2 .$$

The corresponding plot of uncertainty is presented by an annulus (Fig. 2.2).

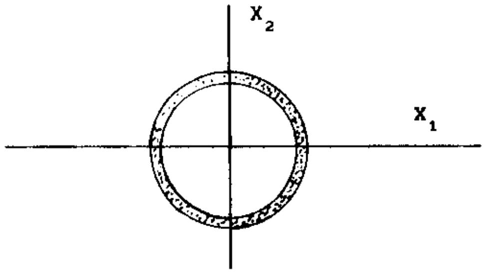


Fig. 2.2. Plot of uncertainty in Fock state.

It should be noted that the uncertainty relation (2.5) imposes a restriction only on the product of two variances. Each factor may have any value including the value less then 1/4 at the expense of increased the fluctuations in the other quadrature. The corresponding states of the field present the great interest for the physical measurements and optical communication and information systems. These are the so-called squeezed stats.

Usually it is defined as a state which has a small value of fluctuation in one quadrature component than the zero-point (coherent state) fluctuation.

The simplest example of a squeezed state can be constructed with the aid of the famous Bogolubov canonical transformation ( Bogolubov N.N. - J. Phys. USSR 11 (1947) 23 ) introduced for the first time in the theory of superfluidity.

Let us consider a monochromatic field and define a formal column

$$A = \begin{pmatrix} a \\ a^+ \end{pmatrix}$$

The Bogolubov transformation has the form

$$B = U A , \qquad (2.7)$$

where the column

$$B = \begin{pmatrix} b \\ b^+ \end{pmatrix}$$

describes some "new" Bose-field and

$$U = \begin{pmatrix} u & v \\ v^* & u^* \end{pmatrix} .$$

From the commutation relation

$$[\, b \,,\, b^+ \,] = 1$$

it follows that

$$\det U = |\, u \,|^2 - |\, v \,|^2 = 1 . \qquad (2.8)$$

One can define vacuum states for the "fields" A and B by formal expressions

$$a \,|\, 0 \rangle_A = 0 \,, \quad b \,|\, 0 \rangle_B = 0 .$$

Because A and B realize two different representations of the same physical field, the state $|\, 0 \rangle_B$ can be expanded over the Fock basis of the "field" A :

$$|\, 0 \rangle_B = \sum_{n=0}^{\infty} \lambda_n \,|\, n \rangle_A . \qquad (2.9)$$

By analogy with the coherent state (1.20), let us substitute (2.9) into the definition of the "B-field" vacuum. We get

$$0 = \sum_{n=1}^{\infty} u \lambda_n \sqrt{n} \,|\, n-1 \rangle_A + \sum_{n=0}^{\infty} v \lambda_n \sqrt{n+1} \,|\, n+1 \rangle_A .$$

For the numerical coefficients $\lambda_n$ it follows that

$$\lambda_n = \begin{cases} \lambda_0 \,(-v/u)^{n/2} \sqrt{n!} \,/\, n!!, & n \text{ is even} \\ 0, & n \text{ is odd} . \end{cases}$$

Using now (1.10), instead of (2.9) we obtain

$$|\, 0 \rangle_B = \sum_{n=0}^{\infty} (-v/2u)^n \frac{(a^+)^{2n}}{n!} \,|\, 0 \rangle_A =$$

$$= \exp\left( -(v/2u)\,(a^+)^2 \right) |\, 0 \rangle_A . \qquad (2.10)$$

The state $|\, 0 \rangle_B$ is the so-called squeezed vacuum of the "A-field". It contains a complete set of the Fock states of the "A-field" with an even number of photons.

To clarify the term "squeezing", let us consider the uncertainty relation (2.5). From (2.7) we derive the transformation for the quadratures of the "A-field"

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\, u + \mathrm{Re}\, v & \mathrm{Im}\, v - \mathrm{Im}\, u \\ \mathrm{Im}\, v + \mathrm{Im}\, u & \mathrm{Re}\, u - \mathrm{Re}\, v \end{pmatrix} \begin{pmatrix} X_1 \\ X_1 \end{pmatrix} \qquad (2.11)$$

where

$$Y_1 = (\, b^+ + b \,)/2 \,, \quad Y_2 = i\,(\, b^+ - b \,)/2 .$$

One can easily obtain

$$\langle 0 \,|\, X_1 \,|\, 0 \rangle_B = 0 \,; \quad \langle 0 \,|\, X_2 \,|\, 0 \rangle_B = 0 \,;$$

$$\langle 0 \,|\, X_1^2 \,|\, 0 \rangle_B = |\, u - v \,|^2/4 \,; \; \langle 0 \,|\, X_1^2 \,|\, 0 \rangle_B = |\, u + v \,|^2/4$$

$$\langle 0 \,|\, (\Delta X_1)^2 \,|\, 0 \rangle_B = |\, u - v \,|^2 / 4,$$

$$\langle 0 \,|\, (\Delta X_2)^2 \,|\, 0 \rangle_B = |\, u + v \,|^2 / 4$$

and the right-hand side of the uncertainty relation(2.5) takes the form

$$\frac{1}{16} \,|\, u - v \,|^2 |\, u + v \,|^2 =$$

$$= (\, |u|^2 + |v|^2 - 2|uv| \cos\Delta \,)(\, |u|^2 + |v|^2 + 2|uv| \cos\Delta \,)/16,$$

$$\Delta = \arg u - \arg v .$$

According to (2.8) it has the minimal value 1/16 when

$$\Delta = k\pi \,, \quad k = 0,1,2,\ldots . \qquad (2.12)$$

In this case we deal with the minimal uncertainty state, but in contrast with the coherent state the plot of uncertainty present an ellipse (Fig. 2.3) owing to asymmetry of the quadrature uncertainties.

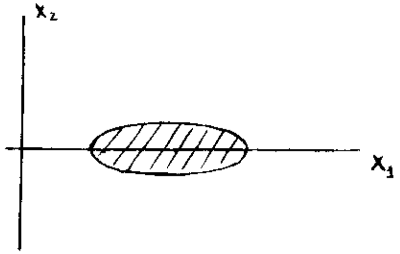


Fig. 2.3. Plot of uncertainty in a squeezed vacuum state.

Acting on the squeezed vacuum state $|\,0\,\rangle_B$ by the operator $(b^+)^n$ one can obtain the squeezed Fock states for the "A-field" which are the standard Fock states of the "B-field".

Let us now consider a coherent squeezed state or a squeezed state of the A-field which is a coherent state of the "B-field". It can be defined as

$$|\,\xi\,,\alpha\,\rangle \;=\; D\,(\,\alpha\,)\;\;S\,(\,\xi\,)\;\;|\,0\,\rangle_A \qquad (2.13)$$

Here $D$ is the displacement operator (1.21) and $S$ is the unitary squeezing operator

$$S\,(\xi) = \exp\,(\frac{1}{2}\;\;\xi^*\,a^2 \;-\; \frac{1}{2}\;\;\xi\,(a^+)^2\;)\;. \qquad (2.14)$$

It was introduced by Stoler ( Stoler D. - Phys. Rev. D1 (1970)3217; D4 (1971) 1925 and 2309). The transformation properties of the squeezing operator are as follows:

$$\begin{cases} S^+(\xi)\;a\;S(\xi) \;=\; a\,u^* \;-\; a^+\,v \\ S^+(\xi)\;a^+\;S(\xi) \;=\; -\,a\,v^* \;+\; a^+\,u \end{cases} \qquad (2.15)$$

It is obvious that the squeezed state (2.13) is an eigenstate of the creation operator $b$ :

$$b\,|\,\xi\,,\alpha\,\rangle \;=\; (\,u\,\alpha \;+\; v\,a^*\,)\,|\,\xi\,,\alpha\,\rangle \qquad (2.16)$$

i.e. it is a coherent state of the "B-field" defined by the Bogolubov transformation (2.7). This transformation is often



represented in the form

$$b \;=\; a\cosh r \;+\; a^+\,e^{i\varphi}\sinh r$$

Then instead of (2.15) we have

$$\begin{cases} S^+(\xi)\;a\;S(\xi) \;=\; a\cosh r \;-\; a^+\,e^{i\varphi}\sinh r \\ S^+(\xi)\;a^+\;S(\xi) \;=\; a^+\cosh r \;-\; a^+\,e^{-i\varphi}\sinh r\;. \end{cases} \qquad (2.17)$$

The most important mean values and variances for a squeezed state in the case of minimal uncertainty (when $\varphi = k\,\pi$ ) are

$$\begin{cases} \langle\,\hat{n}\,\rangle = |\alpha|^2 + \sinh^2 r \\ \langle\,(\Delta\hat{n})^2\,\rangle = |\alpha\cosh r - \alpha^*\sinh r|^2 + 2\cosh^2 r\,\sinh^2 r \\ \langle\,(\Delta X_1)^2\,\rangle = \frac{1}{4}\,e^{-2r} \\ \langle\,(\Delta X_2)^2\,\rangle = \frac{1}{4}\,e^{2r} \end{cases} \qquad (2.18)$$

It follows from the expressions (2.18) that the variance of any quadrature component of the field in the squeezed state can be smaller than in the coherent state or in the vacuum state. This circumstance open the possibility to produce a precision optical measurement beyond the shot-noise limit.

One can say that squeezed state of the electromagnetic field have reduced quantum noise for one observable and preserve the Heisenberg uncertainty relation by an increased quantum noise for the conjugate observable. The information can be extracted from the observable with reduced quantum noise, and thus the standard quantum limit can be overcome. The first successful attempt was realized by Min-Xiao, Ling-An Wu and Kimble (Phys.Rev.Lett. 59 (1987) 278).

The scheme of their experimental arrangement is shown in Fig 2.4.A Mach-Zehnder interferometer is formed by the two beam splitters $m_1$, $m_4$ and the highly reflected mirrors $m_2$, $m_3$. A coherent field $E_1$ is injected into the input port $m_1$, and the fields from the two paths through $P_1$ and $P_2$

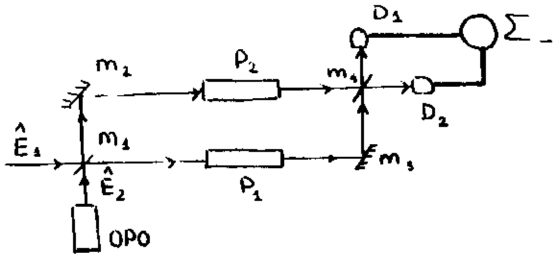


Fig. 2.4. The principal elements of the experimental arrangement for interferometry with squeezed state.

are recombined at the output port $m_4$ to produce interference fringes as a function of phase difference along the two arms. The nonlinear elements in the two arms of the Mach-Zehnder interferometer are deuterated potassiusm dihydrogen phosphate ($KD_2PO_2$) phase modulators. The limit on the minimum detectable phase change is the Shot Noise Limit (SNLL) and presents the best sensitivity possible for input of a coherent state $E_1$ and a vacuum field $E_s$.

To achieve sensitivity beyond the SNL, a squeezed field $E_s$ is injected in place of the vacuum field into the input $m_1$. Two detectors $D_1$ and $D_2$ registered the intensity of the beams which depends on the phase difference for propagation along the two arms of the interferometer. The photocurrents are examined by the subtraction arrangement $\Sigma_-$ which gives the time dependence of the phase fluctuations (Fig. 2.4 ). One can see that in the case of squeezed light the level of fluctuations of the difference photocurrent is smaller than for the case of vacuum field, which is given by the dashed lines. In this experiment an improvement of 3.0dB in signal-to-noise ratio was achieved in (b) related to (a).

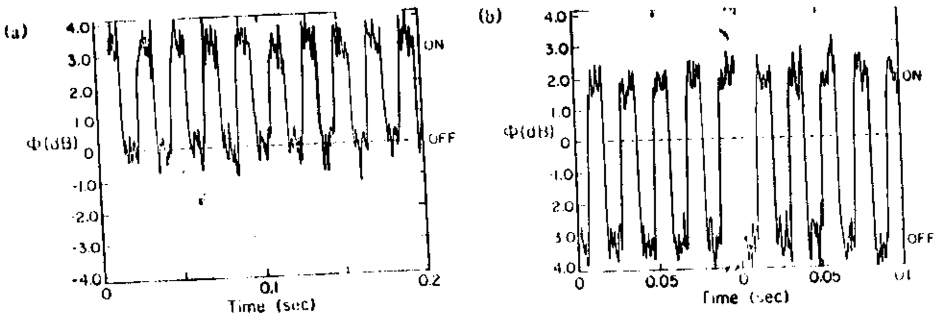


Fig. 2.5. Level of fluctuations vs time: (a) vacuum-state input for the field $\hat{E}_s$ ; (b) squeezed-state input $\hat{E}_s$ . The dashed line gives the vacuum level.



This achievement opens the wide perspectives in the field of the precision quantum measurements, e.g. in the gravitational wave detecting and in the optical transmission of information.

## References

Braginsky V.B., Vorontsov Yu.I. and Thorn K.S. - Science 209 (1980) 547.
Walls D.F. - Nature 306 (1983) 141.
Bondurant R.S. and Shapiro J.H. - Phys.Rev. D30 (1984) 2548.
Caves C.M., Thorne K.S. et al. - Rev.Mod.Phys. 52 (1980) 341.
Yamamoto Y. and Haus H.A. - Rev.Mod.Phys. 58 (1986) 1001.

Lecture 3.

## HOW TO MAKE A SQUEEZED STATE

*"There are some points which are as dark as ever. But we have so much that it will be our own fault if we cannot get the rest."*
*Sir Arthur Connan Doyle -*
*- The second stain.*

The simplest way to obtain a squeezed state consists in the using of a two-photon generation mechanism. For a single mode case this mechanism can be described by the quadratic hamiltonian

$$H = \hbar(\omega a^+ a + f_1^* a^2 + f_1 a^{+2} + f_2^* a + f_2 a^+) \qquad (3.1)$$

where c-numbers f may be time dependent. The $f_1$ terms describe a two-photon mechanism of interaction whereas the $f_2$ terms describe the usual one-photon or linear driving mechanism (see? Yuen H. - Phys.Rev. A13 (1976) 2226).

Instead of (3.1) the general quadratic hamiltonian

$$H = \sum_{k,l} \left\{ A_{kl} a_k^+ a_l^+ + A_{kl} a_k a_l + B_{kl} a_k^+ a_l \right\} \qquad (3.2)$$

can be examined. Here $B_{kl}=B_{lk}^*$ and $A_{kl}=A_{lk}$. Bogolubov canonical transformation of the form

$$a_k = \sum_m (u_{km} b_m + v_{km}^* b_m^+)$$

reduces (3.2) into the diagonal quadratic hamiltonian

$$H = E_0 + \sum_m E_m b_m^+ b_m \, , \qquad E_0 = - \sum_{k,l} E_l |v_{kl}|^2 \, . \qquad (3.4)$$

Here $E_m$ and $u_{km}$, $v_{km}$ are the eigenvalues and eigenfunctions of the system of equations



$$\begin{cases} E_m u_{km} = \sum_l (B_{kl} u_{lm} + 2A_{kl} v_{lm}) \\ - E_m v_{km} = \sum_l (B_{kl}^* v_{lm} + 2A_{kl}^* u_{lm}) \end{cases}$$

The functions $u$ and $v$ obey the following conditions

$$\begin{cases} \sum_k (u_{kl} u_{km}^* - v_{kl} v_{km}^*) = \delta(l-m) \\ \sum_k (u_{kl} v_{km} - u_{km} v_{kl}) = 0 \\ \sum_l (u_{kl} u_{ml}^* - v_{ml} v_{kl}^*) = \delta(k-m) \\ \sum_l (u_{kl} v_{ml} - u_{ml} v_{kl}) = 0 \end{cases}$$

The Heisenberg equations

$$i\hbar \frac{d}{dt} b_m = E_m b_m \, , \qquad i\hbar \frac{d}{dt} b_m^+ = - E_m b_m^+$$

have the simple solutions

$$b_m(t) = b_m(0) \exp(-iE_m t/\hbar) \, , \qquad b_m^+(t) = b_m^+(0) \exp(iE_m t/\hbar) \qquad (3.5)$$

Using (3.5) together with the transformation (3.3) one can define the time dependence of the operators $a_k$, $a_k^*$. Then, averaging the operator constructions with the density matrix of the initial state, it is possible to define the time dependence of the fluctuations (variances).

For example, let us consider a simple particular case of (3.2)

$$H = \hbar\omega_1 a_1^+ a_1 + \hbar\omega_2 a_2^+ a_2 + \hbar G (a_1^+ a_2^+ + a_1 a_2)$$

which describes the parametric interaction processes. Here the coupling constant G is related to the nonlinear susceptibility for the process under consideration. It should be noted that the creation of photons in pairs, described by (3.6), happens in many nonlinear-optical problems (e.g. in downconversion and four-wave mixing).

One can see that the difference in the number operators for the two modes

$$\hat{\Delta} = a_1^+ a_1 - a_2^+ a_2$$

is a time-independent operator. For $a_1(t)$ and $a_2(t)$ one can obtain

$$\begin{cases} a_1(t) = C_1 e^{i\lambda_+ t} + C_2 e^{i\lambda_- t} \\ a_2(t) = C_3 e^{i\lambda_+ t} + C_4 e^{i\lambda_- t} \end{cases}$$

where

$$\lambda_\pm = \frac{1}{2}(\omega_1 - \omega_2) \pm \frac{1}{2}\sqrt{(\omega_1+\omega_2)^2 - 4G^2} \; ; \qquad |G| \le \frac{\omega_1 + \omega_2}{2}$$

and the operators C are defined by the initial conditions.

Making the simple calculations one can obtain now the time dependence of the quadratures and corresponding variances and to define the existence of squeezing in dependence on the initial state.

In connection with the hamiltonian (3.6) it should be noted that the operator $a_1 a_2$ acting on a Fock state, simultaneously annihilates photons of two modes $a_1$ and $a_2$. Thus $a_1 a_2$ can be considered as the pair-annihilation operator and the corresponding pair coherent state can be defined (Agarwal-JOSA, B5 (1988) 1940) as follows

$$\begin{cases} a_1 \, a_2 \, |\xi,q\rangle = \xi \, |\xi,q\rangle \\ \hat{\Delta} \, |\xi,q\rangle = q \, |\xi,q\rangle \end{cases}$$

Its expansion over the Fock basis is

$$|\xi,q\rangle = N_q \sum_{n=0}^{\infty} \frac{\xi^n}{[n! \; (n+q)!]^{1/2}} \, |n+q, \, n\rangle \; ,$$

where $|n',n\rangle = |n'\rangle \otimes |n\rangle$ and $N_q$ is the normalization constant



$$N_q = \left[ \sum_{n=0}^{\infty} \frac{|\xi|^{2n}}{n! \; (n+q)!} \right]^{-1/2} = [(i|\xi|)^q \, J_q(2i|\xi|)]^{-1/2}$$

The generation of these interesting states also lead to the squeezing.

Let us now consider the anharmonic oscillator model. A possible realization of this model is strong light propagation through a nonlinear Kerr medium (Tanas R. - Phys.Lett. A141 (1989) 217).

The two versions of the anharmonic oscillator model that are going to be compared here are defined by the Hamiltonians

$$H = \hbar\omega a^+ a + \frac{1}{2}\hbar k (a^+)^2 a^2 \qquad (3.8)$$

$$H' = \hbar\omega a^+ a + \frac{1}{2}\hbar k (a^+ a)^2 \qquad (3.8a)$$

where k is the nonlinearity parameter, which real and assumed the same in both cases.

The Heisenberg equations of motion for the annihilation operators are then

$$\dot{a} = -\frac{i}{\hbar}[a,H] = -i(\omega + k a^+ a)a, \qquad (3.9)$$

$$\dot{a} = -\frac{i}{\hbar}[a,H] = -i(\omega + \frac{1}{2}k + k a^+ a)a. \qquad (3.10)$$

Since $a^+a$ is a constant of motion in both cases the solutions are the exponentials

$$a(t) = \exp\{-it[\omega + k a^+(0)a(0)]\}a(0), \qquad (3.11)$$

$$a(t) = \exp\{-it[\omega + \frac{1}{2}k + k a^+(0)a(0)]\}a(0). \qquad (3.12)$$

Eqs. (3.11) and (3.12) are the exact operator solutions describing the dynamics of the two versions of the anharmonic oscillator. It is seen that the only difference is the extra phase shift kt/2 which appeared in (3.12).

Since we are interested in squeezing, we define the Hermitian quadrature operator

$$Q_\varphi = a(t)\, e^{i(\omega t-\varphi)} + a^+(t)\, e^{-i(\omega t-\varphi)}, \qquad (3.13)$$

which for $\varphi=0$ corresponds to the in-phase quadrature component of the field and for $\varphi=\pi/2$ to the out-of-phase component.

The variance of such an operator is given by

$$\mathrm{Var}[Q_\varphi] = \langle Q_\varphi^2\rangle - \langle Q_\varphi\rangle^2 =$$

$$= 2\,\mathrm{Re}\,\{\langle a^2(t)\rangle\, e^{2i(\omega t-\varphi)} + \langle a(t)\rangle^2\, e^{2i(\omega t-\varphi)}\} \qquad (3.14)$$

$$+ 2\,\{\langle a^+ a\rangle - \langle a^+(t)\rangle\,\langle a(t)\rangle\} + 1 .$$

For the vacuum state as well as coherent states this variance is equal to unity. If it becomes smaller than unity the state of the field for which this occurs is referred to as squeezed state, and perfect squeezing is obtained if $\mathrm{Var}[Q_\varphi]$. It is convenient to use the normally ordered variance

$$V_\varphi(t) = \langle :Q_\varphi^2(t):\rangle - \langle Q_\varphi(t)\rangle^2 =$$

$$= 2\,\mathrm{Re}\,\{\langle a^2(t)\rangle\, e^{2i(\omega t-\varphi)} + \langle a(t)\rangle^2\, e^{2i(\omega t-\varphi)}\} \qquad (3.15)$$

$$+ 2\,\{\langle a^+(t)a(t)\rangle - \langle a^+(t)\rangle\,\langle a(t)\rangle\} .$$

Negative values of this variance mean squeezing and its value equal to -1 means perfect squeezing.

Assuming that the initial state of the field is a coherent state $|\alpha\rangle$ with the mean number of photons $N=|\alpha|^2$, and using eqs. (3.11) and (3.12), one can easily calculate the normally ordered variances (3.15) for both versions of the nonlinear interactions. The results are as follows:

$$V_\varphi(\tau) = 2N\{\exp[N(\cos 2\tau - 1)] \times \cos[2(\varphi - \varphi_0) + \tau + N \sin 2\tau]$$

$$- \exp[2N(\cos \tau - 1)] \times \cos[2(\varphi - \varphi_0) + 2N \sin \tau]$$

$$+ 1 - \exp[2N(\cos \tau - 1)]\} , \qquad (3.16)$$



$$V'_\varphi(\tau) = 2N\{\exp[N(\cos 2\tau - 1)] \times \cos[2(\varphi - \varphi_0) + 2\tau + N \sin 2\tau]$$

$$- \exp[2N(\cos \tau - 1)] \times \cos[2(\varphi - \varphi_0) + \tau + 2N \sin \tau]$$

$$+ 1 - \exp[2N(\cos \tau - 1)]\} , \qquad (3.17)$$

where we have introduced the notation $\tau = kt$ and $\alpha = \sqrt{N}\, e^{i\varphi_0}$ with $\varphi_0$ being the initial phase of the field.

Hence, in both cases there are the periodic revival of squeezing in the long-time scale with exactly the same period (see Fig. 3.1, 3.2).

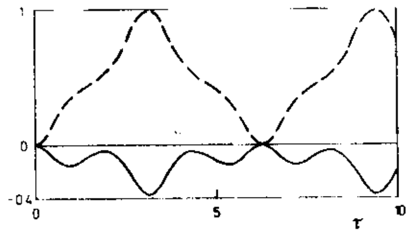


Fig. 3.1. The normally ordered variances $V_\varphi(\tau)$ (eq. (3.16) plotted against $\tau$, for N=0.25: solid line: the in-phase component; dashed line: the out-of-phase component.

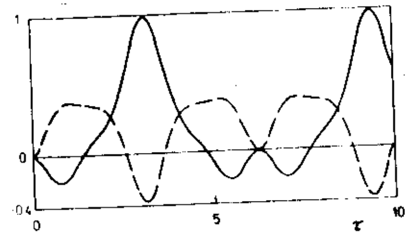


Fig. 3.2. The same as in fig. 1, but for variances $V'_\varphi(\tau)$ (eq. (3.17)).

Let us now consider the squeezing of light via nondegenerate four-wave mixing in a system of three-level atoms (Bogolubov N.N. Jr., Shumovsky A.S. and Tran Kuang - J.Phys. B, 20 (1987) L447).

The N three-level atoms concentrated in a region small compared with the wavelength of all the relevant radiation modes interact with two cavity modes $E_1$, $E_2$ and with two pumping waves $E_3$, $E_4$ with frequencies $\Omega_1$, $\Omega_2$, $\Omega_3$ and $\Omega_4$, respectively (fig. 3.3). The pumping fields $E_3$ and $E_4$ are assumed intense and can be treated classically. For simplicity the pumping field $E_3$ is assumed to be in resonance with the level separation $\omega_2-\omega_1=\omega_{21}$ ($\hbar=1$), and the pumping field $E_4$ is assumed to be in resonance with $\omega_3-\omega_2=\omega_{32}$. Let $a_1$, $a_1^+$ and $a_2$, $a_2^+$ be the annihilation and creation operators of the modes $E_1$ and $E_2$, respectively.

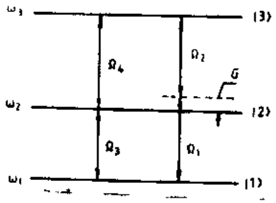


Fig. 3.3. Three-level atoms interacting with two pumping fields , and with two cavity modes , .

The coherence part of the Hamiltonian in the rotating-wave approximation and interaction picture is

$$H_{coh} = \Delta_1 a_1^+ a_1 + \Delta_2 a_2^+ a_2 + G_{21}(J_{21} + J_{12}) + G_{32}(J_{32} + J_{23}) + \qquad (3.18)$$
$$+ g_1(J_{21}a_1 + a_1^+ J_{12}) + g_2(J_{32}a_2 + a_2^+ J_{23})$$

where

$\Delta_1 = \Omega_1 - \omega_{21}$ $\qquad$ $\Delta_2 = \Omega_2 - \omega_{32}$

$G_{21} = - d_{21}E_3$ $\qquad$ $G_{32} = - d_{32}E_4$

$g_1 = - d_{21}(E_1/|E_1|)$ $\qquad$ $g_2 = - d_{32}(E_2/|E_2|)$

d are the electric dipole operators for the atom, $J_{ij}=c_i^+c_j$ $(i,j=1,2,3)$ are the collective angular momenta of the atoms, which have the following form in the Schwinger representation?



$$J_{ij} = c_i^+ c_j \qquad (i,j=1,2,3)$$

where the operators $c_i$ and $c_i^+$ obey the boson commutation relation

$$[c_i, c_i^+] = \delta_{ij}$$

and can be treated as the annihilation and creation operators for the atoms populating the level $|i\rangle$.

Let the signal modes $E_1$ and $E_2$ be initially in coherent states. After Agarwal (1974), by using the Markovian approximation, one finds the master equation for the reduced density matrix $\rho$ for the atomic system alone in the form

$$\partial\rho/\partial t = -i[\tilde{H}_{coh},\rho] - \gamma_{21}(J_{21}J_{12}\rho - J_{12}\rho J_{21} + HC) -$$
$$- \gamma_{32}(J_{32}J_{23}\rho - J_{23}\rho J_{32} + HC) = L\rho \qquad (3.19)$$

where $2\gamma_{ij}$ are the transition rates caused by the atomic reservoirs from level $|i\rangle\to|j\rangle$. The Hamiltonian $\tilde{H}_{coh}$ differs from the Hamiltonian $H_{coh}$ in equation (1) by the substitution of the operators $a_1$ and $a_2$ by their eigenvalues over the initially coherent states (Agarwal 1974).

Further, we investigate only the case of the intense pumping fields $E_3$ and $E_4$, so that

$$G = (G_{21}^2 + G_{32}^2)^{1/2} \gg N\gamma_{ij}, g_{1,2}|E_{1,2}| \qquad (3.20)$$

After the canonical transformation

$$c_3 = -2^{-1/2}\sin\alpha Q_1 + \cos\alpha Q_2 + 2^{-1/2}\sin\alpha Q_3$$
$$c_2 = 2^{-1/2}Q_1 + 2^{-1/2}Q_3 \qquad (3.21)$$
$$c_1 = 2^{-1/2}\cos\alpha Q_1 - \sin\alpha Q_2 + 2^{-1/2}\cos\alpha Q_3$$

and using the secular approximation (Agarwal et al 1978, Bogolubov et al 1985, 1986a), i.e. ignoring the part of the Liouville operator L containing rapidly oscillating terms with frequencies nG(n=1-4), one can find a stationary solution of the master equation in the form (Bogolubov et al 1985, 1986a)

$$\tilde{\rho} = U\rho U^{+} = Z^{-1}\sum_{P=0}^{N} X^{P} \sum_{M=0}^{P} |P,M\rangle\langle M,P| \quad . \qquad (3.22)$$

where

$$X = \gamma_{32}\cos^2\alpha/\gamma_{21}\sin^2\alpha$$

$$Z = [(N+1)X^{N+2} - (N+2)X^{N+1} + 1]/(x-1)^2$$

U is the unitary operator representing the canonical transformation on (4), $|P,M\rangle$ is an eigenstate of the operators $R = R_{11}+R_{33}$, $R_{11}$ and of the operator of the total number of atoms, where $R_{ij}=Q_i^{+}Q_j$ $(i,j=1,2,3)$ are the collective angular momenta of the "dressed" atoms. Now we return to the Hamiltonian (1). Following the laser theory of Haken (1970), one may obtain a quantum Langevin equation for the cavity modes $E_1$, $E_2$ in the form

$$\dot{a}_1(t) = (-i\Delta_1 - \varkappa_1)a_1(t) - ig_1J_{12}(t) + F_1(t)$$

$$\dot{a}_2(t) = (-i\Delta_2 - \varkappa_2)a_2(t) - ig_2J_{23}(t) + F_2(t) \qquad (3.23)$$

where $\varkappa_1$, $\varkappa_2$ and $F_1(t)$, $F_2(t)$ are cavity damping constants and noise operators for the modes $E_1$ and $E_2$ respectively. The noise operators $F_\lambda(t)$ $(\lambda=1,2)$ obey the relations (Haken 1970)

$$\langle F_\lambda(t)\rangle_H = \langle F_\lambda^{+}(t)\rangle_H = 0$$

$$\langle F_\lambda^{+}(t)F_{\lambda'}^{+}(t')\rangle_H = \langle F_\lambda(t)F_{\lambda'}(t')\rangle_H = 0$$

$$\langle F_\lambda^{+}(t)F_{\lambda'}(t')\rangle_H = n_{th,\lambda}(T)2\varkappa_\lambda\delta(t-t')\delta_{\lambda\lambda'} \qquad (3.24)$$

$$\langle F_\lambda(t)F_{\lambda'}^{+}(t')\rangle_H = (n_{th,\lambda}(T) + 1)2\varkappa_\lambda\delta(t-t')\delta_{\lambda\lambda'}$$

where $\langle\ldots\rangle_H$ indicates the thermal average over the states of the heat bath and $n_{th,\lambda}(T)$ is the number of thermal quanta at a temperature T for a field mode $E_\lambda$.

As follows from the calculations, a substantial squeezing in the mixture of two modes $E_1$ and $E_2$ can be obtained if modes $E_1$ and $E_2$ are located near fluorescence spectra at frequencies $\omega_{21} \pm G$ and $\omega_{32} \mp G$, respectively (fig. 1), i.e.

$$|\delta_1|,|\delta_2| \ll G \qquad (3.25)$$



where

$$\delta_1 = \Delta_1 - G \qquad \delta_2 = \Delta_2 + G$$

or

$$|\Delta_1 + G|,|\Delta_2 - G| \ll G$$

With the use of condition (3.25) and the secular approximation, equations (3.23) reduce to

$$\dot{\tilde{a}}_1(t) = (-i\delta_1 - \varkappa_1)\tilde{a}_1(t) + i(g_1/\sqrt{2})\sin\alpha\tilde{R}_{23}(t) + \tilde{F}_1(t)$$
$$\qquad (3.26)$$
$$\dot{\tilde{a}}_2(t) = (-i\delta_2 - \varkappa_2)\tilde{a}_2(t) + i(g_2/\sqrt{2})\sin\alpha\tilde{R}_{32}(t) + \tilde{F}_2(t)$$

where

$$a_1(t) = e^{-iGt}\,\tilde{a}_1(t) \qquad a_2(t) = e^{iGt}\,\tilde{a}_2(t)$$

$$F_1(t) = e^{-iGt}\,\tilde{F}_1(t) \qquad F_2(t) = e^{iGt}\,\tilde{F}_2(t)$$

$$R_{23}(t) = \tilde{R}_{23}(t)e^{-iGt} \qquad R_{32}(t) = \tilde{R}_{32}(t)e^{iGt}$$

I n the secular approximation $\tilde{R}_{23}(t)$ and $\tilde{R}_{32}(t)$ are slowly varying in time (Bogolubov et al 1985, 1986a). For simplicity we consider only the case of $n_{th,\lambda}(T)=0$, i.e. the temperature T=0. In this case, as is easily seen from relations (3.24) and equation (3.260), the noise operators $\tilde{F}_{1,2}(t)$ cannot affect the normally ordered variance of the signal modes $E_1$ and $E_2$, but they give the commutators $[a_1,a_1^{+}]$ and $[a_2,a_2^{+}]$ additional values equal to $1-\exp(-2\varkappa_1t)\to1$ $(t\to\infty)$ and $1-\exp(-2\varkappa_2t)\to1$ $(t\to\infty)$ respectively (Haken 1970).

Deleting the noise operator, one may obtain a stationary solution of equations (3.26) in the form

$$\tilde{a}_1 = \frac{ig_1}{\sqrt{2}}\sin\alpha\,\frac{\tilde{R}_{23}}{i\delta_1+\varkappa_1} \qquad \tilde{a}_2 = \frac{-ig_2}{\sqrt{2}}\sin\alpha\,\frac{\tilde{R}_{32}}{i\delta_2+\varkappa_2} \qquad (3.27)$$

We shall consider the normally ordered variable of fluctuation in the in-phase $(b_1)$ and out-of-phase components $(b_2)$ of the mixture of signal modes $a_1$ and $a_2$

$$b_1 = \frac{1}{2}(b^{+} + b) \qquad b_2 = -\frac{1}{2}i(b^{+} - b)$$

where

$b = a_1 + a_2 \qquad\qquad b^+ = a_1 + a_2$

By using solution (10) and the steady-state density matrix (5), one finds the normally ordered variance of fluctuation of the operators $b_1$ and $b_2$ in the form

$$\langle:(\Delta b_{1,2})^2:\rangle = \frac{1}{4}\ [(g_1^2/x_1^2)\sin^2\alpha\langle R_{32}R_{23}\rangle_s + (g_2^2/x_2^2)\cos^2\alpha\langle R_{23}R_{32}\rangle_s \pm$$

$$\pm (g_1 g_2/x_1 x_2)\sin\alpha\cos\alpha(\langle R_{32}R_{23}\rangle_s + \langle R_{23}R_{32}\rangle_s)] \qquad (3.28)$$

where the statistical moments $\langle R_{32}R_{23}\rangle_s$ and $\langle R_{23}R_{32}\rangle_s$ are calculated over the atomic steady-state density matrix (5) (Bogolubov et al 1985).

In relation (11) and further, for simplicity, we take $\delta_1=\delta_2=0$. The symbol $\langle\ldots\rangle$ indicates the expectation value over the states of the heat bath and the atomic steady-state density matrix (5). Taking into account the noise operators $F_\lambda(t)$ $(\lambda=1,2)$ one can find the commutator of the Hermitian amplitude operators $b_1$ and $b_2$ as

$$\langle[b_1,b_2]\rangle = -\frac{1}{4}i[(g_1^2/x_1^2)\sin^2\alpha - (g_2^2/x_2^2)\cos^2\alpha](N - \frac{3}{2}\langle R\rangle_s) - i \qquad (3.29)$$

The factor of squeezing of the operators $b_1$ and $b_2$ can be defined as (Lakshmi et al 1984)

$$F_{1,2} = \langle:(\Delta b_{1,2})^2:\rangle / \ \frac{1}{2}|\langle[b_1,b_2]\rangle| \qquad (3.30)$$

The squeezing is present if the factors $F_1$ or $F_2$ are less than zero. For the case of X=1 we have $\langle R_{32}R_{23}\rangle_s=\langle R_{23}R_{32}\rangle_s$ and it follows that

$$\langle:(\Delta b_{1,2})^2:\rangle = \frac{1}{2}\ \langle R_{23}R_{32}\rangle_s \left(\frac{g_1\sin\alpha}{x_1} \pm \frac{g_2\cos\alpha}{x_2}\right)^2 \geq 0$$

thus, squeezing is absent in this case. Squeezing is also absent for a separate mode or (i.e. when ). The behaviour of the factor of squeezing as a function of the parameter cot when and =0.8 and as a function of the parameter when cot =0.7, =2 and =1.0is plotted in figs. 3.4 and 3.5, respectively. As can be seen from fig. 3.4, for the one atom case



the squeezing is small. For the case of a large number of atoms, as can be seen from Figs. 3.4 and 3.5, one can find the suitable

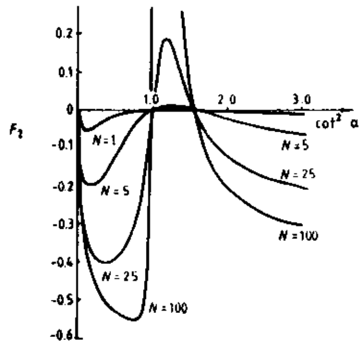


Fig. 3.4. Factor of squeezing as a function of the parameter $\cot^2\alpha$ for the case of $g_1/x_1 = g_2/x_2 = 1$ and $\gamma_{32}/\gamma_{21} = 0.8$.

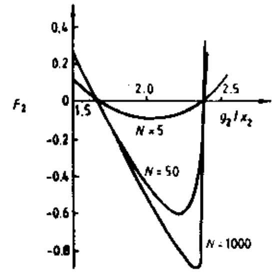


Fig. 3.5. Factor of squeezing as a function of the parameter $g_2/x_2$ for the case of $\cot^2\alpha = 0.7$, $g_1/x_1 = 2$ and $\gamma_{32}/\gamma_{21} = 0.8$.

values of the parameters $\cot^2\alpha$, $\gamma_{32}/\gamma_{21}$, $g_1/x_1$ and $g_2/x_2$ in which subsrantial squeezing is presented. In Fig. 3.4, 92% of the squeezing is obtained for the case of N = 1000. In the collective limit $N \to \infty$ the factor of squeezing tends to limiting value $F_2 = -1$.

REFERENCES


Agarwal G.S. - Springer Tracts in Modern Physics. Berlin, Springer, 1974.

Agarwal G.S. at. al. - Phys. Rev. Lett. 42 (1978) 1260.

Bogolubov N.N. Jr, Shumovsky A.S. and Tran Kuang - Phys. Lett. 112 A (1985) 323.

- - - - - - - - - Phys. Lett. 116 A (1986a) 175.

- - - - - - - - - Phys. Lett. 118 A (1986b) 315.

- - - - - - - - - J. Phys. B 20 (1987) L447.

Bondurant R.S. et. al. - Phys. Rev. A 30 (1984) 343.

Collet M.J. and Gardiner C.W. - Phys. Rev. A 30 (1986) 1386.

Haken H. - Laser theory. Berlin, Springer, 1970.

Klauder J.R., McCall S.L. and Yurke B. - Phys. Rev. A 33 (1986) 3204.

Lakshmi P.A. and Agarwal G.S. - Opt. Comm. 51 (1984) 425.

Ling An Wu et. al. - Phys Rev. Lett. 57 (1986) 2520.

Loudon R. - Opt. Comm. 49 (1984a) 24.

- - - - - - - - - Opt. Comm. 49 (1984b) 47.

Lugiato L.A. and Strini G. - Opt. Comm. 41 (1982a) 374.

- - - - - - - - - Opt. Comm. 41 (1982b) 67.

Mandel L. - Phys. Rev. Lett. 49 (1982) 136.

Milburn G.S. and Walls D.F. - )pt. Comm. 39 (1981) 401.

Reid M.D. and Walls D.F. - Phys. Rev. A 31 (1985a) 1622.

- - - - - - - - Phys. Rev. A 32 (1985b) 396.

- - - - - - - Phys. Rev. Lett. 55 (1985) 1288.

Savage C.M. and Walls D.F. - Phys. Rev. A 33 (1986) 3228.

Schwinger J. - Quantum theory of angular momentum. Ed. L.C.Biederham and H.Van Dam. New York, Academic, 1965.

Slusher R.E. et. al. - Phys. Rev. Lett. 55 (1985) 2409.

Walls D.F. and Zoller P. - Phys. Rev. Lett. 47 (1981) 709.

Yuen H. and Shapiro J.H. - Opt. Lett. 4 (1979) 334.