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Critical point theory and Hamiltonian systems

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Critical Point Theory and Hamiltonian Systems

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2. Consider the linear Hamiltonian systems

$$J\dot{u}(t) + A_1(t)u(t) = 0, \quad J\dot{u}(t) + A_2(t)u(t) = 0,$$

with A_1 and A_2 continuous mappings from $\mathbf{R}$ into the space of positive definite matrices of order $2N$. If

$$A_1(t) \le A_2(t)$$

for all $t \in [0,T]$, then

$$i(A_1,T) \le (A_2,T).$$

3. Let $H \in C^1(\mathbf{R}^{2N}, \mathbf{R})$ be strictly convex and such that

$$H(0) = 0, \quad \nabla H(0) = 0,$$

and let $T > 0$. Assume that there exists $n \in \mathbf{N}$, $\gamma \in]2\pi n/T, 2\pi(n+1)/T[$ such that

$$\nabla H(u) = \gamma u + o(|u|) \quad \text{as} \quad |u| \to \infty$$

and some $k \ge 1$ and $\beta > (2\pi/T)(n+k)$ such that

$$\nabla H(u) = \beta u + o(|u|) \quad \text{as} \quad |u| \to 0.$$

Then the problem

$$J\dot{u} + \nabla H(u) = 0,$$
$$u(0) = u(T)$$

has at least Nk S^1-orbits of nontrivial solutions. Compare this result with the Exercise 7 in Chapter 6.

Hint. Use Theorem 7.2 and the results of Section 7.2.

8

Morse Theory

Introduction

Morse theory's object is the relation between the topological type of critical points of a function φ and the topological structure of the manifold on which the function is defined.

The topological type of a critical point u is described by the *critical groups* of Morse $C_n(\varphi, u)$ (see Section 8.2) which exhibit the following properties:

a) in the nondegenerate case, the critical groups are computable by linearization (see Section 8.6);

b) the critical groups are stable under small perturbations of the function φ (see Sections 8.9 and 8.10).

If $\varphi'(u) = 0$ and $\varphi''(u)$ is invertible, then

$$\dim C_n(\varphi, u) = \delta_{n,k}$$

where k is the *Morse index* of $\varphi''(u)$. Recall that this Morse index is an integer measuring the maximal dimension of the spaces on which $\varphi''(u)$ is negative definite (see Section 8.6). We also present some results in the degenerate case when $\varphi''(u)$ is a Fredholm operator.

The topological structure of the manifold M is described by its *Betti numbers* B_n. Intuitively, B_n is the maximal number of n-dimensional surfaces without boundaries on M which are not the boundaries of a $(n+1)$ dimensional surface on M (see Section 8.1). For example B_0 is the number of path connected components of M. In the case of a sphere, every closed curve is a boundary and $B_1 = 0$. On the other hand, $B_1 = 2$ for the two-dimensional torus.

To illustrate Morse theory, let us consider the classical situation of the function $\varphi(x,y,z) = z$ defined on a two-dimensional torus $M \subset \mathbf{R}^3$ tangent to the plane Oxy (see Figure 8.1). The function φ has a critical point u_1 with Morse index zero, two critical points u_2 and u_3 with Morse index one, and one critical point u_4 with Morse index 2. If M_k denotes the number of critical points of φ with Morse index k, we have $B_k = M_k$ $(k = 0,1,2)$.

In general, for an N-dimensional compact manifold, the following relation is valid

$$\sum^{N} M_k t^k = \sum^{N} B_k t^k + (1+t)Q(t)$$

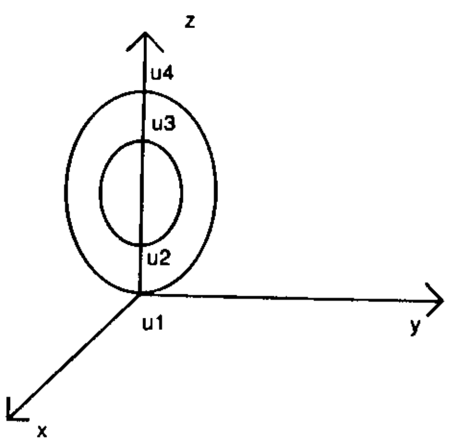

FIGURE 8.1.

where $Q(t)$ is a polynomial with nonnegative integer coefficients (see Section 8.5). In particular, for $t = -1$, we obtain the *Poincaré–Hopf formula* for a gradient vector field

$$\sum_{k=0}^{N}(-1)^k M_k = \sum_{k=0}^{N}(-1)^k B_k.$$

It is important to notice that a single degenerate critical point can contribute to different numbers M_k.

The proof of these results makes use of a deformation technique along the paths of steepest descent along $\nabla\varphi$. The corresponding tools and results are developed in Sections 8.3 and 8.4.

A first application of Morse theory deals with the bifurcation of solutions of equations depending upon a parameter. Loosely speaking, a change of critical groups of the trivial solution implies bifurcation. This result, which corresponds, in the context of Morse theory, to Krasnosel'skii's bifurcation theorem in degree theory, is given in Section 8.9.

8.1 Relative Homology

Let B be a subspace of a topological space A. For every integer n, we denote by $H_n(A,B)$ the nth *singular homology group* of the pair (A,B) over a field F. For $n \le -1$, $H_n(A,B) = \{0\}$. For any map $f : (A,B) \to (A',B')$ (i.e., any continuous map $f : A \to A'$ such that $f(B) \subset B'$) there is a homomorphism

$$f_{n*} : H_n(A,B) \to H_n(A',B')$$

called the *induced homomorphism*. Let $C \subset B$; there is a homomorphism

$$\partial_n : H_n(A,B) \to H_{n-1}(B,C)$$

called the *boundary homomorphism*. We shall frequently write f_* and ∂, omitting the subindex. The *Eilenberg–Steenrod axioms* are satisfied:

(a) $id_* = id$.

(b) $(g \circ f)_* = g_* \circ f_*$.

(c) The following diagram commutes:

$$\begin{array}{ccc} H_n(A,B) & \xrightarrow{f_*} & H_n(A',B') \\ \partial \downarrow & & \downarrow \partial \\ H_{n-1}(B,C) & \xrightarrow{(f|_B)_*} & H_{n-1}(B',C'). \end{array}$$

(d) (*Exactness*). Let

$$i : (B,C) \to (A,C)$$

$$j : (A,C) \to (A,B)$$

be the inclusion maps. The homology sequence

$$\cdots \to H_{n+1}(A,B) \xrightarrow{\partial} H_n(B,C) \xrightarrow{i_*} H_n(A,C) \xrightarrow{j_*} H_n(A,B) \to \cdots$$

is exact (i.e., the image of any homomorphism is equal to the kernel of the next one).

(e) (*Homotopy invariance*). If f and g are homotopic (i.e., $f = F(0,\cdot)$, $g = F(1,\cdot)$ for some continuous mapping $F : [0,1] \times A \to A'$ such that $F([0,1] \times B) \subset B'$), then $f_* = g_*$.

(f) (*Excision*). Assume that C is an open subset of A such that the closure of C is contained in the interior of B. Let $i : (A \setminus C, B \setminus C) \to (A,B)$ be the inclusion map. Then i_* is an isomorphism.

(g) If u is a point, then $H_n(\{u\},\phi) = \delta_{n,0}F$, where $\delta_{n,0}$ is the Kronecker symbol.

We shall also need the following results.

(h) (*Decomposition theorem*). If $(A,B) = \cup_{i=1}^{j}(A_i,B_i)$, where the A_i are closed and disjoint, then

$$H_n(A,B) = \oplus_{i=1}^{j} H_n(A_i,B_i).$$

(i) (*Mayer–Vietoris sequence*). Assume that X_1, X_2 are open in $X = X_1 \cup X_2$ and that $Y_1 \subset X_1$, $Y_2 \subset X_2$ are open in $Y = Y_1 \cup Y_2$. If $X_1 \cap X_2 \ne \phi$, there is an exact sequence

$$\cdots \to H_n(X_1,Y_1) \oplus H_n(X_2,Y_2) \xrightarrow{*} H_n(X,Y) \xrightarrow{\Delta} H_{n-1}(X_1 \cap X_2, Y_1 \cap Y_2)$$
$$\xrightarrow{*} H_{n-1}(X_1,Y_1) \oplus H_{n-1}(X_2,Y_2) \to \cdots$$

called the Mayer–Vietoris sequence of $\{(X_1, Y_1), (X_2, Y_2)\}$.

Since F is a field, the homology groups are vector spaces and the *Betti numbers* $B_n(A, B)$ of the pair (A, B) are defined by

$$B_n(A, B) = \dim H_n(A, B).$$

Let $R_n(A, B, C)$ be the rank of ∂_n. By exactness, we obtain

$$\begin{aligned} B_n(A, B) &= \dim R(j_{n*}) + R_n(A, B, C) \\ B_n(B, C) &= R_{n+1}(A, B, C) + \dim R(i_{n*}) \\ B_n(A, C) &= \dim R(i_{n*}) + \dim R(j_{n*}). \end{aligned} \tag{1}$$

A pair (A, B) is *admissible* if $B_n(A, B)$ is finite for each n and zero for all sufficiently large n. The *Poincaré polynomial* of an admissible pair (A, B) is defined by

$$P(t, A, B) = \sum_{n=0}^{\infty} B_n(A, B)\, t^n.$$

Let us also define

$$Q(t, A, B, C) = \sum_{n=0}^{\infty} R_{n+1}(A, B, C)\, t^n.$$

If (A, B) and (B, C) are admissible, then (1) implies that (A, C) is also admissible and that

$$B_n(A, B) + B_n(B, C) = B_n(A, C) + R_n(A, B, C) + R_{n+1}(A, B, C).$$

Multiplying this equation by t^n and adding over n, we get

$$P(t, A, B) + P(t, B, C) = P(t, A, C) + (1 + t)Q(t, A, B, C), \tag{2}$$

where we have used the fact that $R_0(A, B, C) = 0$.

Assume that $A_1 \supset A_2 \supset \ldots \supset A_j$ are such that (A_i, A_{i+1}) is admissible for $i = 1, \ldots, j-1$. Applying equation (2) to (A_1, A_i, A_{i+1}), we obtain

$$P(t, A_1, A_i) + P(t, A_i, A_{i+1}) = P(t, A_1, A_{i+1}) + (1 + t)Q(t, A_1, A_i, A_{i+1}).$$

Adding those equations, we find

$$\sum_{i=1}^{j-1} P(t, A_i, A_{i+1}) = P(t, A_1, A_j) + (1 + t)Q(t), \tag{3}$$

where $Q(t)$ is a polynomial with nonnegative integer coefficients (by exactness, $P(t, A_1, A_1) = 0$).

A subset A' of A is a *strong deformation retract of* A if there exists $h \in C([0,1] \times A, A)$ such that

$$h(t, u) = u \quad \text{whenever} \quad u \in A' \quad \text{and} \quad t \in [0, 1],$$

$$h(0, u) = u \quad \text{and} \quad h(1, u) \in A' \quad \text{whenever} \quad u \in A.$$

Assume that $A \supset A' \supset C$ with A' a strong deformation retract of A. Define $r : (A, C) \to (A', C')$ by

$$r(u) = h(1, u)$$

and let

$$i : (A', C) \to (A, C)$$

be the inclusion map. By homotopy invariance, we obtain

$$i_* \circ r_* = (i \circ r)_* = id_* = id,$$

and, on the other hand, by definition of r, we have

$$r_* \circ i_* = (r \circ i)_* = id_* = id,$$

so that r_* is an isomorphism between $H_n(A, C)$ and $H_n(A', C)$. In particular, if $C = A'$, we see that

$$H_n(A, A') \approx H_n(A', A') \approx \{0\}.$$

Now, if $A \supset B \supset B'$, with B' a strong deformation retract of B, we have, by the above result and exactness,

$$\{0\} = H_n(B, B') \xrightarrow{i_*} H_n(A, B') \xrightarrow{j_*} H_n(A, B) \xrightarrow{\partial} H_{n-1}(B, B') \approx \{0\},$$

and hence

$$\{0\} = \operatorname{Im} i_* = \ker j_*, \ \operatorname{Im} j_* = \ker \partial = H_n(A, B).$$

Thus $j_* : H_n(A, B') \to H_n(A, B)$ is one to one and

$$H_n(A, B') \approx H_n(A, B).$$

Let A be a subset of $\mathbf{R}^p$ containing 0 and let B^k be the k-ball. Then, for $k \geq 1$,

$$H_n(A \times B^k, (A \times B^k) \setminus \{0\}) \approx H_{n-k}(A, A \setminus \{0\}). \tag{4}$$

Proof. If $k \geq 2$, we obtain, after the identification $B^k \approx [-1, 1]^k$,

$$(A \times B^k, (A \times B^k) \setminus \{0\}) = (A \times B^{k-1} \times [-1, 1], (A \times B^{k-1} \times [-1, 1]) \setminus \{0\}).$$

Thus the result follows by induction from the case $k = 1$. Let us assume that $k = 1$. The suspension ΣA of A is obtained from $A \times [-1, 1]$ by identifying the pair of sets $(A \times \{-1\}, A \times \{1\})$ to a pair of points (w_-, w_+). By excision,

$$H_n(A \times [-1,1], (A \times [-1,1]) \setminus \{0\}) \approx H_n(\Sigma A, \Sigma A \setminus \{0\}).$$

Let us define the sets

$$X_+ = \Sigma A \setminus \{w_-\}, \quad X_- = \Sigma A \setminus \{w_+\},$$

$$Y_+ = X_+ \setminus (\{0\} \times]-1, 0]), \quad Y_- = X_- \setminus (\{0\} \times [0, 1[),$$

so that

$$X_+ \cup X_- = \Sigma A, \quad X_+ \cap X_- = A \times]-1, 1[,$$

$$Y_+ \cup Y_- = \Sigma A \setminus \{0\}, \quad Y_+ \cap Y_- = (A \setminus \{0\}) \times]-1, 1[.$$

Since X_+ and Y_+ (resp. X_- and Y_-) are contractible to w_+ (resp. w_-) we have, by homotopy invariance and exactness

$$H_n(X_\pm, Y_\pm) \approx H_n(\omega_\pm, \omega_\pm) \approx \{0\}.$$

The exactness of the Mayer–Vietoris sequence of $\{(X_+, Y_+), (X_-, Y_-)\}$ implies that

$$H_n(X_+ \cup X_-, Y_+ \cup Y_-) \approx H_{n-1}(X_+ \cap X_-, Y_+ \cap Y_-),$$

i.e.,

$$\begin{aligned} H_n(\Sigma A, \Sigma A \setminus \{0\}) &\approx H_{n-1}(A \times]-1, 1[, (A \setminus \{0\}) \times]-1, 1[) \\ &\approx H_{n-1}(A, A \setminus \{0\}), \end{aligned}$$

and the proof is complete. □

In particular, if $A = \{0\}$, we obtain

$$\begin{aligned} H_n(B_k, B_k \setminus \{0\}) &\approx H_n(\{0\} \times B^k, (\{0\} \times B^k) \setminus \{0\}) \\ &\approx H_{n-k}(\{0\}, \phi) = \delta_{n-k,0} F = \delta_{n,k} F. \end{aligned}$$

Let B^∞ (resp. S^∞) be the unit ball (resp. the unit sphere) in an infinite-dimensional normed space. Then, since S^∞ is a strong deformation retract of B^∞, we have

$$H_n(B^\infty, B^\infty \setminus \{0\}) \approx H_n(B^\infty, S^\infty) \approx H_n(S^\infty, S^\infty) \approx \{0\}.$$

8.2 Manifolds

Let M be a set and V a Banach space. A *chart* is a bijection $x : D(x) \subset M \to R(x) \subset V$ such that $R(x)$ is open. An *atlas of class* C^k ($k \geq 0$) on M is a set $\mathcal{A}$ of charts such that

(AT1) $\cup_{x \in \mathcal{A}} D(x) = M$.

(AT2) $x(D(x) \cap D(y))$ is an open subset of V whenever $x \in \mathcal{A}$ and $y \in \mathcal{A}$.

(AT3) the mapping

$$y \circ x^{-1} : x(D(x) \cap D(y)) \to y(D(x) \cap D(y))$$

is a C^k-diffeomorphism for each $x \in \mathcal{A}$ and $y \in \mathcal{A}$.

A *manifold of class* C^k *modeled on* V (or briefly a C^k*-manifold*) is a pair $(M, \mathcal{A})$ where M is a set and $\mathcal{A}$ is an atlas of class C^k on M. We will use the same symbol M to denote the C^k-manifold $(M, \mathcal{A})$ and the underlying set M. The topology of the manifold M is, by definition, the unique topology on M such that the domain of each chart is open and each chart is an homeomorphism.

Example 8.1. The singleton $\{Id : V \to V\}$ is an atlas of class C^∞ on the Banach space V.

Example 8.2. Let $\mathcal{A}$ be an atlas of class C^k on M and let N be an open subset of the manifold M. The restriction to N of the charts in $\mathcal{A}$ is an atlas of class C^k on N.

Example 8.3. Let G be a discrete subgroup of V and $\pi : V \to V/G$ the canonical projection. Then

$$\{\pi^{-1} : \pi(U) \to U, U \text{ is open and } \pi : U \to V/G \text{ is injective}\}$$

is an atlas of class C^∞ in V/G.

An important example of a manifold is given by the tangent bundle of a C^1-manifold. If x and y are two charts on M whose domains contain a point u, and if $v \in V$ and $w \in V$, let us introduce the equivalence relation (verify it!)

$$(u, x, v) \sim (u, y, w) \Leftrightarrow w = (y \circ x^{-1})'(x(u))v$$

and define the equivalent class

$$[u, x, v] = \{(u, y, w) : u \in D(y) \text{ and } (u, y, w) \sim (u, x, v)\}.$$

The *tangent space of* M at u is the set $T_u M$ of the equivalence classes $[u, x, v]$ such that $u \in D(x)$ and $v \in V$. A vector space structure is defined on $T_u M$ by the formulas

$$[u, x, v] + [u, x, w] = [u, x, v + w],$$

$$s[u,x,v] = [u,x,sv].$$

The chain rule shows that this definition is independent of the chart x.

The *tangent bundle* TM *of* M is defined by

$$TM = \bigcup_{u \in M} T_u M,$$

and the projection $\pi : TM \to M$ is defined by

$$\pi : [u,x,v] \to u.$$

Let M and N be C^k-manifolds modeled on Banach spaces V and W, respectively. A mapping $f : M \to N$ is *locally Lipschitzian* (resp. *of class* C^k) if $y \circ f \circ x^{-1}$ is locally Lipschitzian (resp. of class C^k) for every chart x on M and every chart y on N. If $f : M \to N$ is of class C^1, the *differential* of f is the mapping $df : TM \to TN$ defined by

$$df([u,x,v]) = [f(u), y, (y \circ f \circ x^{-1})'(x(u))v],$$

where x is a chart at u and y a chart at $f(u)$. One can check that this definition is independent of x and y and that the following diagram commutes

$$\begin{array}{ccc} TM & \xrightarrow{df} & TN \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{f} & N. \end{array}$$

If N is a Banach space W, then $TW \approx W^2$ and $df : TM \to W^2$ is defined by (taking $y = Id$ on W)

$$df([u,x,v]) = (f(u), (f \circ x^{-1})'(x(u))v).$$

In particular, if x is a chart on M,

$$dx : \pi^{-1}(D(x)) \to V^2$$

is a chart and $\{dx : x \in \mathcal{A}\}$ is an atlas of class C^{k-1} on TM, such that

$$dx([u,x,v]) = (x(u), v).$$

A *critical point* of $\varphi \in C^1(M, \mathbf{R})$ is a point $u \in M$ such that $d\varphi|_{T_u M} = 0$. The change in topology near an isolated critical point is described by the critical groups. We assume that the C^1-manifold M is regular. (Recall that a topological space is regular if every neighborhood of a point contains a closed neighborhood.) Let u be an isolated critical point of $\varphi \in C^1(M, \mathbf{R})$. The *critical groups* (over a field F) of u are defined by

$$C_n(\varphi, u) = H_n(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}), \quad n = 0, 1, \ldots,$$

where $c = \varphi(u)$ and U is a closed neighborhood of u. By excision, the critical groups are independent of U.

Let us complete the critical groups in a trivial but important case, namely when u is an isolated local minimum point. Then there exists a closed neighborhood U of u such that

$$\varphi(v) > c = \varphi(u)$$

whenever $v \in U \setminus \{u\}$. We obtain therefore

$$C_n(\varphi, u) = H_n(\{u\}, \phi) = \delta_{n,0} F, \quad n = 0, 1, \ldots.$$

8.3 Vector Fields

In this section, M will denote a Hausdorff manifold of class C^2 modeled on a Banach space V. A *vector field* on M is a mapping $f : M \to TM$ such that $\pi \circ f = Id$. If $\sigma :]a,b[\to M$ is a C^1-mapping, then we define $\dot\sigma(t)$ for $t \in]a,b[$ by

$$\dot\sigma(t) = [\sigma(t), x, (x \circ \sigma)'(t)(1)] = d\sigma[t, id, 1],$$

where x is a chart at $\sigma(t)$.

Proposition 8.1. *If f is a locally Lipschitzian vector field on M, then, for every $u \in M$, the Cauchy problem*

$$\begin{cases} \dot\sigma(t) = f(\sigma(t)) \\ \sigma(0) = u \end{cases} \tag{5}$$

has a solution defined on some open interval containing 0. *Moreover, if $\sigma_1 : I_1 \to M$ and $\sigma_2 : I_2 \to M$ is a pair of solutions of* (5) *defined on open intervals I_j $(j = 1, 2)$, then $\sigma_1 = \sigma_2$ on $I_1 \cap I_2$.*

Proof. Let x be a chart at u; near u, the Cauchy problem is equivalent to

$$dx(\dot\sigma(t)) = dx(f(\sigma(t)))$$

$$x(\sigma(0)) = x(u)$$

or

$$\begin{aligned} \dot\eta(t) &= (P \circ dx \circ f \circ x^{-1})(\eta(t)) \\ \eta(0) &= x(u) \end{aligned} \tag{6}$$

where $\eta = x \circ \sigma$ and $P : V \times V \to V$ is defined by $P(v, w) = w$. Since f is locally Lipschitzian, the same is true for $dx \circ f \circ x^{-1}$. The local theory of differential equations in a Banach space implies the existence of a solution $\eta :]-\epsilon, \epsilon[\to V$ of (6), and each solution of (6) defined on $]-\epsilon, \epsilon[$ is equal to η.

Let $I = \{t \in I_1 \cap I_2 : \sigma_1(t) = \sigma_2(t)\}$; this set contains 0 and is closed in $I_1 \cap I_2$ since M is Hausdorff. Using the local uniqueness result, it is easy to verify that I is open in $I_1 \cap I_2$, so that $I = I_1 \cap I_2$, and the proof is complete. □

Proposition 8.1 implies that the union of the graphs of all solutions of (5) defined on open intervals is a solution of (5) defined on an interval $]\omega_-(u), \omega_+(u)[$ with

$$-\infty \leq \omega_-(u) < 0 < \omega_+(u) \leq +\infty.$$

This solution is called the *maximal solution* of (5) and is denoted by $\sigma(\cdot, u)$.

As in the Banach space theory, the set

$$\mathcal{D} = \{(t,u) : \omega_-(u) < t < \omega_+(u)\}$$

is open in $\mathbf{R} \times M$ and the *flow*

$$\sigma : \mathcal{D} \to M, \quad (t,u) \to \sigma(t,u)$$

is continuous.

8.4 Riemannian Manifolds

Let M be a manifold of class C^k $(k \geq 1)$ modeled on a Hilbert space V. A *Riemannian metric of class* C^{k-1} on M is a mapping which associates to each pair (u, x), with $u \in M$ and x a chart at u, a positive definite invertible symmetric operator $M_x(u) : V \to V$ such that the following properties hold.

(RM1) The mapping M_x

$$D(x) \to \mathcal{L}(V) : u \to M_x(u)$$

is of class C^{k-1} for each chart x.

(RM2) If x and y are two charts at $u \in M$, then

$$[(y \circ x^{-1})'(x(u))]^* M_y(u)[(y \circ x^{-1})'(x(u))] = M_x(u).$$

It follows from (RM2) that the relation

$$([u,x,v],[u,x,w]) = (M_x(u)v, w)$$

defines an inner product on $T_u M$, and the corresponding norm is given by

$$|[u,x,v]| = (M_x(u)v, v)^{1/2}.$$

A *Riemannian manifold of class* C^k is a regular connected manifold of class C^k modeled on a Hilbert space and equipped with a Riemannian metric of class C^{k-1}.

Let M be a Riemannian manifold of class C^1. A *piecewise C^1 path from* $u \in M$ *to* $v \in M$ is a piecewise C^1 mapping $\sigma : [a,b] \to M$ such that $\sigma(a) = u$ and $\sigma(b) = v$. We shall denote by C_v^u the set of all piecewise C^1 paths from u to v and define the *length* of $\sigma \in C_v^u$ by

$$L(\sigma) = \int_a^b |\dot{\sigma}(t)|\,dt.$$

Proposition 8.2. *For each $u \in M$ and $v \in M$, the set C_v^u is non-empty.*

Proof. For each $u \in M$, define $A = \{v \in M : C_v^u \neq \phi\}$. Since M is connected and $A \neq \phi$, it suffices to prove that A is open and closed.

If $v \in A$, there is a path $\sigma : [a,b] \to M$ in C_v^u. Let x be a chart at $v = \sigma(b)$. There is a $r > 0$ such that $B = x^{-1}(B(x(v), r))$ is an open subset of $D(x)$, and thus of M. For $w \in B$, the path $\tilde{\sigma} : [a, b+1] \to M$ defined by

$$\tilde{\sigma}(t) = \sigma(t), \quad a \leq t \leq b$$

$$\tilde{\sigma}(t) = x^{-1}((1-(t-b))x(v) + (t-b)x(w)), \quad b \leq t \leq b+1$$

is in C_u^w. Thus $B \subset A$ and A is open.

Now let v be in the closure of A and x be a chart at v. Define B as before; there will exist $w \in A \cap B$ and then a path $\sigma : [a,b] \to M$ in C_u^w. The path $\tilde{\sigma} : [a, b+1]$ defined by

$$\tilde{\sigma}(t) = \sigma(t), \quad a \leq t \leq b$$

$$\tilde{\sigma}(t) = x^{-1}((1-(t-b))x(w) + (t-b)x(v)), \quad b \leq t \leq b+1$$

is in C_u^v. Thus $v \in A$ and A is closed. □

Proposition 8.2 justifies the following definition of the *geodesic distance* d on M

$$d(u,v) = \inf\{L(\sigma) : \sigma \in C_u^v\}.$$

Proposition 8.3. *The geodesic distance d is a distance on M whose topology is compatible with the manifold topology.*

Proof. Clearly d is symmetric and verifies the triangle inequality. Let x be a chart at $u \in M$. By definition, there exists $0 < \alpha \leq \beta$ such that

$$\alpha|h|^2 \leq (M_x(u)h, h) \leq \beta|h|^2, \quad h \in V.$$

By continuity, there exists $r > 0$ such that $B = x^{-1}(B(x(u), r))$ is an open subset of $D(x)$, and hence of M and such that

$$(\alpha/2)|h|^2 \leq (M_x(v)h, h) \leq 2\beta|h|^2, \quad v \in B, \ h \in V.$$

For every piecewise C^1 path $\sigma : [a,b] \to B$, we have

$$L(\sigma) = \int_a^b (M_x(\sigma(t))(x \circ \sigma)'(t), (x \circ \sigma)'(t))^{1/2}\,dt$$

$$\geq (\alpha/2)^{1/2} \int_a^b |(x \circ \sigma)'(t)|\, dt \geq (\alpha/2)^{1/2} \left| \int_a^b (x \circ \sigma)'(t)\, dt \right| \qquad (7)$$
$$= (\alpha/2)^{1/2} |(x \circ \sigma)(b) - (x \circ \sigma)(a)|.$$

For every $v \in B$, the path $\tilde{\sigma}$ defined by

$$\tilde{\sigma}(t) = x^{-1}((1-t)x(u) + tx(v)), \quad 0 \leq t \leq 1,$$

is such that

$$L(\tilde{\sigma}) \leq (2\beta)^{1/2} \int_0^1 |x(v) - x(u)|\, dt = (2\beta)^{1/2} |x(v) - x(u)|. \qquad (8)$$

Let A be a neighborhood of u in M. Since M is regular, there exists a closed neighborhood C of u such that $C \subset A \cap B$. Define $\delta > 0$ by

$$\delta = \inf\{|x(w) - x(u)| \,:\, w \in \partial C\}. \qquad (9)$$

Let $v \in M$. If $\sigma : [a,b] \to M$ belongs to C_u^v, then either $\sigma([a,b]) \subset C$ or there is a $c \in]a,b[$ such that $\sigma([a,c]) \subset C$ and $\sigma(c) \in \partial C$. In the first case, it follows from (7) that

$$L(\sigma) \geq (\alpha/2)^{1/2} |x(v) - x(u)|.$$

In the second case, (7) and (9) imply that

$$L(\sigma) \geq (\alpha/2)^{1/2} |x(\sigma(c)) - x(u)| \geq (\alpha/2)^{1/2} \delta.$$

In particular, $d(u,v) > 0$ for $v \neq u$ and d is a distance.
On the other hand, if $v \in M \setminus C$, then $L(\sigma) \geq (\alpha/2)^{1/2}\delta$ so that

$$\{v \in M \,:\, d(u,v) < (\alpha/2)^{1/2}\delta\} \subset C \subset A.$$

A being arbitrary, this implies that the topology induced by d is stronger than the manifold topology. Now (8) implies that

$$x^{-1}(B(x(u), (2\beta)^{-1/2} R)) \subset \{v \in M \,:\, d(u,v) \leq R\}$$

whenever $R \in]0, (2\beta)^{1/2} r[$, showing that the topology induced by d is weaker than the manifold topology. □

A subset of a Riemannian manifold of class C^1 will be said to be *complete* if it is complete for the geodesic distance.

Let M be a Riemannian manifold of class C^2 and let $\varphi \in C^{2-0}(M, \mathbf{R})$. The *gradient* of φ is the vector field defined on M by

$$\nabla\varphi(u) = [u, x, M_x^{-1}(u)\, J(\varphi \circ x^{-1})'(x(u))]$$

where $J : V^* \to V$ is the inverse duality mapping.

If $\varphi \in C^{2-0}(M, \mathbf{R})$, the Cauchy problem

$$\begin{cases} \dot{\sigma}(t) = -\nabla\varphi(\sigma(t)) \\ \sigma(0) = u \end{cases} \qquad (10)$$

has a unique maximal solution $\sigma(.) = \sigma(., u)$. Since

$$\begin{aligned}
\frac{d}{dt}(\varphi \circ \sigma)(t) &= \frac{d}{dt}(\varphi \circ x^{-1} \circ x \circ \sigma)(t) \\
&= \langle (\varphi \circ x^{-1})'(x(\sigma(t))), \frac{d}{dt}(x \circ \sigma)(t) \rangle \\
&= (M_x(\sigma(t)) M_x^{-1}(\sigma(t)) J(\varphi \circ x^{-1})'(x(\sigma(t))), \frac{d}{dt}(x \circ \sigma)(t)) \\
&= -(\nabla\varphi(\sigma(t)), \nabla\varphi(\sigma(t))) \\
&= -|\nabla\varphi(\sigma(t))|^2,
\end{aligned}$$

where x is a chart at $\sigma(t)$, either $\varphi(\sigma(t)) = \varphi(u)$ for all $t \geq 0$ or $\varphi \circ \sigma$ is decreasing. Moreover we have

$$\varphi(\sigma(t)) = \varphi(\sigma(s)) - \int_s^t |\nabla\varphi(\sigma(r))|^2 dr, \quad \omega_-(u) \leq s \leq t \leq \omega_+(u). \qquad (11)$$

Proposition 8.4. *Under the above assumptions, if $\omega_+(u)$ is finite and the set $\{\sigma(t) \,:\, t \in [0, \omega_+(u)[\}$ is contained in a complete subset of M, then $\varphi(\sigma(t)) \to -\infty$ when $t \to \omega_+(u)$.*

Proof. For $0 \leq s \leq t < \omega_+(u)$, the definition of d and (11) imply that

$$d(\sigma(t), \sigma(s)) \leq \int_s^t |\nabla\varphi(\sigma(r))|\, dr \leq (t-s)^{1/2} \left(\int_s^t |\nabla\varphi(\sigma(r))|^2 dr \right)^{1/2}$$
$$= (t-s)^{1/2} (\varphi(\sigma(s)) - \varphi(\sigma(t)))^{1/2}. \qquad (12)$$

Since $\omega_+(u) < \infty$, $\sigma(t)$ does not converge as $t \to \omega_+(u)$, and hence does not verify the corresponding Cauchy condition. Since $\varphi \circ \sigma$ is non-increasing, (12) implies that $\varphi(\sigma(t)) \to -\infty$ as $t \to \omega_+(u)$. □

8.5 Morse Inequalities

Let us consider the following framework:

i) M is a Riemannian manifold of class C^2 and $\varphi \in C^{2-0}(M, \mathbf{R})$;

ii) $X \subset M$ is positively invariant for the flow σ defined by (10) (i.e., $\sigma(t,u) \in X$ whenever $u \in X$ and $t \in]0, \omega_+(u)[$);

iii) $a < b$ are real numbers such that the critical points of φ in $\varphi^{-1}([a,b]) \cap X$ are isolated and contained in the interior of $\varphi^{-1}([a,b]) \cap X$;

iv) $\varphi^{-1}([a,b]) \cap X$ is complete;

v) the Palais–Smale condition over $\varphi^{-1}([a,b]) \cap X$ is satisfied, i.e., every sequence (u_j) in $\varphi^{-1}([a,b]) \cap X$ such that $|\nabla\varphi(u_j)| \to 0$ contains a convergent subsequence.

More generally, we shall say that φ satisfies the *Palais–Smale condition over a closed subset* S of M if every sequence $(u_j) \subset S$ such that $(\varphi(u_j))$ is bounded and $|\nabla\varphi(u_j)| \to 0$ contains a convergent subsequence.

Lemma 8.1. *Let M be a Riemannian manifold of class C^2 and let v be an isolated critical point of $\varphi \in C^{2-0}(M,\mathbf{R})$. If the Palais–Smale condition is satisfied over a closed neighborhood A of v, then there exists $\epsilon > 0$ and a neighborhood B of v such that, if $u \in B$, either $\sigma(t,u)$ stays in A for $0 < t < \omega_+(u)$, or $\sigma(t,u)$ stays in A until $\varphi(\sigma(t,u))$ becomes less than $\varphi(v) - \epsilon$.*

Proof. Let $\rho > 0$ be such that $B[v,\rho] \subset A$, φ is bounded on $B[v,\rho]$, and $C = \{u \in M : \rho/2 \le d(u,v) \le \rho\}$ is free of critical points. The Palais–Smale condition implies that

$$\delta = \inf\{|\nabla\varphi(u)| : u \in C\} > 0.$$

Let us define $B = B[v,\rho/2] \cap \varphi^{c+\delta\rho/4}$ where $c = \varphi(v)$. If $u \in B$ is such that $\sigma(t,u)$ does not stay in A for all $0 < t < \omega_+(u)$, then there exists $0 \le t_1 < t_2 < \omega_+(u)$ such that $\sigma(t,u) \in C$ for $t_1 \le t \le t_2$, $d(\sigma(t_1,u),v) = \rho/2$ and $d(\sigma(t_2,u),v) = \rho$. It follows from (11) that

$$\begin{aligned}
\varphi(\sigma(t_2,u)) &\le \varphi(\sigma(t_1,u)) - \delta\int_{t_1}^{t_2} |\nabla\varphi(\sigma(r,u))|\,dr \\
&\le \varphi(u) - \delta\int_{t_1}^{t_2} |\dot\sigma(r,u)|\,dr \\
&\le \varphi(u) - \delta d(\sigma(t_1,u),\sigma(t_2,u)) \\
&\le c + \delta\rho/4 - \delta(d(\sigma(t_2,u),v) - d(\sigma(t_1,u),v)) \\
&= c + \delta\rho/4 - \delta\rho/2 \\
&= \varphi(v) - \delta\rho/4,
\end{aligned}$$

and the proof is complete with $\epsilon = \delta\rho/4$. □

Lemma 8.2. *If Assumptions (A) hold, then, for every $u \in \varphi^{-1}([a,b]) \cap X$, either there is a (unique) $t \ge 0$ such that $\varphi(\sigma(t,u)) = a$ or $\omega_+(u) = +\infty$ and there is a critical point v of φ in $\varphi^{-1}([a,b]) \cap X$ such that $\sigma(t,u) \to v$ when $t \to +\infty$.*

Proof. If $\varphi(\sigma(t,u)) > a$ for all $t \in]0,\omega_+(u)[$, Proposition 8.4 implies that $\omega_+(u) = +\infty$, and hence $\varphi(\sigma(t,u)) \to c \ge a$ when $t \to +\infty$. By (11),

$$\int_0^\infty |\nabla\varphi(\sigma(r,u))|^2 dr < \infty.$$

Consequently, $\liminf_{t\to+\infty} |\nabla\varphi(\sigma(t,u))|^2 = 0$ and the (PS) condition implies the existence of a sequence (t_j) tending to $+\infty$ and of a critical point v such that $\sigma(t_j,u) \to v$ as $j \to \infty$. In particular, $v \in X$ and $c = \varphi(v)$. It follows then from Lemma 8.1 that $\sigma(t,u) \to v$ as $t \to +\infty$. □

Let us define, for $c \in [a,b]$,

$$X^c = \{u \in X : \varphi(u) \le c\}$$

$$K_c = \{u \in X : \varphi(u) = c, d\varphi(u) = 0\}.$$

Lemma 8.3. *Under assumptions (A), let $a \le \alpha < \beta \le b$ be such that $\varphi^{-1}(]\alpha,\beta[) \cap X$ is free of critical points. Then X^α is a strong deformation retract of $X^\beta \setminus K_\beta$. Moreover, φ is non-increasing during the deformation.*

Proof. By Lemma 8.2, if $u \in X^\beta \setminus K_\beta$ and $\varphi(u) > \alpha$, either there is a unique $t(u)$ such that $\varphi(\sigma(t(u),u)) = \alpha$ or $\varphi(\sigma(t,u)) \to \alpha$ as $t \to +\infty$. If $\psi(t,u) = \varphi(\sigma(t,u))$, then $D_t\psi(t(u),u) = -|\nabla(\sigma(t(u),u)|^2 \ne 0$, and $t(u)$ is continuous by the implicit function theorem. Define the function ρ by

$$\begin{aligned}
\rho(t,u) &= \sigma(t,u) && \text{if } 0 \le t \le t(u) \\
&= \sigma(t(u),u) && \text{if } t(u) < t < \infty
\end{aligned}$$

in the first case and by

$$\rho(t,u) = \sigma(t,u), \quad 0 \le t < +\infty$$

in the second case. Moreover define ρ by

$$\rho(t,u) = u, \quad 0 \le t < +\infty$$

whenever $u \in X^\alpha$. The continuity of the flow σ implies the continuity of ρ. Now define the deformation on $[0,1] \times (X^\beta \setminus K_\beta)$ by

$$\eta(t,u) = \rho\left(\frac{t}{1-t},u\right), \quad 0 \le t < 1$$

$$\eta(1,u) = \lim_{t\to\infty} \rho(t,u).$$

The continuity of η follows from Lemma 8.1 and from the continuity of ρ. By construction, $\varphi(\eta(.,u))$ is non-increasing, and the proof is complete. □

It is easy to verify that, under assumptions (A), $\varphi^{-1}([a,b]) \cap X$ contains at most a finite number of critical points $u_1,\ldots,u_j$. The *Morse numbers* of the pair (X^b, X^a) are defined by

$$M_n(X^b,X^a) = \sum_{i=1}^{j} \dim C_n(\varphi,u_i), \quad n = 0,1,\ldots.$$

If $M_n(X^b, X^a)$ is finite for every n and is equal to zero for n sufficiently large, the *Morse polynomial* of the pair (X^b, X^a) is defined by

$$M(t, X^b, X^a) = \sum_{n=0}^{\infty} M_n(X^b, X^a) t^n.$$

Theorem 8.1. *Under assumptions* (A) *if, every critical point in* $\varphi^{-1}([a,b] \cap X$ *corresponds to the same critical value* $c \in]a,b[$, *then*

$$M_n(X^b, X^a) = B_n(X^b, X^a), \quad n = 0, 1, \ldots.$$

Proof. Lemma 8.3 implies that

$$H_n(X^b, X^a) \approx H_n(X^c, X^a) \approx H_n(X^c, X^c \setminus K_c).$$

Since $K_c = \{u_1, \ldots, u_j\}$ is contained in the interior of $\varphi^{-1}([a,b]) \cap X$, the critical points have disjoint closed neighborhoods $U_1, \ldots, U_j$ such that

$$U = \bigcup_{i=1}^{j} U_i \subset \varphi^{-1}([a,b]) \cap X.$$

Therefore we obtain, by the excision and decomposition properties,

$$\begin{aligned} H_n(X^c, X^c \setminus K_c) &\approx H_n(X^c \cap U, (X^c \setminus K_c) \cap U) \\ &= H_n(\varphi^c \cap U, (\varphi^c \setminus K_c) \cap U) \\ &\approx \bigoplus_{i=1}^{j} H_n(\varphi^c \cap U_i, \varphi^c \cap U_i \setminus \{u_i\}) \\ &= \bigoplus_{i=1}^{j} C_n(\varphi, u_i), \end{aligned}$$

and the result follows from the definitions. □

Theorem 8.2. *Under assumptions* (A), *if* $M_n(X^b, X^a)$ *is finite for every* n *and equal to zero for* n *sufficiently large, then there exists a polynomial* $Q(t)$ *with nonnegative integer coefficients such that*

$$M(t, X^b, X^a) = P(t, X^b, X^a) + (1+t)Q(t).$$

Proof. Let $a < c_1 < \ldots < c_j < b$ be the critical values corresponding to the critical points in $\varphi^{-1}([a,b]) \cap X$. If we take real numbers a_i such that

$$a = a_0 < c_1 < a_1 < c_2 < \ldots < a_{j-1} < c_j < a_j = b,$$

Theorem 8.1 implies that the pairs $(X^{a_{i+1}}, X^{a_i})$ are admissible and that

$$\sum_{i=0}^{j-1} B_n(X^{a_{i+1}}, X^{a_i}) = \sum_{i=0}^{j-1} M_n(X^{a_{i+1}}, X^{a_i}) = M_n(X^b, X^a).$$

It follows then from formula (3) that

$$M(t, X^b, X^a) = \sum_{i=0}^{j-1} P(t, X^{a_{i+1}}, X^{a_i}) = P(t, X^b, X^a) + (1+t)Q(t),$$

where $Q(t)$ is a polynomial with nonnegative integer coefficients. □

Remarks.

1. Theorem 8.2 implies that

$$M_n(X^b, X^a) \geq B_n(X^b, X^a), \quad n = 0, 1, \ldots,$$

and that

$$\sum_{n=0}^{\infty} (-1)^n M_n(X^b, X^a) = \sum_{n=0}^{\infty} (-1)^n B_n(X^b, X^a).$$

The second relation is an extension of the Poincaré–Hopf formula.

2. If $M_n(X^b, X^a)$, $M_{n+1}(X^b, X^a) \equiv 0$ for every n, then necessarily

$$M(t, X^b, X^a) = P(t, X^b, X^a).$$

The above observation is called the *Morse lacunary principle*.

Let us now extend Theorem 8.2 to the case of an unbounded interval $[a, +\infty[$.

Lemma 8.4. *Let* M *be a Riemannian manifold of class* C^2, *let* $\varphi \in C^{2-0}(M, \mathbf{R})$, *and let* X *be a subset of* M *positively invariant with respect to the flow* σ *defined by* (10). *If for every* $d > b$, $\varphi^{-1}([b,d]) \cap X$ *is complete and free of critical points, and if* φ *satisfies* (PS) *over* $\varphi^{-1}([b,d]) \cap X$, *then* X^b *is a strong deformation retract of* X.

Proof. Let $u \in X$ be such that $\varphi(u) > b$. If $\varphi(\sigma(t,u)) > b$ for every $t \in]0, \omega^+(u)[$ then, as in the first part of the proof of Lemma 8.2, there exists a critical point v of φ in X such that $\varphi(v) \geq b$. But this is not possible by assumption. Thus there exists a unique $t(u)$ such that $\varphi(\sigma(t(u), u)) = b$. The deformation can then be given on $[0,1] \times X$ by

$$\eta(s,u) = \sigma(t(u)s, u), \quad 0 \leq s \leq 1$$

if $u \in X \setminus X^b$ and by

$$\eta(s,u) = u, \quad 0 \leq s \leq 1$$

if $u \in X^b$. □

Let us suppose that, in addition to assumption (A), the following condition holds.

(B) For every $d \geq b$, $\varphi^{-1}([b,d[) \cap X$ is complete and free of critical points and φ satisfies (PS) over $\varphi^{-1}([b,d[) \cap X$.

The *Morse numbers* of the pair (X, X^a) are defined by

$$M_n(X, X^a) = M_n(X^b, X^a).$$

If $M_n(X, X^a)$ is finite for every n and equal to zero for n sufficiently large, the *Morse polynomial* of the pair (X, X^a) is defined by

$$M(t, X, X^a) = M(t, X^b, X^a).$$

Corollary 8.1. *Under assumptions* (A) *and* (B), *if $M_n(X, X^a)$ is finite for every n and equal to zero for n sufficiently large, there exists a polynomial $Q(t)$ with nonnegative integer coefficients such that*

$$M(t, X, X^a) = P(t, X, X^a) + (1+t)Q(t).$$

Proof. By Lemma 8.4, X^b is a strong deformation retract of X so that

$$P(t, X, X^a) = P(t, X^b, X^a).$$

The result then follows from Theorem 8.2 and from the definition of $M(t, X, X^a)$. □

Corollary 8.2. *Let M be a complete Riemannian manifold of class C^2 and let $\varphi \in C^{2-0}(M, \mathbf{R})$. If*

i) *φ satisfies the Palais–Smale condition over M,*

ii) *φ is bounded from below on M,*

iii) *φ has only a finite number of critical points $u_1, \dots, u_j$ and $\dim C_n(\varphi, u_i)$ is finite for every n and zero for n sufficiently large, $i = 1, \dots, j$,*

then there exists a polynomial $Q(t)$ with nonnegative integer coefficients such that

$$\sum_{n=0}^{\infty}\sum_{i=1}^{j} \dim C_n(\varphi, u_i)t^n = P(t, M, \phi) + (1+t)Q(t).$$

Proof. Let $a < \inf_M \varphi$ and $b > \sup\{\varphi(u) : \nabla\varphi(u) = 0\}$. It suffices to apply Corollary 8.1 with $X = M$. □

8.6 The Generalized Morse Lemma

The generalized Morse lemma, also called the splitting theorem, is the basic tool for the effective computation of critical groups. The theory of Fredholm operators provides a natural setting for this lemma.

A linear continuous operator L between two Banach spaces is called a *Fredholm operator* if the dimension of $\ker L$ and the codimension of $R(L)$ are finite. This implies that $R(L)$ is closed.

Let V be a Hilbert space, U an open neighborhood of $u \in V$, and let $\varphi \in C^2(U, \mathbf{R})$. Define implicitly the linear operator $L : V \to V$ by

$$(Lv, w) = \varphi''(u)(v, w).$$

Then L is self-adjoint and we shall identify L with $\varphi''(u)$. If $\varphi''(u)$ is a Fredholm operator, V is the orthogonal sum of $R(\varphi''(u))$ and $\ker(\varphi''(u))$.

Assume now that u is a critical point of φ. The *Morse index of* u is defined as the supremum of the dimensions of the vector subspaces of V on which $\varphi''(u)$ is negative definite. The *nullity* of u is defined as the dimension of $\ker \varphi''(u)$. Finally, the critical point u will be said to be *non-degenerate* if $\varphi''(u)$ is invertible.

Theorem 8.3. *Let U be an open neighborhood of 0 in a Hilbert space V and let $\varphi \in C^2(U, \mathbf{R})$. Suppose that 0 is a critical point of φ with positive nullity and that $L = \varphi''(0)$ is a Fredholm operator, so that V is the orthogonal direct sum of $\ker(L)$ and $R(L)$. Let $w + v$ be the corresponding decomposition of $u \in V$. Then there exists an open neighborhood A of 0 in V, an open neighborhood B of 0 in $\ker(L)$, a local homeomorphism h from A into U, and a function $\hat{\varphi} \in C^2(B, \mathbf{R})$ such that*

$$h(0) = 0, \quad \hat{\varphi}'(0) = 0, \quad \hat{\varphi}''(0) = 0$$

and

$$\varphi(h(u)) = (1/2)(Lv, v) + \hat{\varphi}(w)$$

on the domain of h.

Proof. 1) Let $Q : V \to V$ be the orthogonal projection onto $R(L)$. By the implicit function theorem, we can find $r_1 > 0$ and a C^1-mapping

$$g : B(0, r_1) \cap \ker L \to R(L)$$

such that $g(0) = 0$, $g'(0) = 0$ and

$$Q\nabla\varphi(w + g(w)) = 0. \tag{13}$$

Let us define $\hat{\varphi}$ on $B = B(0, r_1) \cap \ker L$ by

$$\hat{\varphi}(w) = \varphi(w + g(w))$$

so that, by direct computation and (13),

$$\nabla\hat{\varphi}(w) = (I - Q)\nabla\varphi(w + g(w))$$

and

$$\hat{\varphi}''(w) = (I - Q)\varphi''(w + g(w))(Id + g'(w)).$$

In particular

$$\nabla\hat{\varphi}(0) = (I - Q)\nabla\varphi(0) = 0$$

and

$$\hat{\varphi}''(0) = (I-Q)\varphi''(0) = (I-Q)L = 0.$$

Let us define, near $[0,1] \times \{0\}$, the function

$$\Phi(t,v,w) = (1-t)(\hat{\varphi}(w) + (1/2)(Lv,v)) + t\varphi(v+w+g(w))$$

and the vector field

$$\begin{aligned} f(t,v,w) &= 0, && \text{if } v = 0, \\ &= -\Phi_t(t,v,w)\,|\Phi_v(t,v,w)|^{-2}\Phi_v(t,v,w), && \text{if } v \neq 0. \end{aligned}$$

If $\eta(t) = \eta(t,v,w)$ is a solution of the Cauchy problem

$$\dot{\eta} = f(t,\eta,w)$$

$$\eta(0) = v$$

we have

$$\begin{aligned} \frac{d}{dt}\Phi(t,\eta(t),w) &= \Phi_t(t,\eta(t),w) + (\Phi_v(t,\eta(t),w),\dot{\eta}(t)) \\ &= 0 \end{aligned}$$

and, in particular,

$$\begin{aligned} \hat{\varphi}(w) + (1/2)(Lv,v) &= \Phi(0,v,w) \\ &= \Phi(1,\eta(1,v,w),w) \\ &= \varphi(\eta(1,v,w)+w+g(w)). \end{aligned}$$

Let us assume that the flow $\eta(t,v,w)$ is well defined and continuous on $[0,1] \times A$, where A is an open neighborhood of 0 in V. Then the local homeomorphism h is given by

$$h(u) = h(v,w) = w + g(w) + \eta(1,v,w).$$

The local invertibility of h follows from the local invertibility of $\eta(1,\cdot,w)$.

2) It remains to prove that η is well defined and continuous. Let us define Ψ by

$$\Psi(v,w) = \varphi(v+w+g(w)) - \hat{\varphi}(w) - (1/2)(Lv,v).$$

We obtain, using (13),

$$\Psi(0,w) = 0, \quad \Psi_v(0,w) = 0, \quad \Psi_v''(0,0) = 0;$$

and, consequently

$$\Psi(v,w) = \int_0^1 (1-s)(\Psi_v''(sv,w)v,v)\,ds$$

$$\Psi_v(v,w) = \int_0^1 \Psi_v''(sv,w)v\,ds.$$

Thus, for each $\epsilon > 0$, there exists $\delta(\epsilon) \in \,]0,r_1[$ such that

$$|\Psi(v,w)| \le \epsilon|v|^2, \quad |\Psi_v(v,w)| \le \epsilon|v| \tag{14}$$

whenever $|v+w| \le \delta(\epsilon)$. Since $L : R(L) \to R(L)$ is continuous and invertible, there exists $c > 0$ such that

$$c^{-1}|v| \le |Lv| \le c\,|v| \tag{15}$$

for $v \in R(L)$. We have, for $v \neq 0$,

$$f(t,v,w) = -\Psi(v,w)|Lv + t\Psi_v(v,w)|^{-2}(Lv + t\Psi_v(v,w)).$$

Let $\epsilon = (2c)^{-1}$. Using (14) and (15), we obtain, for $|v+w| \le \delta(\epsilon)$,

$$|f(t,v,w)| \le 2c(c+\epsilon)\epsilon\,|v|. \tag{16}$$

Since $f(t,0,w) = 0$, f is continuous. Let $\rho \in \,]0,\delta(\epsilon)[$ be such that

$$|\Psi_v''(v,w)| \le 1 \tag{17}$$

for $|v+w| \le \rho$ and $v \neq 0$. Using (14), (15), and (17), it is easy to verify the existence of $c_1 > 0$ such that

$$|f_v(t,v,w)| \le c_1$$

for $|v+w| \le \rho$ and $v \neq 0$. By the mean value theorem and (16), there exists $c_2 > 0$ such that

$$|f(t,v_1,w) - f(t,v_2,w)| \le c_2|v_1 - v_2|$$

for $|v_i + w| \le \rho$, $i = 1,2$. Thus the flow η is locally well defined and continuous. Moreover, since $\eta(t,0,w) = 0$, η is well defined on $[0,1] \times A$ where A is an open neighborhood of 0 in V. □

Remarks. 1) It is easy to verify that h restricted to $R(L)$ is a local diffeomorphism since $f_v(t,v,0)$ is continuous.

2) A similar but simpler proof gives the following result, which is called the *Morse lemma*.

Theorem 8.3bis. *Let U be a neighborhood of 0 in a Hilbert space V and let $\varphi \in C^2(U,\mathbf{R})$ be such that 0 is a non-degenerate critical point of φ. Then there exists an open neighborhood A of 0 in V and a local diffeomorphism h from A into U such that $h(0) = 0$ and*

$$\varphi(h(u)) = \varphi(0) + (1/2)(\varphi''(0)u,u).$$

Let now M be a regular C^2-manifold modelled on a Hilbert space V and let u be an isolated critical point of $\varphi \in C^2(M, \mathbf{R})$. The *Morse index* (resp. the nullity) of u is defined as the Morse index (resp. the *nullity*) of $x(u)$ as a critical point of $\varphi \circ x^{-1}$, where x is a chart at u. The critical point u is called *non-degenerate* if $x(u)$ is a non-degenerate critical point of $\varphi \circ x^{-1}$.

Remarks. 1. If y is another chart at u, then, since

$$\varphi \circ x^{-1} = \varphi \circ y^{-1} \circ y \circ x^{-1}$$

on $D(x) \cap D(y)$, it is easy to verify that

$$(\varphi \circ x^{-1})''(x(u)) = (\varphi \circ y^{-1})''(y(u))[(y \circ x^{-1})'(x(u)), (y \circ x^{-1})'(x(u))].$$

The invertibility of $(y \circ x^{-1})'(x(u))$ implies that the above definitions are independent of the chart x.

2. In the non-degenerate case, the Morse index is the supremum of the dimensions of the subspaces along which φ is decreasing near the critical point u.

3. By the implicit function theorem (or by the Morse lemma) any non-degenerate critical point is isolated.

We now show that the critical groups of a non-degenerate critical point depend only upon its Morse index.

Corollary 8.3. *Let M be a regular C^2 manifold modeled on a Hilbert space V and let u be a non-degenerate critical point of $\varphi \in C^2(M, \mathbf{R})$ with Morse index k. Then*

$$C_n(\varphi, u) = \delta_{n,k} F, \quad n = 0, 1, \ldots.$$

Proof. 1) Let x be a chart at u and let $U \subset D(x)$ be a closed neighborhood of u. Since, by definition

$$C_n(\varphi, u) = H_n(\varphi^c \cap U, \varphi^c \cap U \setminus \{0\})$$

with $c = \varphi(u)$, it is sufficient to consider the case where M is an open subset of V.

2) We can assume without loss of generality that $u = 0$ and $c = 0$. By Theorem 8.3bis, there exists an open neighborhood A of 0 in M and a local homeomorphism h from A into V such that $h(0) = 0$ and

$$\varphi(h(u)) = \psi(u) \equiv (1/2)(\varphi''(0)u, u)$$

whenever $u \in A$. Let $B \subset A$ be a closed ball centered at 0. We have

$$\begin{aligned} C_n(\varphi, 0) &= H_n(\varphi^0 \cap h(B), \varphi^0 \cap h(B) \setminus \{0\}) \\ &\approx H_n(\psi^0 \cap B, \psi^0 \cap B \setminus \{0\}). \end{aligned}$$

From the invertibility of $\varphi''(0)$ it follows that V is the orthogonal sum of V^- and V^+ with ψ negative (resp. positive) definite on V^- (resp. V^+). Let $v = v^- + v^+$ be the corresponding decomposition of any $v \in V$. Define the deformation η of B by

$$\eta : [0,1] \times B \to B, \quad (t, v) \to v^- + (1-t)v^+$$

so that

$$\psi(\eta(t, v)) = \psi(v^-) + (1-t)^2 \psi(v^+).$$

Thus $V^- \cap B \setminus \{0\}$ is a deformation retract of $\psi^0 \cap B \setminus \{0\}$ and $V^- \cap B$ is a deformation retract of $\psi^0 \cap B$. Since, by definition, $k = \dim V^-$, we obtain, for $k \geq 1$,

$$\begin{aligned} H_n(\psi^0 \cap B, \psi^0 \cap B \setminus \{0\}) &\approx H_n(\psi^0 \cap B, V^- \cap B \setminus \{0\}) \\ &\approx H_n(V^- \cap B, V \cap B \setminus \{0\}) \approx H_n(B^k, S^{k-1}) \approx \delta_{n,k} F, \end{aligned}$$

and for $k = 0$,

$$H_n(\psi^0 \cap B, \psi^0 \cap B \setminus \{0\}) \approx H_n(\{0\}, \phi) = \delta_{n,0} F. \quad \square$$

Remarks. 1. If the Morse index of a nondegenerate critical point u is infinite, then all the critical groups of φ at u are isomorphic to 0.

2. Under the assumptions of Theorem 8.2, if the critical points of φ in $\varphi^{-1}([a, b]) \cap X$ are non-degenerate, then $M_n(X^b, X^a)$ is equal to the number of critical points of φ with Morse index n in $\varphi^{-1}([a, b]) \cap X$.

8.7 Computation of the Critical Groups

The use of Morse inequalities depends on the effective computation of the critical groups in the degenerate case.

Lemma 8.5. *Let U be an open neighborhood of v in a Hilbert space V and let $\varphi \in C^{2-0}(U; \mathbf{R})$. If v is the only critical point of φ, and if the Palais–Smale condition is satisfied over a closed ball $B[v, r] \subset U$, then there exists $\epsilon > 0$ and $X \subset U$ such that:*

i) *X is a neighborhood of v, closed in U;*

ii) *X is positively invariant for the flow σ defined by* (10);

iii) *$\varphi^{-1}([c - \epsilon, c + \epsilon]) \cap X$ is complete, where $c = \varphi(v)$;*

iv) *the Palais–Smale condition is satisfied over $\varphi^{-1}([c - \epsilon, c + \epsilon]) \cap X$.*

Proof. Let $\epsilon > 0$ and $B \subset U$ be given by Lemma 8.1 applied to $A = B[v, r]$ and let X be the closure in U of the set

$$Y = \{\sigma(t, u) : u \in B, 0 \leq t < \omega_+(u)\}.$$

By construction, X satisfies i) and ii). Lemma 8.1 implies that

$$\varphi^{-1}([c-\epsilon, c+\epsilon]) \cap Y \subset B[v, r].$$

Since $B[v,r]$ is closed in U, the set $\varphi^{-1}([c-\epsilon, c+\epsilon]) \cap X$ is contained in $B[v,r]$ and closed in $B[v,r]$; hence it is complete. By our Palais–Smale assumption, iv) then follows from iii). □

We shall prove that, in the setting of Theorem 8.3, the critical groups depend on the Morse index and on the "degenerate part" of the functional. Thus the computation of the critical groups is reduced to a finite dimensional problem. This result is called the Shifting theorem.

Theorem 8.4. *Under the assumptions of Theorem 8.3, if 0 is the only critical point of φ, and if the Morse index k of 0 is finite, then*

$$C_n(\varphi, 0) \approx C_{n-k}(\hat\varphi, 0), \quad n = 0, 1, \ldots.$$

Proof. 1) With the notations of Theorem 8.3, let $C \subset A$ be a closed neighborhood of 0. Setting $c = \varphi(0) = \hat\varphi(0)$ and $\psi(u) = \psi(v+w)$ $(1/2)(Lv, v) + \hat\varphi(w)$, we obtain

$$\begin{aligned} C_n(\varphi, 0) &= H_n(\varphi^c \cap h(C), \varphi^c \cap h(C) \setminus \{0\}) \\ &\approx H_n(\psi^c \cap C, \psi^c \cap C \setminus \{0\}) = C_n(\psi, 0). \end{aligned}$$

2) By assumption, $0 \in \ker L$ is the only critical point of $\hat\varphi \in C^2(B, \mathbf{R})$. Since $\dim \ker L$ is finite, the Palais–Smale condition is satisfied over any closed ball $B[0,r] \subset B$. Let $\epsilon > 0$ and $X \subset B$ be given by Lemma 8.5 applied to $\hat\varphi$. Lemma 8.3 implies that X^c is a strong deformation retract of $X^{c+\epsilon}$. Moreover, $\hat\varphi$ is non-increasing during the corresponding deformation η. Define the deformation Δ over $D = R(L) \times X^{c+\epsilon}$ by

$$\Delta(t, v, w) = v^- + (1-t)v^+ + \eta(t, w).$$

It is easy to verify that $V^- \times X^c$ is a strong deformation retract of $\psi^c \cap D$ and that $(V^- \times X^c) \setminus \{0\}$ is a strong deformation retract of $\psi^c \cap D \setminus \{0\}$. Therefore we obtain

$$\begin{aligned} C_n(\psi, 0) &= H_n(\psi^c \cap D, \psi^c \cap D \setminus \{0\}) \\ &\approx H_n(V^- \times X^c, (V^- \times X^c) \setminus \{0\}). \end{aligned}$$

3) If $k = \dim V^- = 0$, we have

$$\begin{aligned} C_n(\psi, 0) &= H_n(X^c, X^c \setminus \{0\}) \\ &= H_n(\hat\varphi^c \cap X, \hat\varphi^c \cap X \setminus \{0\}) = C_n(\hat\varphi, 0), \end{aligned}$$

and the proof is complete. If $k \geq 1$, relation (4) implies that

$$\begin{aligned} C_n(\psi, 0) &\approx H_n(\mathbf{R}^k \times X^c, (\mathbf{R}^k \times X^c) \setminus \{0\}) \\ &\approx H_n(B^k \times X^c, (B^k \times X^c) \setminus \{0\}) \\ &\approx H_{n-k}(X^c, X^c \setminus \{0\}) = C_{n-k}(\hat\varphi, 0). \end{aligned}$$ □

Lemma 8.6. *Let U be an open subset of $\mathbf{R}^p$ and let v be the only critical point of $\varphi \in C^2(U, \mathbf{R})$. Then, for every $\rho > 0$, there exists $\tilde\varphi \in C^2(U, \mathbf{R})$ such that the following hold:*

a) *The critical points of $\tilde\varphi$, if any, are finite in number and non-degenerate.*

b) *If $|u - v| \geq \rho$, then $\tilde\varphi(u) = \varphi(u)$.*

c) *If $u \in U$, then*

$$|\tilde\varphi(u) - \varphi(u)| + |\tilde\varphi'(u) - \varphi'(u)| + |\tilde\varphi''(u) - \varphi''(u)| \leq \rho.$$

Proof. We can assume that the closed ball $B[v, \rho]$ is contained in U. Let $\omega \in C^2(U, \mathbf{R})$ be such that

$$\omega(u) = \begin{cases} 1 & \text{if } |u - v| \leq \rho/2 \\ 0 & \text{if } |u - v| \geq \rho \end{cases}$$

and let $e \in \mathbf{R}^p$. The function $\tilde\varphi \in C^2(U, \mathbf{R})$ defined by

$$\tilde\varphi(u) = \varphi(u) - \omega(u)(u, e)$$

satisfies b). It is easy to verify the existence of $\alpha > 0$ such that c) is satisfied for $|e| \leq \alpha$. Since

$$\nabla\tilde\varphi(u) = \nabla\varphi(u) - \omega(u)e - \nabla\omega(u)(u, e),$$

we obtain

$$|\nabla\tilde\varphi(u)| \geq |\nabla\varphi(u)| - |e|\,|\omega(u)| - |\nabla\omega(u)|\,|u|\,|e|.$$

But

$$\delta = \inf\{|\nabla\varphi(u)| : \rho/2 \leq |u - v| \leq \rho\} > 0.$$

Thus there exists $\beta \in \,]0, \alpha]$ such that, for $|e| \leq \beta$,

$$\inf\{|\nabla\tilde\varphi(u)| : \rho/2 \leq |u - v| \leq \rho\} \geq \delta/2.$$

By Sard's theorem, we can assume that e is a regular value of $\nabla\varphi$ such that $|e| \leq \beta$. If $|u - v| \geq \rho$, $\tilde\varphi(u) = \varphi(u)$, so that $\nabla\tilde\varphi(u) \neq 0$. If $\rho/2 \leq |u - v| \leq \rho$, we have $|\nabla\tilde\varphi(u)| \geq \delta/2$. If $|u - v| < \rho/2$, then, by definition

$$\nabla\tilde\varphi(u) = 0 \quad \text{if and only if} \quad \nabla\varphi(u) = e.$$

Since e is a regular value of $\nabla\varphi$, the critical points of $\tilde\varphi$ are non degenerate and, consequently, isolated. Being contained in $B[v, \rho/2]$, they must be finite in number. □

Let U be an open subset of $\mathbf{R}^p$ and let v be an isolated zero of $f \in C(\overline{U}, \mathbf{R}^p)$. Assume that $r > 0$ is such that the ball $B[v,r]$ is contained in $\overline{U}$ and v is the unique zero of f in $B[v,r]$. Then the *topological index* $i(f,v)$ of f at v is defined by

$$i(f,v) = d(f, B(v,r)).$$

By the excision property of the topological degree, the right-hand member is independent of r.

The following theorem gives a relation between the topological index and the critical groups.

Theorem 8.5. *Let U be an open subset of $\mathbf{R}^p$ and let v be an isolated critical point $\varphi \in C^2(U,\mathbf{R})$. Then $\dim C_n(\varphi, v)$ is finite for every n and is zero for $n \geq p+1$. Moreover*

$$i(\nabla\varphi, v) = \sum_{n=0}^{p}(-1)^n \dim C_n(\varphi, v).$$

Proof. 1) By diminishing U if necessary, we can assume that v is the only critical point of φ lying in U. Moreover, the Palais–Smale condition is satisfied over any closed ball $B[v,r] \subset U$. Let $\epsilon > 0$ and $X \subset U$ be given by Lemma 8.5. The definition of the Morse numbers and Theorem 8.1 imply that

$$\dim C_n(\varphi, v) = M_n(X^{c+\epsilon}, X^{c-\epsilon}) = B_n(X^{c+\epsilon}, X^{c-\epsilon}) \tag{18}$$

where $c = \varphi(v)$.

2) There exists $\rho \in]0, \epsilon/3]$ such that

$$B[v, 2\rho] \subset \varphi^{-1}\left(\left[c - \frac{\epsilon}{3}, c + \frac{\epsilon}{3}\right]\right) \cap X.$$

Let $\tilde{\varphi} \in C^2(U,\mathbf{R})$ be given by Lemma 8.6. Properties b) and c) of $\tilde{\varphi}$ imply that $\tilde{\varphi}^{c\pm\epsilon} = \varphi^{c\pm\epsilon}$. Thus $\tilde{\varphi}^{-1}([c-\epsilon, c+\epsilon]) \cap X = \varphi^{-1}([c-\epsilon, c+\epsilon]) \cap X$ is complete. In particular, $\tilde{\varphi}$ satisfies the Palais–Smale condition over $\tilde{\varphi}^{-1}([c-\epsilon, c+\epsilon]) \cap X$. Since $B[v,\rho]$ is contained in the interior of X, property b) of $\tilde{\varphi}$ implies that X is positively invariant for the flow $\tilde{\sigma}$ defined by

$$\dot{\tilde{\sigma}}(t) = -\nabla\tilde{\varphi}(\tilde{\sigma}(t))$$

$$\tilde{\sigma}(0) = u.$$

By a), $\tilde{\varphi}$ has only a finite number of critical points $u_1, \ldots, u_j$, all non degenerate. By b), the critical points are contained in $B[v,\rho]$, and, hence, in the interior of $\tilde{\varphi}^{-1}([c-\epsilon, c+\epsilon]) \cap X$.

3) Let $k_i \in \{0, 1, \ldots, p\}$ be the Morse index of u_i, $i = 1, \ldots, j$. If we denote by $\tilde{M}_n(X^{c+\epsilon}, X^{c-\epsilon})$ the Morse numbers corresponding to $\tilde{\varphi}$, Corollary 8.2 implies that

$$\tilde{M}_n(X^{c+\epsilon}, X^{c-\epsilon}) = \sum_{i=1}^{j} \delta_{n,k_i}. \tag{19}$$

In particular, $\tilde{M}_n(X^{c+\epsilon}, X^{c-\epsilon})$ is finite for every n and equal to zero for $n \geq p+1$. It follows from Theorem 8.2 that

$$\tilde{M}_n(X^{c+\epsilon}, X^{c-\epsilon}) \geq B_n(X^{c+\epsilon}, X^{c-\epsilon})$$

and that

$$\sum_{n=0}^{p}(-1)^n \tilde{M}_n(X^{c+\epsilon}, X^{c-\epsilon}) = \sum_{n=0}^{p}(-1)^n B_n(X^{c+\epsilon}, X^{c-\epsilon}). \tag{20}$$

In particular, $\dim C_n(\varphi, v)$ is finite for every n and equal to zero for $n \geq p+1$.

4) By definition of the topological index and of the topological degree, we have

$$i(\nabla\tilde{\varphi}, u_i) = (-1)^{k_i}.$$

It follows from (19) and from the additivity of the topological degree that

$$\sum_{n=0}^{p}(-1)^n \tilde{M}_n(X^{c+\epsilon}, X^{c-\epsilon}) = \sum_{n=0}^{p}(-1)^n\left(\sum_{i=1}^{j}\delta_{n,k_i}\right) = \sum_{i=1}^{j}(-1)^{k_i}$$

$$= \sum_{i=1}^{j} i(\nabla\tilde{\varphi}, u_i) = d(\nabla\tilde{\varphi}, B(v, 2\rho)). \tag{21}$$

By continuity of the topological degree, we have

$$d(\nabla\tilde{\varphi}, B(v,2\rho)) = d(\nabla\varphi, B(v,2\rho)) = i(\nabla\varphi, v). \tag{22}$$

Theorem 8.5 then follows from (18), (20), (21), and (22). □

Theorem 8.6. *Let U be an open subset of $\mathbf{R}^p$ and let v be an isolated critical point $\varphi \in C^2(U,\mathbf{R})$. If v is neither a local minimum nor a local maximum, then*

$$C_0(\varphi, v) = C_p(\varphi, v) = 0.$$

Proof. 1) By diminishing U if necessary, we can assume that v is the only critical point of φ located in U. Moreover, the Palais–Smale condition is satisfied over any closed ball $B[v,r] \subset U$. Let $\epsilon > 0$ and $X \subset U$ be given by Lemma 8.5. Then, by Lemma 8.3, X^c is a deformation retract of $X^{c+\epsilon}$, so that

$$C_n(\varphi, v) = H_n(X^c, X^c \setminus \{0\}) = H_n(X^{c+\epsilon}, X^c \setminus \{v\}).$$

Let $\eta \in C([0,1] \times X^{c+\epsilon}, X^{c+\epsilon})$ be the corresponding deformation.

2) In order to prove that $H_0(X^{c+\epsilon}, X^c \setminus \{0\}) = \{0\}$, it suffices to show that every point $u \in X^{c+\epsilon}$ is connected to a point in $X^c \setminus \{v\}$ by a continuous path contained in $X^{c+\epsilon}$. Let $\rho > 0$ be such that $B[v,\rho] \subset X^{c+\epsilon}$. Since v is not a local minimum, ...

v is connected to the point $w \in X^c \setminus \{v\}$ by a continuous path contained in $X^{c+\epsilon}$. Now, every point $u \in X^{c+\epsilon}$ is connected by a continuous path contained in $X^{c+\epsilon}$ to $\eta(1,u)$, which either is v or belongs to $X^c \setminus \{v\}$.

3) Any continuous map

$$f : S^{p-1} \to B[v,\rho] \cap \varphi^c \setminus \{v\}$$

has a continuous extension $g_1 : B^p \to B[v,\rho]$. It follows from Lemmas 8.3 and 8.5 that f has a continuous extension $g_2 : B^p \to \varphi^c$. Since v is not a local maximum, v is not an interior point of $g_2(B^p)$. Thus f has a continuous extension $g_3 : B^p \to \varphi^c \cup S_\delta \setminus B_\delta$ where $\delta > 0$ is small, $S_\delta = S(v,\delta)$, and $B_\delta = B(v,\delta)$. Using the argument of Lemma 6.5, we obtain a continuous extension $g_4 : B^p \to \varphi^c \setminus \{v\}$ of f. Thus $H_{p-1}(\varphi^c \cap B[v,\rho] \setminus \{v\}) \approx 0$. Since $H_p(\varphi^c \cap B[v,\rho]) \approx 0$, we obtain by exactness $C_p(\varphi, v) \approx 0$. □

Corollary 8.4. *Under the assumptions of Theorem 8.3, if 0 is an isolated critical point of φ with finite Morse index k and nullity ν, then the following are true.*

i) $\dim C_n(\varphi,0)$ *is finite for every n and is equal to zero if $n \notin \{k, k+1, \ldots, k+\nu\}$;*

ii) *if 0 is a local minimum of $\tilde{\varphi}$, then*

$$C_n(\varphi,0) = \delta_{n,k}F;$$

iii) *if 0 is a local maximum of $\tilde{\varphi}$, then*

$$C_n(\varphi,0) = \delta_{n,k+\nu}F;$$

iv) *if 0 is neither a local minimum nor a local maximum of $\tilde{\varphi}$, then*

$$C_k(\varphi,0) = C_{k+\nu}(\varphi,0) = 0;$$

v) *if there exist integers $n_1 \neq n_2$ such that $C_{n_1}(\varphi,0) \neq 0$ and $C_{n_2}(\varphi,0) \neq 0$, then*

$$|n_1 - n_2| \le \nu - 2.$$

Proof. By Theorem 8.4, $C_n(\varphi,0) \approx C_{n-k}(\tilde{\varphi},0)$, so that $C_n(\varphi,0) = 0$ if $n \le k-1$. It follows from Theorem 8.5 and $\dim C_n(\varphi,0)$ is finite for every n and is equal to zero for $n \ge k+\nu+1$. It is easy to obtain ii) and iii) by a direct calculation. Theorem 8.6 implies iv). Finally, v) follows from i) to iv). □

8.8 Critical Groups at a Point of Mountain Pass Type

Interesting multiplicity results can be obtained by combining the minimax theorems and the Morse theory. Let us illustrate this fact by the mountain pass theorem situation.

Theorem 8.7. *Let X be a Hilbert space and let $\varphi \in C^2(X,\mathbf{R})$. Assume that there exists $u_0 \in X$, $u_1 \in X$ and a bounded open neighborhood Ω of u_0 such that $u_1 \in X \setminus \overline{\Omega}$ and*

$$\inf_{\partial\Omega} \varphi > \max(\varphi(u_0), \varphi(u_1)).$$

Let $\Gamma = \{g \in C([0,1],X) : g(0) = u_0, g(1) = u_1\}$ *and*

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} \varphi(g(s)).$$

If φ satisfies the Palais–Smale condition over X, and if each critical point of φ in K_c is isolated in X, then there exists $u \in K_c$ such that $\dim C_1(\varphi,u) \ge 1$.

Proof. Let $\epsilon > 0$ be such that $c - \epsilon > \max(\varphi(u_0), \varphi(u_1))$ and c is the only critical value of φ in $[c-\epsilon, c+\epsilon]$. Consider the exact sequence

$$\cdots \to H_1(\varphi^{c+\epsilon}, \varphi^{c-\epsilon}) \xrightarrow{\partial} H_0(\varphi^{c-\epsilon}, \phi) \xrightarrow{i_*} H_0(\varphi^{c+\epsilon}, \phi) \to \cdots$$

where i_* is induced by the inclusion mapping $i : (\varphi^{c-\epsilon}, \phi) \to (\varphi^{c+\epsilon}, \phi)$. The definition of c implies that u_0 and u_1 are path connected in $\varphi^{c+\epsilon}$ but not in $\varphi^{c-\epsilon}$. Thus, $\ker i_* \neq \{0\}$ and, by exactness, $H_1(\varphi^{c+\epsilon}, \varphi^{c-\epsilon}) \neq \{0\}$. It follows from Theorem 8.1 that

$$M_1(\varphi^{c+\epsilon}, \varphi^{c-\epsilon}) = B_1(\varphi^{c+\epsilon}, \varphi^{c-\epsilon}) = \dim H_1(\varphi^{c+\epsilon}, \varphi^{c-\epsilon}) \ge 1.$$

Thus $\varphi^{-1}([c-\epsilon, c+\epsilon])$ contains a critical point u such that $\dim C_1(\varphi,u) \ge 1$ and, necessarily, $u \in K_c$. □

Corollary 8.5. *Besides the above assumptions, assume moreover that each $u \in K_c$ satisfies the following conditions:*

a) *$\varphi''(u)$ is a Fredholm operator;*

b) *the nullity of u is less than 2 provided the Morse index of u is equal to 0.*

Then there exists $u \in K_c$ such that

$$\dim C_n(\varphi,u) = \delta_{n,1}, \quad n \in \mathbf{N}.$$

Proof. 1) Let $u \in K_c$, with Morse index k and nullity ν, be such that dim $C_1(\varphi, u) \geq 1$. We can assume that $u = 0$. By Corollary 8.4, $k \leq 1$ and $\nu \geq 1$ if $k = 0$.

2) If $k = 0$, assumption b) implies that $\nu = 1$. It then follows from Corollary 8.4 that 0 is a local maximum of $\hat{\varphi}$ and

$$C_n(\varphi, 0) = \delta_{n,k+\nu} F = \delta_{n,1} F.$$

3) If $k = 1$, then, by Corollary 8.4, either $\nu = 0$ or 0 is a local minimum of $\hat{\varphi}$. In both cases,

$$C_n(\varphi, 0) = \delta_{n,k} F = \delta_{n,1} F. \quad \square$$

Corollary 8.6. *Under the assumptions of Corollary 8.5, if $X = \mathbf{R}^p$, there exists $u \in K_c$ such that*

$$i(\nabla\varphi, u) = -1.$$

Proof. By Corollary 8.5, there exists $u \in K_c$ such that $\dim C_n(\varphi, u) = \delta_{n,1}$. Theorem 8.5 implies that

$$i(\nabla\varphi, u) = \sum_{n=0}^{p} (-1)^n \dim C_n(\varphi, u) = -1. \quad \square$$

8.9 Continuity of the Critical Groups and Bifurcation Theory

The critical groups are continuous with respect to the C^1 topology.

Theorem 8.8. *Let U be an open neighborhood of v in a Hilbert space V and let $\varphi, \psi \in C^{2-0}(U, \mathbf{R})$. Assume that φ and ψ have v as the only critical point and satisfy the Palais-Smale condition over a closed ball $B[v, r] \subset U$. Then there exists $\eta > 0$, depending only upon φ, such that the condition*

$$\sup_{u \in U} (|\psi(u) - \varphi(u)| + |\nabla\psi(u) - \nabla\varphi(u)|) \leq \eta \tag{23}$$

implies

$$\dim C_n(\psi, v) = \dim C_n(\varphi, v), \quad n \in \mathbf{N}. \tag{24}$$

Proof. 1) Let $\epsilon > 0$ and $X \subset U$ be given by Lemma 8.5 applied to φ. The definition of the Morse numbers and Theorem 8.1 imply that

$$\dim C_n(\varphi, v) = M_n(X^{c+\epsilon}, X^{c-\epsilon}) = B_n(X^{c+\epsilon}, X^{c-\epsilon}) \tag{25}$$

where $c = \varphi(v)$.

2) Let $\rho > 0$ be such that

$$B[v, 2\rho] \subset \varphi^{-1}\left(\left[c - \frac{\epsilon}{3}, c + \frac{\epsilon}{3}\right]\right) \cap X. \tag{26}$$

By the Palais–Smale condition,

$$\delta = \inf\{|\nabla\varphi(u)| : \rho/2 \leq |u - v| \leq \rho\} > 0. \tag{27}$$

Let $\omega \in C^2(U, \mathbf{R})$ be such that

$$\begin{gathered} \omega(u) = 1 \quad \text{if} \quad |u - v| \leq \rho/2 \\ \omega(u) = 0 \quad \text{if} \quad |u - v| \geq \rho \\ 0 \leq \omega(u) \leq 1 \\ \gamma = \sup_{u \in U} |\nabla\omega(u)| < \infty, \end{gathered} \tag{28}$$

and let

$$\eta = \min(\epsilon/3, \delta/2(1 + \gamma)).$$

Assume that ψ satisfies the assumptions of the theorem and define $\tilde{\psi}$ on U by

$$\tilde{\psi}(u) = \varphi(u) + \omega(u)(\psi(u) - \varphi(u)).$$

It follows from (23), (27), and (28) that, for $\rho/2 \leq |u - v| \leq \rho$,

$$\begin{aligned} |\nabla\tilde{\psi}(u)| &\geq |\nabla\varphi(u)| - \omega(u)|\nabla\psi(u) - \nabla\varphi(u)| - |\nabla\omega(u)|\,|\psi(u) - \varphi(u)| \\ &\geq \delta - (1 + \gamma)\eta \geq \delta/2. \end{aligned} \tag{29}$$

We obtain from (23) and (28) that, for $u \in U$,

$$|\tilde{\psi}(u) - \varphi(u)| = \omega(u)\,|\psi(u) - \varphi(u)| \leq \eta \leq \epsilon/3. \tag{30}$$

3) Since $\tilde{\psi}(u) = \varphi(u)$ if $|u - v| \geq \rho$, relations (26) and (30) imply that $\tilde{\psi}^{c\pm\epsilon} = \varphi^{c\pm\epsilon}$. Thus $\tilde{\psi}^{-1}([c-\epsilon, c+\epsilon]) \cap X = \varphi^{-1}([c-\epsilon, c+\epsilon]) \cap X$ is complete. It follows easily from (29) that $\tilde{\psi}$ satisfies the Palais–Smale condition over $\tilde{\psi}^{-1}([c - \epsilon, c + \epsilon]) \cap X$. Moreover, $B[v, \rho]$ is contained in the interior of X, so that X is positively invariant for the flow $\tilde{\sigma}$ defined by

$$\dot{\tilde{\sigma}}(t) = -\nabla\tilde{\psi}(\tilde{\sigma}(t))$$

$$\tilde{\sigma}(0) = u.$$

Finally, the definition of $\tilde{\psi}$ and (29) imply that v is the only critical point of $\tilde{\psi}$. If we denote by $\tilde{M}_n(X^{c+\epsilon}, X^{c-\epsilon})$ the Morse numbers corresponding to $\tilde{\psi}$, we have

$$\dim C_n(\tilde{\psi}, v) = \tilde{M}_n(X^{c+\epsilon}, X^{c-\epsilon}) = B_n(X^{c+\epsilon}, X^{c-\epsilon}). \tag{31}$$

But, by the definition of $\tilde{\psi}$,

$$C_n(\tilde{\psi}, v) = C_n(\psi, v), \tag{32}$$

and (24) follows from (25), (31), and (32). $\square$

The preceding theorem is useful in *bifurcation theory*.

Let V, W be Banach spaces, let U be an open neighborhood of 0 in V, and let Λ be an open interval. Consider a mapping $f \in C(\Lambda \times U, W)$ such that $f(\lambda, 0) = 0$ for every $\lambda \in \Lambda$. A point $(\lambda_0, 0) \in \Lambda \times U$ is a *bifurcation point* for the equation

$$f(\lambda, u) = 0 \tag{33}$$

if every neighborhood of $(\lambda_0, 0)$ in $\Lambda \times U$ contains at least one solution (λ, u) of (32) such that $u \neq 0$.

If f is a C^1 mapping, the implicit function theorem implies that a necessary condition for $(\lambda_0, 0)$ to be a bifurcation point is the non-invertibility of $D_u f(\lambda_0, 0)$. However, this condition is not sufficient in general, as shown by the simple example with $V = W = \mathbf{R}^2$ and

$$f(\lambda, u_1, u_2) = (u_1 - \lambda u_1 + u_2^3, u_2 - \lambda u_2 - u_1^3)$$

for which

$$D_u f(1, 0, 0) = 0$$

is not invertible and which, however, has no bifurcation point, as $f(\lambda, u_1, u_2) = 0$ implies

$$0 = u_2(u_1 - \lambda u_1 + u_2^3) - u_1(u_2 - \lambda u_2 - u_1^3) = u_2^4 + u_1^4 = 0$$

and hence $(u_1, u_2) = 0$. Notice however that f is not a gradient mapping with respect to u. We shall describe a rather wide class of gradient mappings for which the necessary condition above is sufficient.

The proof of the following simple lemma is left to the reader.

Lemma 8.7. *Let $K \subset \Lambda$ be a non-empty compact interval such that $K \times \{0\}$ contains no bifurcation point for (33). Then there exists $\rho > 0$ such that $B[0, \rho] \subset U$ and each solution (λ, u) of (33) in $K \times B[0, \rho]$ satisfies $u = 0$.*

Theorem 8.9. *Let U be an open neighborhood of 0 in a Hilbert space V, let Λ be an open interval and let $f(\lambda, u)$ be the gradient with respect to u of $\varphi \in C^2(\Lambda \times U, \mathbf{R})$. Assume that the following conditions are satisfied:*

α) 0 is a critical point of $\varphi_\lambda = \varphi(\lambda, .)$ for every $\lambda \in \Lambda$ and 0 is an isolated critical point of φ_a and φ_b for some reals $a < b$ in Λ.

β) φ_λ satisfies the Palais-Smale condition over a closed ball $B[0, r] \subset U$ for every $\lambda \in [a, b]$.

γ) There exists $n \in \mathbf{N}$ such that

$$\dim C_n(\varphi_a, 0) \neq \dim C_n(\varphi_b, 0).$$

Then there exists a bifurcation point $(\lambda_0, 0) \in [a, b] \times \{0\}$ for (33).

Proof. If $[a, b] \times \{0\}$ contains no bifurcation point for (33), then, by Lemma 8.7, there exists $\rho > 0$ such that $B[0, \rho] \subset U$ and each solution (λ, u) of (33) in $[a, b] \times B[0, \rho]$ satisfies $u = 0$. We can assume, without loss of generality, that $\varphi(\lambda, 0) = 0$. Since φ is of class C^2, we can choose ρ small enough so that

$$|D_\lambda \varphi(\lambda, u)| + |D_{\lambda u} \varphi(\lambda, u)| \leq 1$$

whenever $\lambda \in [a, b]$ and $u \in B[0, \rho]$. Thus φ_λ and $\nabla \varphi_\lambda$ depend continuously on $\lambda \in [a, b]$, uniformly on $B[0, \rho]$. By Theorem 8.8, $\dim C_n(\varphi_\lambda, 0)$ is locally constant, and hence constant, on $[a, b]$, for every $n \in \mathbf{N}$. In particular,

$$\dim C_n(\varphi_a, 0) = \dim C_n(\varphi_b, 0), \quad n \in \mathbf{N},$$

a contradiction with assumption γ). □

8.10 Lower Semi-Continuity of the Betti Numbers

We shall prove in this section a lower semi-continuity property for the Betti numbers $B_n(\varphi^b, \varphi^a)$ with respect to the C^0 topology. It is interesting to notice that this property is weaker than the corresponding continuity property of the topological degree whenever both concepts are defined.

Lemma 8.8. *Let $B \subset F \subset B' \subset A \subset E \subset A'$ be topological spaces. Suppose that*

$$H_n(B', B) \approx H_n(A', A) \approx \{0\}, \quad n = 0, 1, \ldots. \tag{34}$$

Then

$$B_n(A, B) \leq B_n(E, F), \quad n = 0, 1, \ldots.$$

Proof. Let us consider the following diagrams:

$$\begin{array}{ccccccc} H_{n+1}(A', A) & \to & H_n(A, B) & \xrightarrow{i_*} & H_n(A', B) & \to & H_n(A', A) \\ & & f_* \searrow & & \nearrow g_* & & \\ & & & H_n(E, B) & & & \end{array}$$

$$\begin{array}{ccccccc} H_n(B', B) & \to & H_n(E, B) & \xrightarrow{j_*} & H_n(E, B') & \to & H_{n-1}(B', B) \\ & & f'_* \searrow & & \nearrow g'_* & & \\ & & & H_n(E, F) & & & \end{array}$$

By exactness, assumption (34) implies that i_* and j_* are isomorphisms. But $i_* = g_* \circ f_*$ and $j_* = g'_* \circ f'_*$, so that f_* and f'_* are injections. Thus

$$h_* = f'_* \circ f_* : H_n(A, B) \to H_n(E, F)$$

is an injection. □

Theorem 8.10. *Let M be a complete Riemannian manifold of class C^2 and let $\varphi \in C^{2-0}(M,\mathbf{R})$. Suppose that there exists $\hat{\varphi} : M \to \mathbf{R}$, $c \in \mathbf{R}$ and $\epsilon > 0$ such that*

i) $\sup_{u \in M} |\varphi(u) - \hat{\varphi}(u)| \le \epsilon/3$;

ii) *c is the only critical value of φ in $[c-\epsilon, c+\epsilon]$;*

iii) *φ satisfies the* (PS) *condition over $\varphi^{-1}([c-\epsilon, c+\epsilon])$.*

Then

$$B_n(\varphi^{c+\epsilon}, \varphi^{c-\epsilon}) \le B_n(\hat{\varphi}^{c+\epsilon/2}, \hat{\varphi}^{c-\epsilon/2}), \quad (n = 0,1,\ldots).$$

Proof. By assumption i), we have

$$\varphi^{c-\epsilon} \subset \hat{\varphi}^{c-\epsilon/2} \subset \varphi^{c-\epsilon/6} \subset \varphi^{c+\epsilon/6)} \subset \hat{\varphi}^{c+\epsilon/2} \subset \varphi^{c+\epsilon}.$$

It follows from Lemma 8.3 and assumptions ii) and iii) that $\varphi^{c-\epsilon}$ (resp. $\varphi^{c+\epsilon/6}$ is a strong deformation retract of $\varphi^{c-\epsilon/6}$ (resp. $\varphi^{c+\epsilon}$). Hence

$$H_n(\varphi^{c-\epsilon/6}, \varphi^{c-\epsilon}) \approx H_n(\varphi^{c+\epsilon}, \varphi^{c+\epsilon/6}) \approx = 0 \quad (n = 0,1,\ldots).$$

Applying Lemma 8.8, we obtain

$$B_n(\varphi^{c+\epsilon/6}, \varphi^{c-\epsilon}) \le B_n(\hat{\varphi}^{c+\epsilon/2}, \hat{\varphi}^{c-\epsilon/2}) \quad (n = 0,1,\ldots).$$

Since

$$B_n(\varphi^{c+\epsilon/6}, \varphi^{c-\epsilon}) = B_n(\varphi^{c+\epsilon}, \varphi^{c-\epsilon}),$$

the proof is complete. □

8.11 Critical Groups at a Saddle Point

Let X be a Hilbert space and assume that $\varphi \in C^2(X,\mathbf{R})$ satisfies the Palais–Smale condition over X. Assume also, as in the saddle point theorem, that X splits into a direct sum of closed subspaces X^- and X^+ with

$$2 \le m := \dim X^- < +\infty,$$

$$b := \sup_{S_R^-} < d := \inf_{X^+} \varphi$$

where $S_R^- = \{u \in X^- : |u| = R\}$. Regard the identity mappings $\sigma : S_R^- \to S_R^-$ as the generator of the homology $H_{m-1}(S_R^-, \phi)$ and define

$$c = \inf_{\partial\tau = \sigma} \sup_{u \in |\tau|} \varphi(u),$$

where τ is any chain of m dimensional singular simplices on X such that $\partial\tau = \sigma$. In this way we obtain a natural variant of the saddle point theorem.

Theorem 8.11. *Under the above assumptions, c is a critical value of φ. Moreover, if each critical point of φ in K_c is isolated in X, then there exists $u \in K_c$ such that*

$$\dim C_m(\varphi, u) \ge 1.$$

Proof. 1) Let us show that $|\tau| \cap X^+$ is nonempty for any chain τ with $\partial\tau = \sigma$. Consider the exact sequence

$$\ldots \to H_m(X, X \setminus X^+) \xrightarrow{\partial} H_{m-1}(X \setminus X^+, \phi) \approx H_{m-1}(S_R^-, \phi) \to \ldots.$$

Since $\partial[\tau] = [\sigma] \ne 0$, necessarily $[\tau] \ne 0$. Thus $|\tau| \not\subset X \setminus X^+$. In particular, we have that

$$\sup_{u \in |\tau|} \varphi(u) \ge d$$

for any chain τ with $\partial\tau = \sigma$. Hence $c \ge d$.

2) Using Lemma 6.5, one can easily prove, by contradiction, that K_c is nonempty.

3) Assume now that each critical point of φ in K_c is isolated in X. Let $\epsilon > 0$ be such that $c-\epsilon > b$ and c is the only critical value of φ in $[c-\epsilon, c+\epsilon]$. Consider the exact sequence

$$\ldots \to H_m(\varphi^{c+\epsilon}, \varphi^{c-\epsilon}) \xrightarrow{\partial} H_{m-1}(\varphi^{c-\epsilon}, \phi) \xrightarrow{i_*} H_{m-1}(\varphi^{c+\epsilon}, \phi) \to \ldots.$$

There exists a chain τ such that $\sigma = \partial\tau$ and $|\tau| \subset \varphi^{c+\epsilon}$. Thus $[\sigma] = 0$ in $H_{m-1}(\varphi^{c+\epsilon}, \phi)$. On the other hand, if $[\sigma] = 0$ in $H_{m-1}(\varphi^{c-\epsilon}, \phi)$, there exists a chain τ such that $\sigma = \partial\tau$ and $|\tau| \subset \varphi^{c-\epsilon}$. But this contradicts the definition of c. Thus $[\sigma]$ is a nonzero element of Ker i_*. By exactness, $H_m(\varphi^{c+\epsilon}, \varphi^{c-\epsilon}) \ne \{0\}$. The conclusion then follows from Theorem 8.1. □

Using Corollary 8.3, we obtain the following result.

Corollary 8.7. *Under the assumptions of Theorem 8.11, if each critical point of φ in K_c is nondegenerate, then there exists $u \in K_c$ such that*

$$\dim C_n(\varphi, u) = \delta_{n,m}, \quad n \in \mathbf{N}.$$

Historical and Bibliographical Notes

The reader can consult [Wal₁] for a brief and lucid exposition of singular homology. Surveys of the mathematical work of Morse are given by [Bot₁], [Cai₁], and [Tho₂], and surveys of Morse theory are given by [Bot₂], [Cha₁], and [Rot₅].

Morse's first paper on this theory [Mrs₁] already includes such essential ingredients as the Morse lemma, gradient deformations, and Morse inequalities for a nondegenerate function on a smooth domain in $\mathbf{R}^N$. It was aimed as a generalization of the Birkhoff minimax theory [Bir₁]. The theory is extended to compact smooth manifolds in [Mrs₂], which also contains the

Morse index theorem and applications to the calculus of variations by the method of "broken extremals."

The Morse theory was extended to Hilbert spaces by Rothe [Rot_6] and to infinite-dimensional Riemannian manifolds by Palais and Smale ([Pal_1], [PaS_1], [Sma_1]). As in the Leray–Schauder theory, the compactness of the domain is replaced by a compactness condition on the function (the PS condition).

The classical Morse lemma for nondegenerate critical points was extended by Palais [Pal_1] to Hilbert spaces. Because of the loss of two orders of differentiability, the Palais method is only applicable to functions of class C^3. Using the Lyapunov-Schmidt method and the Palais approach, Gromoll–Meyer [GrM_1] succeeded in treating the case of degenerate critical points when the second differential of the function is a compact perturbation of the identity. On the other hand, Kuiper [Kui_1] and Cambini [Cam_1] independently gave a proof of the Morse lemma for a nondegenerate critical point of a C^2 function. This result was extended to the degenerate case by Hofer [Hof_3] when the second differential is a compact perturbation of identity. Hofer's proof uses deformations by a gradient flow (see [GoM_1] for extensions). Theorem 8.3 generalizes the previous results. We follow the proof of Hofer [Hof_3] (see [MaW_4] for another proof).

The shifting theorem is due to Gromoll–Meyer [GrM_1]. A new proof is given here. Theorem 8.5 was first proved by Rothe [Rot_7], to whom we also owe the first results on the continuity of the critical groups and the lower semicontinuity of the Betti numbers in the Hilbert space case [Rot_8].

Lemma 8.6, Lemma 8.8 and Theorem 8.10 are contained in the important paper of Marino–Prodi [MaP_1] on perturbation methods in Morse theory.

Of course the genericity of the non-degenerate case is known and has been used since Morse. Theorem 8.6 is due to Dancer [Dan_2] and Theorem 8.7 to Ambrosetti [$\mathrm{Amb}_{2,3}$] in the nondegenerate case and to Hofer [Hof_3] in the general case.

Minimax methods were introduced in bifurcation theory by Krasnosel'skii [Kra_2] and Morse theory by Marino–Prodi [MaP_2] (see the surveys [Cha_1], [Rab_6]).

The results of Section 8.11 are due to Liu [Liu_1].

For Morse theory on Banach manifolds, the reader can consult [Sk_1], [Tr_1], and [U_1]. The completeness of the Morse inequalities is studied in [Joh_1] and [Sma_2]. Degenerate critical points are considered in [CGR_1], [Dan_9], and [Ro_7]. More results on bifurcation through variational methods can be found in [$\mathrm{Boh}_{1,2}$], [Ch_7], [Clk_4], [Dan_6], [Mar_1], [Prod_1], [BenP_1], and [Wil_{12}].

Exercises

1. Let M be a complete C^2-Riemannian manifold and assume that $\varphi \in C^2(M, \mathbf{R})$ satisfies the Palais-Smale condition on M. If φ has a global minimum and $\chi(M)$ nondegenerate critical points with finite Morse index, then φ has at least $\chi(M) + 2$ critical points.

2. Let U be an open subset of $\mathbf{R}^2$ and let v be an isolated critical point of $\varphi \in C^2(U, \mathbf{R})$. Then $i(\nabla\varphi, v) \leq 1$.

3. Let U be an open subset of $\mathbf{R}^p$ and let v be an isolated local minimum point of $\varphi \in C^2(U, \mathbf{R})$. Then $i(\nabla\varphi, v) = 1$.

4. Let M be a complete C^2-Riemannian manifold. Assume that $\varphi \in C^1([0,1] \times M, \mathbf{R})$ and $a < b$ are such that the following conditions hold:

   i) $\varphi(\lambda, u)$ is continuous with respect to λ uniformly in $u \in M$.

   ii) For every $\lambda \in [0,1]$, $\varphi_\lambda \in C^{2-0}(M, \mathbf{R})$ and $\nabla\varphi_\lambda(u) \neq 0$ whenever $\varphi_\lambda(u) \in \{a, b\}$.

   iii) Every sequence (λ_j, u_j) such that $(\varphi(\lambda_j, u_j))$ is bounded and $\nabla\varphi_{\lambda_j}(u_j) \to 0$ contains a convergent subsequence.

   Then
   $$P(t, \varphi_1^b, \varphi_1^a) = P(t, \varphi_0^b, \varphi_0^a).$$

   Hint. Find $0 < c < (b-a)/2$ and, for $\lambda_0 \in [0,1]$, find $\eta > 0$ such that for $|\lambda - \lambda_0| \leq \eta$ one has
   $$\varphi_{\lambda_0}^{a-c} \subset \varphi_\lambda^{a+c/\epsilon} \subset \varphi_{\lambda_0}^{a+c/\epsilon} \subset \varphi_\lambda^{a+c} \subset \varphi_\lambda^{b-c} \subset \varphi_{\lambda_0}^{b-c/\epsilon} \subset \varphi_\lambda^{b+c/\epsilon} \subset \varphi_\lambda^{b+c}.$$
   Use Lemma 8.3 and compactness of $[0,1]$.

5. (Principle of symmetric criticality, [Pal_4]). Let $\{T(g)\}_{g\in G}$ be an isometric representation of the topological group G over a Hilbert space X. Let $\psi \in C^2(X, \mathbf{R})$ be an invariant functional. If $u \in V = \mathrm{Fix}(G)$ is a critical point of $\psi|_V$, then u is a critical point of ψ.

   Hint. Prove that $\nabla\psi(u) \in V$.

6. ([Wil_{12}]). Let $\{T(g)\}_{g\in G}$ and X be as in Exercise 5. Let $\psi \in C^2(\Lambda \times X, \mathbf{R})$ where Λ is an open interval. Assume that

   i) $\psi_\lambda = \psi(\lambda, .)$ is invariant for every $\lambda \in \Lambda$,

   ii) ψ restricted to $\Lambda \times \mathrm{Fix}(G)$ satisfies the assumptions of Theorem 8.8, where $V = \mathrm{Fix}\, G$.

Then there exists $[\lambda_0, 0] \in [a, b] \times \{0\}$ such that every neighborhood of $[\lambda_0, 0]$ in $\Lambda \times \mathrm{Fix}(G)$ contains at least one solution (λ, u) of

$$\nabla\psi_\lambda(u) = 0$$

such that $u \neq 0$.

9

Applications of Morse Theory to Second Order Systems

Introduction

The *Liapunov center theorem* is the classical result which follows easily from the equivariant Crandall–Rabinowitz bifurcation theorem. Consider the second order autonomous system

$$\ddot{u} + f(u) = 0$$

and assume that 0 is a solution and that

$$\beta_1^2 < \beta_2^2 < \ldots < \beta_s^2$$

($\beta_r \geq 0, 0 \leq r \leq s$) are the non-negative eigenvalues of $f'(0)$. The Liapunov theorem insures that if the geometric multiplicity of β_i^2 is one and if $\beta_r/\beta_i \notin \mathbb{N}$ for $r \neq i$, then this system has a family of periodic solutions with minimal period tending to $2\pi/\beta_i$ and with amplitude tending to 0.

In many applications, the multiplicity of β_i^2 is bigger than one. We prove in Section 9.2 that, in the variational case ($f = \nabla F$), it suffices to assume that $\beta_r/\beta_i \notin \mathbb{N}$ for $r \neq i$ in order to obtain a sequence (u_k) of solutions with minimal period tending to $2\pi/\beta_i$ and with amplitude tending to zero when $k \to \infty$.

The application given in Section 9.3 concerns *asymptotically linear non-autonomous systems* of the form

$$\ddot{u}(t) + \nabla F(t, u(t)) = 0.$$

Since the problem is no more S^1-invariant as in the autonomous case, the results of Chapter 7 are no more applicable. Using the Morse inequalities, one can prove the existence of one or two nontrivial solutions when $\nabla F(t, 0) = 0$. The basic condition, namely a distinct behavior of ∇F at the origin and at infinity, is an extension of the "twist" condition of the famous Poincaré–Birkhoff geometric fixed point theorem.

Finally, in Section 9.4, a strong multiplicity result for non-autonomous second order systems with periodic potential and non-degenerate periodic solutions is given, which corresponds, in the more difficult case of Hamiltonian systems with periodic Hamiltonian, to a famous solution by Conley–Zehnder of a *conjecture of Arnold*.

For some of the above results, it was necessary to analyze in more detail the Morse index of the action associated to non-autonomous linear second order system (this index is finite here because the system has order two), and it is the object of Section 9.1.

9.1 The Index of a Linear Second Order Differential System

Let A be a continuous mapping from $\mathbf{R}$ into the space of symmetric matrices of order N. We consider the periodic boundary value problem

$$\begin{gathered}\ddot{u}(t) + A(t)u(t) = 0 \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(t) = 0\end{gathered} \tag{1}$$

where $T > 0$ is fixed. The corresponding action is defined on H_T^1 by

$$\chi_T(u) = \int_0^T (1/2)[|\dot{u}(t)|^2 - (A(t)u(t), u(t))]\,dt.$$

Definition 9.1. The *index* $j(A,T)$ is the Morse index of χ_T.

Let us define the linear operator K on H_T^1 (with its usual norm and inner product) by the formula

$$((Ku, v)) = \int_0^T (u(t) + A(t)\,u(t), v(t))\,dt.$$

It is easy to check that K is self-adjoint and compact, and that

$$2\chi_T(u) = ((u - Ku, u)).$$

The space H_T^1 can be written as the orthogonal direct sum of $\ker(I-K)$, H^+ and H^- with $I-K$ positive (resp. negative) definite on H^+ (resp. H^-). Since K has at most finitely many eigenvalues (having, moreover, finite multiplicity) greater than one,

$$j(A,T) = \dim H^- < \infty,$$

i.e. *the index $j(A,T)$ is finite.*

Definition 9.2. The *nullity* $\nu(A,T)$ is the dimension of $\ker(I-K)$.

It is easy to verify that the nullity is equal to the number of linearly independent solutions of (1), so that *the nullity $\nu(A,T)$ is less or equal to $2N$.* The linear operator $I-K$ is a Fredholm operator of index zero and hence is invertible if and only if $\nu(A,T) = 0$.

In the autonomous case, it is easy to compute the index and the nullity after diagonalization of A.

Proposition 9.1. *Let $\alpha_1 \le \alpha_2 \le \dots \le \alpha_N$ be the eigenvalues of the constant matrix A. Then*

$$j(A,T) = \#\{k : \alpha_k > 0\} + 2\sum_{k=1}^{N} \#\left\{j \in \mathbf{N}^* : \frac{4\pi^2 j^2}{T^2} < \alpha_k\right\}$$

$$\nu(A,T) = \#\{k : \alpha_k = 0\} + 2\sum_{k=1}^{N} \#\left\{j \in \mathbf{N}^* : \frac{4\pi^2 j^2}{T^2} = \alpha_k\right\}.$$

Let us now consider the functional φ defined on H_T^1 by

$$\varphi(u) = \int_0^T [(1/2)|\dot{u}(t)|^2 - F(t, u(t))]\,dt$$

where $F \in \mathbf{C}^2([0,T] \times \mathbf{R}^N, \mathbf{R})$. Since

$$\varphi''(u_0)(u,v) = \int_0^T [(\dot{u}(t), \dot{v}(t)) - (D_u^2 F(t, u_0(t))u(t), v(t))]\,dt,$$

every critical point u_0 of φ satisfies the following properties:

i) The Morse index of u_0 is equal to $j(A,T)$ where $A(t) = D_u^2 F(t, u_0(t))$.

ii) The nullity of u_0 is equal to $\nu(A,T)$.

iii) $\varphi''(u_0)$ is a Fredholm operator.

iv) u_0 is non-degenerate if and only if $\nu(A,T) = 0$.

By Corollary 8.4, if u_0 is an isolated critical point of φ, then $\dim C_n(\varphi, u_0)$ is finite for every n and equal to zero except if $n \in \{j(A,T), j(A,T)+1, \dots, j(A,T)+\nu(A,T)\}$.

9.2 Periodic Solutions of Autonomous Second Order Systems Near an Equilibrium

We consider the existence of small non-trivial periodic solutions for the autonomous system

$$\ddot{u}(t) + \nabla F(u(t)) = 0 \tag{2}$$

where $F \in \mathbf{C}^2(\mathbf{R}^N, \mathbf{R})$ is such that

$$\nabla F(u) = Au + o(|u|)$$

as $|u| \to 0$. Let

($\beta_r \geq 0, 0 \leq r \leq s$) be the non-negative eigenvalues of the symmetric matrix A.

Theorem 9.1. *If β_i is such that $\beta_r/\beta_i \notin \mathbf{N}$ for all $r \neq i$, then there exists a sequence (u_k) of periodic solutions of (2), with minimal period T_k such that $|u_k|_\infty \to 0$ and $T_k \to 2\pi/\beta_i$ as $k \to \infty$.*

It is easy to verify that u is a periodic solution of (2) with minimal period $2\pi\lambda$ if and only if $u(t) = v(t/\lambda)$ where v is a solution of

$$\begin{aligned} \ddot{v}(t) + \lambda^2 \nabla F(v(t)) = 0 \\ v(0) - v(2\pi) = \dot{v}(0) - \dot{v}(2\pi) = 0 \end{aligned} \tag{3}$$

with minimal period 2π.

Let us define φ on $\mathbf{R} \times H^1_{2\pi}$ by

$$\varphi(\lambda, u) = \varphi_\lambda(u) = \int_0^{2\pi} [(1/2)|\dot{u}(t)|^2 - \lambda^2 F(u(t))]\, dt$$

so that the solutions of (3) are the critical points of φ_λ. For each $\lambda \in \mathbf{R}$, 0 is a critical point of φ_λ. Let us also define the operators K and N on $H^1_{2\pi}$ by the formulas

$$((Ku, v)) = \int_0^{2\pi} (u(t), v(t))\, dt$$

$$((Nu, v)) = \int_0^{2\pi} (\nabla F(u(t)), v(t))\, dt,$$

so that

$$\langle \varphi'_\lambda(u), v\rangle = ((u - Ku - \lambda^2 Nu, v)).$$

The proof of Theorem 9.1 requires the following lemma.

Lemma 9.1. *Let $\lambda \in \mathbf{R}$ and $r > 0$. The functional φ_λ satisfies the Palais–Smale condition over $B[0, r]$.*

Proof. Let (u_j) be a sequence in $B[0, r]$ such that $\nabla\varphi_\lambda(u_j) \to 0$, i.e.

$$u_j - Ku_j - \lambda^2 Nu_j = f_j, \quad j \in \mathbf{N}^*,$$

with $f_j \to 0$ in $H^1_{2\pi}$. Going if necessary to a subsequence, we can assume that $u_j \rightharpoonup u$ in $H^1_{2\pi}$ and that $u_j \to u$ uniformly on $[0, 2\pi]$. This implies that $Ku_j \to Ku$ and $Nu_j \to Nu$. Therefore, $u_j \to Ku + \lambda^2 Nu$.

Proof of Theorem 1. 1) Let us first prove that $(1 \,|\, \beta_i, 0)$ is a bifurcation point for the equation

$$\nabla\varphi_\lambda(u) = 0. \tag{4}$$

By assumption, if $\epsilon \in]0, \beta_i/2[$ is sufficiently small, we have $\lambda\beta_r \notin \mathbf{N}$ for $r \neq i$ whenever

$$\lambda \in [1/(\beta_i + \epsilon), 1/(\beta_i - \epsilon)].$$

We shall apply Theorem 8.8 with $a = 1/(\beta_i + \epsilon)$ and $b = 1/(\beta_i - \epsilon)$. Proposition 9.1 implies that

$$\nu(a^2 A, 2\pi) = \nu(b^2 A, 2\pi) = 0$$

and

$$j(b^2 A, 2\pi) - j(a^2 A, 2\pi) = 2m$$

where m is the multiplicity of β_i^2 as an eigenvalue of A. Thus, 0 is an isolated critical point of φ_a and φ_b and, if $n = j(a^2 A, 2\pi)$, Corollary 8.3 implies that

$$\dim C_n(\varphi_a, 0) = 1 \neq 0 = \dim C_n(\varphi_b, 0).$$

By Lemma 9.1 and Theorem 8.8, there exists a bifurcation point $(\lambda_0, 0) \in [a, b] \times \{0\}$ for (4). Letting $\epsilon \to 0$, we obtain the desired conclusion.

2) By the first part of the proof, there exists a sequence (λ_k, v_k) of solutions of (3) such that $\lambda_k \to 1/\beta_i$, $v_k \neq 0$ and $v_k \to 0$ in $H^1_{2\pi}$. Since $v_k \to 0$ uniformly on $[0, 2\pi]$, we have

$$\frac{\|\nabla F(v_k) - Av_k\|_\infty}{\|v_k\|_\infty} \to 0 \quad \text{as} \quad k \to \infty. \tag{5}$$

In particular, there exists $C > 0$ such that

$$\frac{\|\nabla F(v_k)\|_\infty}{\|v_k\|_\infty} \leq C, \quad k \in \mathbf{N}^*. \tag{6}$$

Let $w_k = v_k/\|v_k\|_\infty$. It follows from (3) and (6) that $(\|\ddot{w}_k\|_\infty)$ is bounded, and so is $(\|\dot{w}_k\|_\infty)$ by the Sobolev inequality. Using the Ascoli–Arzela theorem, we can assume, going if necessary to a subsequence, that $w_k \to w$ and $\dot{w}_k \to \dot{w}$ uniformly on $[0, 2\pi]$. It follows then from (5) that

$$\left\| \frac{\nabla F(v_k)}{\|v_k\|_\infty} - Aw \right\|_\infty \to 0 \quad \text{as} \quad k \to \infty. \tag{7}$$

By (3) and (7), we obtain

$$\ddot{w}(t) + \frac{1}{\beta_i^2} Aw = 0$$

$$w(0) - w(2\pi) = \dot{w}(0) - \dot{w}(2\pi) = 0.$$

Since $\|w\|_\infty = 1$ and, by assumption, $\beta_r/\beta_i \notin \mathbf{N}$ for $r \neq i$, 2π is the minimal period of w. Thus, for k sufficiently large, 2π is also the minimal period of w_k and, hence, of v_k, which completes the proof. □

Remarks. 1) If the multiplicity of β_i^2 as an eigenvalue of A is equal to one, then Theorem 9.1 follows from the classical Liapunov Center Theorem.

2) By Theorem 9.1, the small periodic solutions of system (2) are related to the periodic solutions of the linearized system $\ddot{u} + Au = 0$. This is not

the case in general, as shown by the following examples where, respectively, the nonlinearity is not a gradient and the differential operator is not of the second order.

Example 9.1. Assume that $u = (u_1, u_2)$ is a T-periodic solution of the system

$$\ddot{u}_1 + u_1 + u_2^3 = 0$$
$$\ddot{u}_2 + u_2 - u_2^3 = 0.$$

After multiplying the first equation by u_2, the second by u_1, integrating from 0 to T and subtracting, we obtain

$$\int_0^T [u_2^4(t) + u_1^4(t)]\, dt = 0,$$

i.e. $u = 0$. On the other hand, all the solutions of the linearized system are 2π-periodic.

Example 9.2. Consider the Hamiltonian

$$\begin{aligned} H(u) &= H(u_1, u_2, u_3, u_4) \\ &= (1/2)(u_1^2 + u_3^2 - u_2^2 - u_4^2) + (u_1^2 + u_2^2 + u_3^2 + u_4^2)(u_3u_4 - u_1u_2). \end{aligned}$$

If u is a solution of the corresponding system

$$J\dot{u} + \nabla H(u) = 0,$$

then

$$\frac{d}{dt}(u_1u_4 + u_2u_3) = 4(u_3u_4 - u_1u_2)^2 + 2u_1^2u_2^2 + 2u_3^2u_4^2.$$

Since the right-hand side is positive for $u \neq 0$, we conclude that $u = 0$ is the unique periodic solution of the system. But, in this case also, all the solutions of the linearized system are 2π-periodic.

9.3 Periodic Solutions of Asymptotically Linear Non-Autonomous Second Order Systems

We consider the existence of multiple solutions of the periodic boundary value problem

$$\begin{aligned} \ddot{u}(t) + \nabla F(t, u(t)) = 0 \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0 \end{aligned} \qquad (8)$$

where $F \in \mathbf{C}^2([0,T] \times \mathbf{R}^N, \mathbf{R})$ satisfies the conditions

$$\nabla F(t, u) = A_0(t)u + o(|u|) \quad \text{as} \quad |u| \to 0 \qquad (9)$$

and

$$\nabla F(t, u) = A_\infty(t)u + o(|u|) \quad \text{as} \quad |u| \to \infty, \qquad (10)$$

uniformly in $t \in [0,T]$. We shall write $j_0 = j(A_0, T)$ and $j_\infty = j(A_\infty, T)$.

Theorem 9.2. *Assume that $T > 0$ is such that the following conditions hold.*

A_1. $\nu(A_0, T) = 0$

A_2. $\nu(A_\infty, T) = 0$

A_3. $j_0 \neq j_\infty$.

Then the problem (8) has at least one non-zero solution. Assume, moreover, that

A_4. $|j_0 - j_\infty| \geq 2N$.

Then the problem (8) has at least two non-zero solutions.

The solution of (8) are the critical points of the functional φ defined on H_T^1 by

$$\varphi(u) = \int_0^T [(1/2)|\dot{u}(t)|^2 - F(t, u(t))]\, dt.$$

Let us also define the operator L and the functional ψ on H_T^1 by

$$((Lu, v)) = \int_0^T (\dot{u}(t) - A_\infty(t)u(t), v(t))\, dt$$

$$\psi(u) = \varphi(u) - (1/2)((Lu, u)).$$

Assumption A_2 implies that L is invertible. Since

$$\begin{aligned} |((\nabla\psi(u), v))| &= \left| \int_0^T (A_\infty(t)u(t) - \nabla F(t, u(t)), v(t))\, dt \right| \\ &\leq \|A_\infty u - \nabla F(., u)\|_{L^2} \|v\|_{L^2} \\ &\leq \|A_\infty u - \nabla F(., u)\|_{L^2} \|v\|, \end{aligned}$$

it follows from (10) that for every $\epsilon > 0$, there exists $c(\epsilon) \geq 0$ such that

$$\|\nabla\psi(u)\| \leq \epsilon\|u\| + c(\epsilon) \qquad (11)$$

for every $u \in H_T^1$.

Proposition 9.2. *Under assumptions (10) and A_2, there exists $\rho > 0$ and $\hat{\sigma} \in C^\infty(H_T^1, \mathbf{R})$ satisfying the following conditions:*

a) $\nabla\varphi(u) = 0$ *implies* $\|u\| < \rho$.

b) *$\hat\sigma(u) = 1$ if $0 \le \|u\| \le \rho$ and $\hat\sigma(u) = 0$ if $\|u\| \ge 2\rho$.*

c) *the functional $\hat\varphi(u) = (1/2)((Lu,u)) + \hat\sigma(u)\psi(u)$ is such that $\|\nabla\hat\varphi(u)\| \ge 1$ if $\rho \le \|u\| \le 2\rho$.*

Proof. Taking $\epsilon = \|L^{-1}\|^{-1}/9$, there will exist by (11) $c_1(\epsilon)$ such that

$$\|\nabla\psi(u)\| \le \epsilon\|u\| + c_1 \tag{12}$$

on H_T^1. Therefore, by the mean value theorem, we have

$$\begin{aligned}|\psi(u)| &\le \int_0^T ((\nabla\psi(su),u))\,ds + |\psi(0)| \\ &\le (\epsilon/2)\|u\|^2 + c_1\|u\| + |\psi(0)|.\end{aligned}$$

Thus, there exists $c_2 > 0$ such that

$$|\psi(u)| \le \epsilon\|u\|^2 + c_2. \tag{13}$$

Let

$$\rho = 1 + \left(1 + c_1 + \frac{3c_2}{2}\right)/\epsilon.$$

It follows from (12) that

$$\|\nabla\varphi(u)\| \ge 9\epsilon\|u\| - \epsilon\|u\| - c_1.$$

Thus, the critical points of φ satisfy the a priori estimate

$$\|u\| \le c_1/8\epsilon < \rho$$

and (a) is verified.

Let $\sigma \in C^\infty(\mathbf{R},\mathbf{R})$ be such that

$$\begin{aligned}\sigma(s) &= 1 \quad \text{for } s \le 0\\ &= 0 \quad \text{for } s \ge 1\\ -3/2 \le \sigma'(s) &\le 0 \quad \text{for } s \in \mathbf{R}.\end{aligned}$$

The function $\hat\sigma$ defined on H_T^1 by

$$\hat\sigma(u) = \sigma\left(\frac{\|u\| - \rho}{\rho}\right)$$

satisfies (b). If $\rho \le \|u\| \le 2\rho$, we deduce from (12) and (13) that

$$\begin{aligned}\|\nabla\hat\varphi(u)\| &= \left\|Lu + \sigma\left(\frac{\|u\|-\rho}{\rho}\right)\nabla\psi(u) + \sigma'\left(\frac{\|u\|-\rho}{\rho}\right)\frac{\psi(u)}{\rho\|u\|}u\right\| \\ &\ge 9\epsilon\rho - 2\epsilon\rho - c_1 - (3/2\rho)(4\epsilon\rho^2 + c_2) \ge \epsilon\rho - c_1 - (3/2)c_2 \ge 1,\end{aligned}$$

and the proof is complete. □

By Proposition 9.2, $\nabla\varphi(u) = 0$ if and only if $\nabla\hat\varphi(u) = 0$. Thus, in order to solve problem (8), it suffices to find the critical points of $\hat\varphi$.

Lemma 9.2. *Under the assumptions (20) and A_2, every sequence (u_j) in H_T^1 such that $\nabla\hat\varphi(u_j) \to 0$ contains a convergent subsequence.*

Proof. Assumption A_2 and Proposition 9.2 imply that $\|u_j\| < \rho$ $(j \in \mathbf{N})$. Thus,

$$\nabla\hat\varphi(u_j) = \nabla\varphi(u_j), \quad (j \in \mathbf{N}).$$

Arguing as in Lemma 9.1, we can conclude that (u_j) contains a convergent subsequence. □

Lemma 9.3. *Under the Assumption (10) and A_2, there exist $a < b$ such that the critical points of $\hat\varphi$ belong to $\hat\varphi^{-1}(]a,b[)$ and*

$$P(t,\hat\varphi^b,\hat\varphi^a) = t^{j_\infty}.$$

Proof. Define

$$a = \inf_{B[0,2\rho]} \hat\varphi - 1, \quad b = \sup_{B[0,2\rho]} \hat\varphi - 1,$$

and $\varphi_\infty(u) = (1/2)((Lu,u))$. Proposition 9.2 implies that $\hat\varphi^{-1}(]a,b[)$ contains the critical points of $\hat\varphi$ and that $\hat\varphi^a = \varphi_\infty^a$, $\hat\varphi^b = \varphi_\infty^b$. Hence

$$P(t,\hat\varphi^b,\hat\varphi^a) = P(t,\varphi_\infty^b,\varphi_\infty^a).$$

Since, by assumption A_2, 0 is the only critical point of the quadratic functional φ_∞, it follows from Theorem 8.1 and Corollary 8.3 that

$$P(t,\varphi_\infty^b,\varphi_\infty^a) = t^{j_\infty}. \quad □$$

Proof of Theorem 9.2. We can assume that problem (8) has only a finite number of solutions, i.e. that $\hat\varphi$ has only a finite number of critical points. By Lemma 9.2, $\hat\varphi$ satisfies the Palais–Smale condition over H_T^1. Theorem 8.2 and Lemma 9.3 imply the existence of a polynomial $Q(t)$ with non-negative integer coefficients such that

$$M(t,\hat\varphi^a,\hat\varphi^b) = t^{j_\infty} + (1+t)Q(t). \tag{14}$$

Assumptions (9) and A_1 and Corollary 8.3 imply that

$$\dim C_n(\hat\varphi,0) = \delta_{n,j_0}. \tag{15}$$

Since $j_0 \ne j_\infty$, we obtain from (14) and (15) the existence of at least one non-zero critical point.

Now assume that $|j_0 - j_\infty| \ge 2N$ and that u is the only non-zero critical point of φ_1. Since, by (14),

$$t^{j_0} + \sum_{n=0}^{\infty} \dim C_n(\varphi_1,u)t^n = t^{j_\infty} + (1+t)Q(t),$$

we necessarily have

$$\dim C_{j_\infty}(\hat{\varphi}, u) \geq 1$$

and either $\dim C_{j_0-1}(\hat{\varphi}, u) \geq 1$ or $\dim C_{j_0+1}(\hat{\varphi}, u) \geq 1$. Let us consider the case where $\dim C_{j_0-1}(\hat{\varphi}, u) \geq 1$, the other one being similar. By assumption, $j_0 - 1 \neq j_\infty$. Since the nullity of u is less or equal to $2N$, Corollary 8.4 implies that

$$|j_0 - 1 - j_\infty| \leq 2N - 2.$$

Hence, we obtain $|j_0 - j_\infty| \leq 2N - 1$, which is impossible since $|j_0 - j_\infty| \geq 2N$. □

9.4 Multiple Solutions of Lagrangian Systems

We consider the periodic boundary value problem

$$\begin{aligned} &\frac{d}{dt} D_y L(t, u(t), \dot{u}(t)) = D_x L(t, u(t), \dot{u}(t)) \\ &u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0 \end{aligned} \tag{16}$$

where $L = L(t, x, y)$ satisfies the assumptions (L_1) to (L_4) of Section 4.2.

Theorem 9.3. *Under the above assumptions, if all the weak solutions of* (16) *are non-degenerate, then* (16) *has at least* 2^N *geometrically distinct weak solutions.*

Proof. The weak solutions of (16) are the critical points of the functional φ defined on H_T^1 by

$$\varphi(u) = \int_0^T L(t, u(t), \dot{u}(t))\, dt.$$

By Proposition 4.1, φ is bounded from below and continuously differentiable. Since, by assumption (L_3),

$$\varphi(u + T_i e_i) = \varphi(u), \quad 1 \leq i \leq N,$$

it is natural to define φ on the manifold $M = T^N \times \tilde{H}_T^1$, where T^N is the N-dimensional torus and

$$\tilde{H}_T^1 = \left\{ u \in H_T^1 : \int_0^T u(t)\, dt = 0 \right\}.$$

By Proposition 4.1, φ satisfies the Palais–Smale condition over M. Without loss of generality, we can assume that φ has only a finite number of critical points $u_1, \ldots, u_j$. By a classical result of algebraic topology,

$$P(t, M, \phi) = P(t, T^N, \phi) = \sum_{n=0}^{N} \binom{N}{n} t^n.$$

By Corollary 8.2, there exists a polynomial $Q(t)$, with non-negative integer coefficients, such that

$$\sum_{n=0}^{\infty} \sum_{i=1}^{j} \dim C_n(\varphi, u_i) t^n = \sum_{n=0}^{N} \binom{N}{n} t^n + (1+t)Q(t). \tag{17}$$

Since the critical points of φ are non-degenerate, Corollary 8.3 implies that

$$\dim C_n(\varphi, u_i) = \delta_{n,k_i}, \tag{18}$$

where k_i is the Morse index of u_i. It follows from (17) and (18) that φ has at least

$$\sum_{n=0}^{N} \binom{N}{n} = 2^N$$

critical points in M, so that (16) has at least 2^N geometrically distinct weak solutions. □

Historical and Bibliographical Notes

Theorem 9.1 was proved by Berger [Ber$_1$] using the Lyapunov-Schmidt method. Example 9.2 is due to Moser. The existence of a non-zero solution in Theorem 9.2 follows from Amann–Zehnder [AmZ$_1$] and the existence of two non-zero solutions from Dancer [Dan$_2$]. These authors use a finite-dimensional reduction which is in fact Cesari's method [Ces$_2$] and a wide generalization of Morse theory, the Conley index (see [Con$_1$, CoZ$_1$]). By the same method, Conley and Zehnder were able to solve the Arnold's conjecture for a torus ([CoZ$_2$]) and to consider general first order asymptotically linear Hamiltonian systems [CoZ$_1$]. But, since those problems are variational, it suffices to apply Morse theory or minimax methods. Chang [Ch$_1$] uses Proposition 9.2 and some deformation arguments which are in fact superfluous. The simple approach of Section 9.3 is also applicable to first order Hamiltonian systems and to semi-linear wave equations after a finite-dimensional reduction.

An earlier application of homology to obtain multiple critical points was made by Castro–Lazer [CaL$_1$]. See also [Cha$_2$] and [Cot$_{1,2}$] for applications of Morse theory to related problems.

Other applications of Morse theory are given in [Amb$_8$], [AmbL$_1$], [Ch$_{6,8}$], [TsW$_1$], [MerP$_1$], [Ts$_1$].

We have not considered here the important concept of Conley's index and its generalizations. See [Ben$_{7,8}$], [Ryb$_{2,3,6,8}$], [RybZ$_1$] and, for applications to Hamiltonian systems [Bart$_1$], [BentZ$_1$], [CZ$_3$], to boundary value problems [Dan$_{3,4,5,8}$], [Ryb$_{4,5,7}$].

The Arnold conjecture is considered in [Chap$_1$], [Hof$_6$].

Concerning singular dynamical systems, the reader can consult [AmC$_2$], [Cot$_{3,4,5}$] for a treatment by Morse theory and [Ben$_4$], [CGS$_1$], [CaS$_3$], [Gor$_{3,4}$], [Gre$_{2,3,4}$], [PiT$_3$], [DGM$_1$] for a treatment by minimax methods.

Exercises

1. Assume that $F \in C^2([0,T] \times \mathbf{R}^N, \mathbf{R})$ satisfies conditions (9) and (10) of Section 9.3. If

   A1. $\nu(A_0, T) = 0$

   A2. $\nu(A_\infty, T) = 0$

   A3. $j_\infty = 0 \neq j_0$,

   then the problem (8) has at least two non-zero solutions.

2. Assume that $F \in C^2([0,T] \times \mathbf{R}^N, \mathbf{R})$ satisfies condition (9) and (10) of Section 9.3. If

   A1. $\nu(A_0, T) = 0$

   A2. $\nu(A_\infty, T) = 0$

   A3. $j_0 = 0, \quad j_\infty = 1$,

   then problem (8) has at least two non-zero solutions.

3. Assume that $F \in C^2([0,T] \times \mathbf{R}^N, \mathbf{R})$ satisfies conditions (9) and (10) of Section 9.3. If

   A1. $\nu(A_\infty, T) = 0$

   A2. $j_\infty \notin [j_0, j_0 + \nu(A_0, T)]$,

   then problem (8) has at least one non-zero solution.

10

Nondegenerate Critical Manifolds

Introduction

After recalling some preliminary notions from differential geometry, this chapter presents the local and global aspects of the theory of *nondegenerate critical manifolds*. These manifolds are a natural extension of the notion of non-degenerate critical point.

The theory is applied to proving the existence of infinitely many periodic solutions of the *forced superlinear second order equation*

$$\ddot{u} + |u|^{p-2}u = f(t), \quad p \in]2, \infty[.$$

The periodic solutions of the forced equation are obtained from the periodic orbits of the corresponding autonomous equation

$$\ddot{u} + |u|^{p-2}u = 0$$

by a global perturbation argument. This approach depends upon a precise description of the solutions of the autonomous equation.

The last section is devoted to the existence of T-periodic solutions of the *perturbed second order equation*

$$\ddot{u}(t) + g(u(t)) = \epsilon f(t)$$

near a T-periodic orbit of the autonomous equation

$$\ddot{u}(t) + g(u(t)) = 0.$$

Since this equation is conservative, 1 is a Floquet multiplier with multiplicity 2 of its variational equation, so that classical perturbation arguments are not applicable. The periodic solutions are obtained here by combining the *Liapunov–Schmidt method* with an elementary variational argument. An application is given to the *subharmonics of the forced pendulum equation*.

10.1 Submanifolds

We define a class of sets locally diffeomorphic to a subspace of a Banach space.