INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

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SECOND COLLEGE ON VARIATIONAL PROBLEMS IN ANALYSIS

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Rearrangements in variational problems and applications (III)

G.R. Burton
School of Mathematical Sciences
University of Bath
Claverton Down
Bath BA2 7AY
U.K.

These are preliminary lecture notes, intended only for distribution to participants

MAIN BUILDING STRADA COSTIERA, 11 TEL 22401 TELEFAX 224163 TELEX 460392 ADRIATICO GUEST HOUSE VIA GRIGNANO, 9 TEL. 224241 TELEFAX 224531 TELEX 460449
MICROPROCESSOR LAB. VIA BEIRUT, 31 TEL 224471 TELEFAX 224163 TELEX 460392 GALILEO GUEST HOUSE VIA BEIRUT, 7 TEL 22401 TELEFAX 224559 TELEX 460392

§2 Some of Rearrangements.

The order, presentation and proofs in this section are often different from those in the literature.

<u>8 Thm</u> If μ is a (positive) Borel measure on a complete separable metric space X, $0<\mu(X)=\omega<\infty$, and $\mu\{x\}=0$ $\forall x\in X$, then there is a bijection $\sigma: X \to [0,\omega]$, such that A is μ-measurable iff $\sigma(A)$ is Lebesgue measurable, and $\mu(A) = \mu_1(\sigma(A))$ for all μ-measurable A.

Consequently, a bounded open set in $\mathbb{R}^n$ has the same measure-theoretic structure as an interval in $\mathbb{R}^1$. So for many purposes, we may as well deal with intervals only. The above theorem, in a slightly modified form, can be found in H.L. Royden's Real Analysis, for example.

<u>Definition</u> If f is a measurable function (possibly two-signed) on $[0,\omega]$, then f^* denotes the decreasing rearrangement of f on $[0,\omega]$ (not symmetric!!) and $R(f)$ denotes the set of all rearrangements of f on $[0,\omega]$.

The inequality

$$\int_0^\omega fg \le \int_0^\omega f^*g^*, \quad f\in L^p[0,\omega],\ g\in L^q[0,\omega]$$

p,q conjugate exponents

can be proved by similar methods to those discussed in §1.
Also $\|f^*-g^*\|_p \le \|f-g\|_p$.

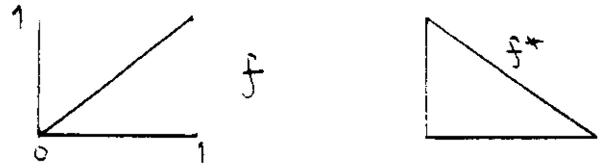
<u>Notation</u> for $f\in L^p[0,\omega]$, $g\in L^q[0,\omega]$, p,q conjugate exponents, write

$$\langle f,g\rangle = \int_0^\omega fg.$$

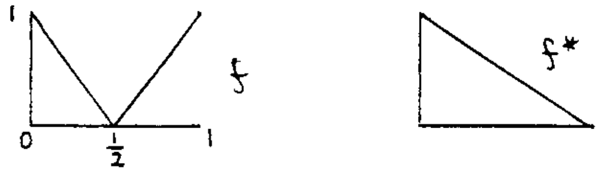
<u>9 Theorem</u> (J.V. Ryff, 1970) If f is a measurable function on $[0,\omega]$ then $f=f^*\circ\sigma$ for some measure-preserving transformation $\sigma:[0,\omega]\to[0,\omega]$ (i.e. $\forall$ μ_1-measurable $A\subset[0,\omega]$, $\sigma^{-1}(A)$ is measurable and $\mu_1\sigma^{-1}(A)=\mu_1(A)$).

(The results of Ryff have been rediscovered by several authors including GRB)

Examples



$\sigma(s) = \frac{1}{2} - s$.



$$\sigma(s) = \begin{cases} 2s , & 0 \le s \le 1/2 \\ 2-2s , & \frac{1}{2} \le s \le 1. \end{cases}$$

Notice that in the second example σ is not injective, and $\mu_1(\sigma(A)) \neq \mu_1(A)$ in general – indeed $\sigma[0,\frac{1}{2}] = [0,1]$. It can be shown that in this example, there is no measure-preserving transformation τ such that $f^* = f \circ \tau$.

Proof of Th 9 Define

$$\sigma(s) = \mu_1\{x \mid f(x) > f(s)\} + \mu_1\{x \mid f(x) = f(s),\ x \le s\}.$$

For the detailed verification, see J. V. Ryff, Measure preserving transformations and rearrangements, J. Math. Analysis & Applications 31 (1970), 449-458.

10 Theorem Let $f_0 \in L^p[0,\omega]$, $1 \le p < \infty$. Then every bounded linear functional on L^p attains its supremum relative to $R(f_0)$

Proof Any bounded linear functional on L^p can be represented as $\langle \cdot, g \rangle$ for some $g \in L^q$, q the conjugate exponent of p.

We have

$$\langle f, g\rangle \le \langle f^*, g^* \rangle = \langle f_0^*, g^* \rangle \quad \forall f \in R(f_0).$$

It will be sufficient to find an f for which equality occurs. Now $g = g^* \circ \sigma$ for some measure-preserving transformation σ. Take $f = f_0^* \circ \sigma$. Then f_0 is a rearrangement of f, and

$$\langle f, g \rangle = \int_0^\omega (f_0^* \circ \sigma)(g^* \circ \sigma) = \int_0^\omega (f_0^* g^*) \circ \sigma = \int_0^\omega f_0^* g^*$$

since $(f_0^* g^*)\circ\sigma$ is a rearrangement of $f_0^* g^*$.

<u>11 Theorem</u> Let $f_0 \in L^p[0,\omega]$, $1 \le p \le \infty$, $g \in L^q$ (q,p conjugate exponents) and suppose there exists an increasing function φ such that $\varphi\circ g = f_1 \in R(f_0)$. Then

(a) $\langle f_1, g\rangle = \langle f_0^*, g^*\rangle$

(b) $\langle f, g\rangle < \langle f_0^*, g^*\rangle$ $\quad f_1 \neq f \in R(f_0)$.

Proof.

(a)
$$\langle f_1, g\rangle = \int_0^\omega (\varphi\circ g) g = \int_0^\omega \psi\circ g \qquad (\psi(s) = \varphi(s)s)$$

$$= \int_0^\omega \psi\circ g^* \qquad (\psi\circ g^* \text{ is a rearrangement of } \psi\circ g)$$

$$= \int_0^\omega (\varphi\circ g^*) g^*$$

$$= \int_0^\omega (\varphi\circ g)^* g^* \qquad (\text{since } \varphi \text{ is increasing})$$

$$= \int_0^\omega f_1^* g^* = \int_0^\omega f_0^* g^*.$$

(b) For simplicity consider only the case $f_0 \ge 0$.
Suppose $f \in R(f_0)$ and $\langle f, g\rangle = \langle f_1, g\rangle$.

<u>First consider two special cases</u>

(i) $\varphi = 1_{(t,\infty)}$ for some t. Then $f_1 = 1_A$ where $A = g^{-1}(t,\infty)$, and $f = 1_B$ where $\mu_1(B) = \mu_1(A)$.
Now $g > t$ on $A\setminus B$ and $g \le t$ on $B\setminus A$ so
$$0 = \int 1_A g - \int 1_B g = \int_{A\setminus B}(g-t) + \int_{B\setminus A}(t-g) \ge \int_{A\setminus B}(g-t)$$

hence $\mu_1(A\setminus B) = 0$. Since $\mu_1(A) = \mu_1(B)$ we have $\mu_1(B\setminus A) = 0$.

(ii) $\varphi = 1_{[\alpha,\infty)}$ is similar.

<u>Case of general $f_0 \ge 0$.</u>

$$0 = \langle f_1, g\rangle - \langle f, g\rangle = \int_0^\omega f_1 g - \int_0^\omega f g$$

$$= \int_0^\omega \int_0^{f_1(x)} dt\, g(x)dx - \int_0^\omega \int_0^{f(x)} dt\, g(x)dx$$

$$= \int_0^\infty \int_{f_1(x)\geq t} g(x)\, dx\, dt - \int_0^\infty \int_{f(x)\geq t} g(x) dx\, dt$$

$$= \int_0^\infty \int_0^\omega 1_{A(t)} g\, dx\, dt - \int_0^\infty \int_0^\omega 1_{B(t)} g\, dx\, dt \qquad (*)$$

where $A(t) = f_1^{-1}[t,\infty)$, $B(t) = f^{-1}[t,\infty)$. Now

$$x \in A(t) \iff \varphi \circ g(x) \geq t \iff g(x) \in \varphi^{-1}[t,\infty).$$

Then $J = \varphi^{-1}[t,\infty)$ has the form (α,∞), $[\alpha,\infty)$, $\mathbb{R}$ or $\emptyset$ since φ is increasing; hence $1_{A(t)} = \psi\circ g$ where $\psi = 1_J$ which is increasing. By (a) we now have

$$\int 1_{A(t)} g \geq \int 1_{B(t)} g$$

so it follows from $(*)$ that

$$\int 1_{A(t)} g = \int 1_{B(t)} g \qquad \text{for almost all } t.$$

It now follows from the special cases (i), (ii) that for almost all t, $A(t) = B(t)$ apart from a set of zero measure. Finally

$$\int_0^\omega |f - f_1| = \int_{f(x)\geq f_1(x)} \int_{f_1(x)}^{f(x)} dt\, dx + \int_{f_1(x)>f(x)} \int_{f(x)}^{f_1(x)} dt\, dx$$

$$= \int_0^\infty \int_{\substack{f(x)\geq t\\ f_1(x)<t}} dx\, dt + \int_0^\infty \int_{\substack{f_1(x)\geq t\\ f(x)<t}} dx\, dt$$

$$= \int_0^\infty \mu_1(B(t)\setminus A(t)) + \mu_1(A(t)\setminus B(t))\, dt$$

$$= 0$$

So $f = f_1$ a.e.

12 Theorem Let $f_0 \in L^p[0,\omega]$, $1 \le p \le \infty$, $g \in L^q$ (q conjugate exponent of p) and suppose there is an $f_1 \in R(f_0)$ such that
$$\langle f, g\rangle < \langle f_0^*, g^*\rangle \quad \text{for } f_1 \ne f \in R(f_0).$$
Then there exists an increasing function φ such that $f_1 = \varphi \circ g$.

Proof. Consider the possibility that some level set S of g^* is a nontrivial interval. We claim that then f_0^* is constant on S. Suppose not. Then by rearranging f_0^* on S we can obtain $\tilde{f} \in R(f_0)$ s.t. $\langle \tilde{f}, g^*\rangle = \langle f_0^*, g^*\rangle$, $\tilde{f} \ne f_0^*$. Let σ be a measure preserving transformation such that $g = g^* \circ \sigma$. Then $\tilde{f}\circ\sigma$, $f_0^*\circ\sigma \in R(f_0)$ and $\langle \tilde{f}\circ\sigma, g\rangle = \langle f_0^*\circ\sigma, g\rangle = \langle f_0^*, g^*\rangle$. Since $\tilde{f}\circ\sigma \ne f_0^*\circ\sigma$ we have a contradiction.

Thus f_0^* is constant on level sets of g^*. Since f_0^*, g^* are decreasing, it is not difficult to see that $f_0^* = \varphi\circ g^*$ for some increasing function φ. Now
$$\langle \varphi\circ g, g\rangle = \langle \varphi\circ g^*, g^*\rangle = \langle f_0^*, g^*\rangle$$
so $\varphi\circ g = f_1$.

13 Theorem Let $f_0 \in L^p[0,\omega]$, $1 \le p < \infty$, and let $\Phi : L^p \to \mathbb{R}$ be convex and weakly sequentially continuous. Then Φ attains its supremum relative to $R(f_0)$.

Proof. $R(f_0)$ is weakly relatively compact in L^p; this clear if $p > 1$ since $R(f_0)$ is bounded and L^p is reflexive. In the case $p = 1$, the equi-integrability of the rearrangements of f_0 ensures $R(f_0)$ is weakly relatively compact in L^1.

Write $m = \sup \Phi(R(f_0))$, and let u be the weak limit of a maximising sequence for Φ. Then $\Phi(u) = m$.

Weak sequential continuity implies strong continuity; together with convexity this implies subdifferentiability of Φ. Choose $h \in \partial\Phi(u)$ (which we regard as a subset of L^q, q the conjugate exponent of p). Then
$$\langle u, h\rangle \le k = \sup\{\langle f, h\rangle \mid f \in R(f_0)\}$$

by weak continuity of $\langle \cdot , h \rangle$. Choose (Th 10) $f \in R(f_0)$ such that

$$\langle f, h \rangle = k.$$

Then

$$\begin{aligned} \Phi(f) &\geq \Phi(u) + \langle f - u, h \rangle \\ &= m + k - \langle u, h \rangle \\ &\geq m. \end{aligned}$$

Hence f maximises Φ relative to $R(f_0)$.

<u>14. Theorem</u> Let $f_0 \in L^p[0,\omega]$, $1 \leq p < \infty$, suppose $\Phi : L^p \to \mathbb{R}$ is strictly convex, suppose Φ possesses a maximiser $\hat{f}$ relative to $R(f_0)$, and suppose there exists $h \in \partial\Phi(\hat{f})$ ($\subset L^q$, q conjugate exponent of p). Then

$$\hat{f} = \varphi \circ h$$

for some increasing function φ.

<u>Proof</u> If $f \in R(f_0)$, $f \neq \hat{f}$, then

$$\Phi(\hat{f}) \geq \Phi(f) > \Phi(\hat{f}) + \langle f - \hat{f}, h \rangle$$

so

$$\langle f, h \rangle < \langle \hat{f}, h \rangle.$$

It follows from Th 12 that there is an increasing function φ s.t. $\hat{f} = \varphi \circ h$

<u>Example</u> Ω domain in $\mathbb{R}^2$ bounded by a simple closed curve, $K = (-\Delta)^{-1}$ with zero Dirichlet boundary conditions on Ω, $\Phi(u) = \frac{1}{2}\int_\Omega uKu$, $f_0 \in L^2(\Omega)$.

Then Φ attains its supremum relative to $R(f_0)$ at $\hat{f}$ say, and $\hat{f} = \varphi \circ (K\hat{f})$ for some increasing function φ.

Put $\psi = K\hat{f}$. Then

$$-\Delta\psi = \varphi \circ \psi$$
$$\psi \in W_0^{1,2}(\Omega)$$

Physical Interpretation

If ζ represents the vorticity of an ideal fluid flowing in Ω, then $\psi = K\zeta$ is the stream function, $\left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}\right)$ is the velocity, and $\Phi(\zeta)$ is kinetic energy.

The condition $\psi = 0$ on $\partial\Omega$ is the condition for the velocity to be tangential at the boundary. $\zeta = \varphi\circ\psi$ is the condition for the flow to be steady (i.e. velocity is independent of time)

We have thus shown that maximising the kinetic energy subject to the constraint that the vorticity is a rearrangement of a given function. This constraint is a natural one, to the extent that in an unsteady flow (without forcing) the vorticity always continues to be a rearrangement of its initial configuration.

Definition For $f \in L^1[0,\omega]$ let

$$\overline{R}(f) = \left\{h \in L^1 \,\middle|\, \int_0^\omega h = \int_0^\omega f,\ \int_0^s h^* \le \int_0^s f^* \text{ for } 0<s<\omega\right\}.$$

15 Theorem (Ryff) Let $f \in L^p[0,\omega]$, $1 \le p < \infty$. Then

(a) $\|h\|_p \le \|f\|_p \quad \forall h \in \overline{R}(f)$

(b) $\overline{R}(f)$ is closed in L^p

(c) $\overline{R}(f)$ is convex

(d) $\overline{R}(f)$ is weakly compact in L^p.

Proof For simplicity suppose $f \ge 0$.

(a) If $p = 1$ then $\|h\|_1 = \int_0^\omega h = \int_0^\omega f = \|f\|_1$. Consider therefore the case $1 < p < \infty$. Let $h \in \overline{R}(f)$ and write

$$g = |h|^{p-1} / \|h\|_p^{p-1}.$$

Then for $q^{-1} + p^{-1} = 1$

$$\|g\|_q^q = \int_0^\omega |h|^{q(p-1)} / \|h\|_p^{q(p-1)} = \|h\|_p^p / \|h\|_p^p = 1$$

and

$$\langle h, g \rangle = \int_0^\omega |h|^p \Big/ \|h\|_p^{p-1} = \|h\|_p \, .$$

Moreover

$$\langle h,g\rangle = \int_0^\omega hg \qquad\qquad G(t) = g^{*-1}[t,\infty)$$

$$\leq \int_0^\omega h^* g^* \quad = \int_0^\infty \int_{G(t)} h^*$$

$$\leq \int_0^\infty \int_{G(t)} f^* \qquad \text{since } G(t) = [0,s] \text{ for some } s$$

$$= \langle f^*, g^* \rangle \leq \|f^*\|_p \|g^*\|_q = \|f\|_p \, .$$

(b) $\overline{R}(f)$ is closed in L^p by continuity of $*$.

(c) To show $\overline{R}(f)$ is convex, it will be enough to show each functional $\Phi_s : h \mapsto \int_0^s h^*$ is convex.

Now

$$\Phi_s(h) = \int_0^\omega 1_{[0,s]} h^*$$

$$= \sup_{g\in X} \int_0^\omega gh \qquad\qquad X = R(1_{[0,s]}) \quad (\text{Th } 10)$$

$$= \chi_X^*(h) \qquad \text{conjugate convex functional of}$$

$$\chi_X(u) = \begin{cases} 0 & u \in X \\ \infty & u \in L^q \setminus X. \end{cases}$$

Thus Φ_s is convex.

(e) If $p>1$ then $\overline{R}(f)$ is closed, bounded and convex, so $\overline{R}(f)$ is weakly compact.

If $p = 1$ then for $h \in \overline{R}(f)$

$$\int_0^s h^* \leq \int_0^s f \to 0 \text{ as } s \to 0$$

so $\int_0^s h^* \to 0$ as $s \to 0$ uniformly over $h \in \overline{R}(f)$. It easily follows that

$$\lim_{M\to\infty} \int_{h\geq M} h = 0 \quad \text{uniformly over } h \in \overline{R}(f)$$

This uniform integrability condition, together with the fact that $\bar{R}(f)$ is weakly closed, gives the weak compactness of $\bar{R}(f)$ in L^1

<u>16 Theorem</u> Let $f \in L^p[0,\omega]$, $1 \le p < \omega$ and let L be a closed affine subspace of $L^p[0,\omega]$ having finite codimension. Then every extreme point of $\bar{R}(f) \cap L$ belongs to $R(f)$.

(The case $L = L^p$ is due to Ryff; the general case is due to GRB + E.P. Ryan).

<u>Proof.</u> Write

$$L = \{ u \in L^p \mid \int_0^\omega u g_i = \alpha_i,\ i = 1, \dots, n \}$$

where $g_1, \dots, g_n \in L^q$ (q conjugate exponent of p), $\alpha_1, \dots, \alpha_n$ some reals.

Consider $h \in \bar{R}(f) \cap L$, and write

$$H(s) = \int_0^s h^\#, \qquad F(s) = \int_0^s f^\# \qquad 0 \le s \le \omega$$

where $\#$ denotes <u>increasing</u> rearrangement. Thus H, F are continuous convex functions on $[0,\omega]$, and $H \ge F$ since $h \in \bar{R}(f)$

Suppose $h \notin R(f)$. Then $H(s) > F(s)$ for some s, and let (s_0, s_1) be a maximal open interval such that

$$H(s) > F(s) \quad \text{for} \quad s_0 < s < s_1. \tag{1}$$

Then

$$H(s_0) = F(s_0) \tag{2}$$

$$H(s_1) = F(s_1). \tag{3}$$

Since $H \ge F$ we now have

$$\partial H(s_0) \subset \partial F(s_0) \quad \text{if} \quad s_0 > 0 \tag{4}$$

$$\partial H(s_1) \subset \partial F(s_1) \quad \text{if} \quad s_1 < \omega \tag{5}$$

Let ℓ_0, ℓ_1 be the one-sided limits $\ell_0 = f^\#(s_0+)$, $\ell_1 = f^\#(s_1-)$. Then by (1) and convexity of H, F we have, for $s_0 < s < s_1$,

$$\ell_0 \le \frac{F(s) - F(s_0)}{s - s_0} < \frac{H(s) - H(s_0)}{s - s_0} \le h^\#(s)$$

$$\ell_1 \ge \frac{F(s_1) - F(s)}{s_1 - s} > \frac{H(s_1) - H(s)}{s_1 - s} \ge h^\#(s).$$

Thus

$$\ell_0 < h^\#(s) < \ell_1 \quad \text{for} \quad s_0 < s < s_1. \tag{6}$$

From (4),(5) we also have

$$h^{\#}(s_0-) \le l_0 \quad \text{if} \quad s_0 > 0 \tag{7}$$

$$h^{\#}(s_1+) \ge l_1 \quad \text{if} \quad s_1 < \omega. \tag{8}$$

By (6) and the defintions of l_0 and l_1 we can choose, in the following order, numbers $\tau_0, \tau_1, a, b, \varepsilon, \sigma_0, \sigma_1$ satisfying

$$\begin{gathered} s_0 < \sigma_0 < \tau_0 < \tau_1 < \sigma_1 < s_1 \\ l_0 \le f^{\#}(\sigma_0) < a < a+\varepsilon < h^{\#}(\tau_0) \le h^{\#}(\tau_1) < b-\varepsilon < b < f^{\#}(\sigma_1) \le l_1 \end{gathered} \tag{9}$$

Let

$$\Sigma = h^{-1}(a+\varepsilon, b-\varepsilon)$$

so $\mu_1(\Sigma) \ge \tau_1 - \tau_0 > 0$. By Lyapounov's Theorem [the precise result we require is 16.1.i of L. Cesari, "Optimization — Theory and Applications"] we can partition Σ into two measurable subsets Σ_1, Σ_2 of equal measure, such that

$$\int_{\Sigma_1} g_i = \int_{\Sigma_2} g_i, \quad i = 1, \dots, n \quad . \tag{10}$$

Let

$$u(s) = \begin{cases} 1 & s \in \Sigma_1 \\ -1 & s \in \Sigma_2 \\ 0 & s \in [0,\omega] \setminus (\Sigma_1 \cup \Sigma_2) \end{cases} .$$

By (10) we have $h \pm \xi u \in L$ for all positive ξ. We now show that $h \pm \xi u \in \bar{R}(f)$ provided ξ is sufficiently small and positive; then $h = \frac{1}{2}(h+\xi u) + \frac{1}{2}(h-\xi u)$ is a non-extreme point of $\bar{R}(f) \cap L$.

Since Σ_1 and Σ_2 have equal measure we have

$$\int_0^\omega (h \pm \xi u) = \int_0^\omega h = \int_0^\omega f . \tag{11}$$

Write $t_0 = \mu_1 h^{-1}(-\infty, a]$ and $t_1 = \mu_1 h^{-1}(-\infty, b]$, so by (7,8,9)

$$s_0 \le t_0 \le t_1 \le s_1 .$$

Now suppose $0 < \xi < \varepsilon$. Then $(h \pm \xi u)^{-1}(a,b) = h^{-1}(a,b)$, and $(h \pm \xi u)(s) = h(s)$ for $s \in [0,\omega] \setminus (a,b)$. Therefore $(h \pm \xi u)^{\#}(s) = h^{\#}(s)$ if $0 < s < t_0$ or $t_1 < s <$; from this and from (11) we obtain

$$\int_0^s (h \pm \xi u)^{\#} = \int_0^s h^{\#} \ge \int_0^s f^{\#} \quad \text{if } 0 \le s \le t_0 \text{ or } t_1 \le s \le \omega . \tag{12}$$

If $t_0 < s < \sigma_0$, then $(h \pm \xi u)^{\#}(s) > a > f^{\#}(s)$. With (12) this gives

$$\int_0^s (h \pm \xi u)^{\#} = \int_0^{t_0} h^{\#} + \int_{t_0}^s (h \pm \xi u)^{\#} \geq \int_0^s f^{\#} \quad \text{if } t_0 \leq s \leq \sigma_0. \quad (13)$$

If $\sigma_1 < s < t_1$ then $(h \pm \xi u)^{\#}(s) < b < f^{\#}(s)$. With (12) this gives

$$\int_0^s (h \pm \xi u)^{\#} = \int_0^{t_1} h^{\#} - \int_s^{t_1} (h \pm \xi u)^{\#} \geq \int_0^s f^{\#} \quad \text{if } \sigma_1 \leq s \leq t_1. \quad (14)$$

If $\sigma_0 \leq s \leq \sigma_1$, then the functional $v \mapsto \int_0^s v^{\#}$ is concave downwards (see the proof of Th 15(c)), so

$$\int_0^s (h \pm \xi u)^{\#} \geq \int_0^s h^{\#} + \xi \int_0^s (\pm u)^{\#} \geq \int_0^s h^{\#} - \xi \mu_1(\Sigma_1),$$

therefore

$$\int_0^s (h \pm \xi u)^{\#} \geq \int_0^s f^{\#} \quad \text{if } \sigma_0 \leq s \leq \sigma_1$$

provided that $\xi \mu_1(\Sigma_1) \leq \delta = \min\{ H(t) - F(t) \mid \sigma_0 \leq t \leq \sigma_1 \}$.

Now (11)–(15) show that $h \pm \xi u \in \overline{R}(f) \cap L$ provided that $0 < \xi < \min\{ \varepsilon, \delta / \mu_1(\Sigma_1)\}$.

<u>17 THEOREM</u> Let $f_0 \in L^p[0,\omega]$, $1 \leq p < \infty$, $g \in L^q[0,\omega]$, p, q conjugate exponents, and suppose there exists an increasing function φ such that $\varphi \circ g = \hat{f} \in R(f_0)$. Then $\hat{f}$ is the unique maximiser of $\langle \cdot, g \rangle$ relative to $\overline{R}(f_0)$.

<u>Proof</u> Write

$$k = \sup\{ \langle f, g \rangle \mid f \in \overline{R}(f_0) \}$$

$$K = \{ f \in \overline{R}(f_0) \mid \langle f, g \rangle = k \}.$$

Then $K \neq \emptyset$ by the weak compactness of $\overline{R}(f_0)$. Moreover K is convex and weakly compact. Consider any extreme point f_1 of K.
By theorem 16 $f_1 \in R(f_0)$. Thus f_1 is then a maximiser of $\langle \cdot, g \rangle$ relative to $R(f_0)$; but <u>by</u> Theorem 11 $\hat{f}$

is the only such maximiser, so $f_1 = \hat{f}$. It follows that $\hat{f}$ is the only extreme point of K. But by the Krein-Milman theorem K is the closed convex hull of its extreme points, so $K = \{\hat{f}\}$ as required.

<u>18 Corollary</u> Let $f_0 \in L^p[0,\omega]$, $1 \le p < \infty$. Then every point of $R(f_0)$ is an exposed point of $\bar{R}(f_0)$.

<u>Proof</u> Let $f \in R(f_0)$ and write $g = \tan^{-1} f$. Then $g \in L^q[0,\omega]$ (q conjugate exponent of p) indeed $g \in L^\infty$! Moreover $f = \varphi \circ g$ where $\varphi = \tan$. By Th 17 f is the unique maximiser of $\langle \cdot, g \rangle$ relative to $\bar{R}(f)$.

<u>19 Corollary</u> Let $f_0 \in L^p[0,\omega]$, $1 \le p < \infty$. Then $\bar{R}(f_0)$ is the closed convex hull of $R(f_0)$ in L^p.

<u>Proof.</u> $\bar{R}(f_0)$ is weakly compact and convex, so by Krein-Milman $\bar{R}(f_0)$ is the closed convex hull of its set of extreme points, which is $R(f_0)$.

<u>20 Theorem</u> (G.R.B & E.P.Ryan) Let $f_0 \in L^p[0,\omega]$, $1 \le p < \infty$ and let $T : L^p \to \mathbb{R}^n$ by a bounded linear map. Then $T(R(f_0)) = T(\bar{R}(f_0))$ which is compact and convex.

<u>Proof</u> Let $y \in T(\bar{R}(f_0))$. We must show $y \in T(R(f_0))$. Write $L = T^{-1}(y)$. Then L is a closed affine subspace of L^p, having finite codimension. Now $L \cap \bar{R}(f_0)$ is a nonempty weakly compact convex set, so by Krein-Millman $L \cap \bar{R}(f_0)$ has an extreme point f. By Theorem 16 $f \in R(f_0)$. Thus $y \in T(R(f_0))$.

21 Corollary (Ryff, by a different method) Let $f_0 \in L^p[0,w]$, $1 \le p < \infty$. Then $R(f_0)$ is weakly dense in $\overline{R}(f_0)$.

Proof. Let $f \in \overline{R}(f_0)$ and let U be a weak neighbourhood of f. Then U contains a set of the form

$$V = \{ u \in L^p \mid |w_i^*(u) - w_i^*(f)| < \varepsilon,\ i = 1, \dots, n \}$$

where $\varepsilon > 0$ and $w_1^*, \dots, w_n^*$ are some bounded linear functionals on L^p. Write

$$L = \{ u \in L^p \mid w_i^*(u) = w_i^*(f),\ i = 1, \dots, n \}.$$

Then L is a closed affine subspace of finite codimension, and

$$f \in L \subset V \subset U.$$

By Krein-Milman $\overline{R}(f_0) \cap L$ has an extreme point $\hat{f}$, and by Theorem 17 $\hat{f} \in R(f_0)$. Thus $U \cap R(f_0) \neq \emptyset$. This proves that $R(f_0)$ is weakly dense in $\overline{R}(f_0)$.

Sometimes one wishes to have the conclusions of Theorem 12, without the hypothesis that the maximiser is unique. A result that can sometimes be applied is the following:

22 Lemma Let Ω be a bounded open set in $\mathbb{R}^n$, let $f_0 \in L^p(\Omega)$, $1 \le p < \infty$, let q be the conjugate exponent of p. Suppose $f_1 \in \overline{R}(f_0) \subset L(\Omega)$, $u \in L^q(\Omega) \cap W^{2,1}_{loc}(\Omega)$, and that

- f_1 maximises $\langle \cdot, u \rangle$ relative to $\overline{R}(f_0)$
- $-\Delta u \ge f_1$ a.e. (or $\Delta u \ge f_1$ a.e.)
- $f_0 \ge 0$ a.e.

Then $f_1 \in R(f_0)$ and $f_1 = \varphi \circ u$ for some increasing function φ.

Proof We consider only the special case $f_0 > 0$ a.e. Then every element of $\overline{R}(f_0)$ is positive a.e.

A $W^{1,1}_{loc}$-function has zero derivative almost everywhere on each of its level sets, and so a $W^{2,1}_{loc}$-function has zero second derivatives almost everywhere on any level set. Since $-\Delta u \ge \hat{f} > 0$ almost everywhere, it follows that every level set of u has zero measure. (similarly if $\Delta u = f_1$ a.e.)

The decreasing rearrangement u^* of u, defined on $[0, \omega]$ where $\omega = \mu_n(\Omega)$, therefore has no level sets of positive measure. That is, u^* is strictly decreasing, and therefore injective. So u^* possesses a left inverse ψ: Range $u^* \to [0,\omega]$, that is $\psi \circ u^* = id_{[0,\omega]}$, and ψ is a decreasing function also. Now $f_0^* = f_0^* \circ \psi \circ u^* = \varphi \circ u^*$ where $\varphi = f_0^* \circ \psi$ is an increasing function.

Then $\hat{f} = \varphi \circ u$ is a rearrangement of $\varphi \circ u^* = f_0^*$, so $\hat{f} \in R(f_0)$, and by Theorem 17, $\hat{f}$ is the unique maximiser of $\langle \cdot, u \rangle$ relative to $\overline{R}(f_0)$. Thus $f_1 = \hat{f}$. Therefore $f_1 \in R(f_0)$ and $f_1 = \varphi \circ u$.

The following result has an interpretation in terms of hydrodynamics, in the case when $n = 2$ and $\partial\Omega$ is a simple closed curve.

23 Theorem GRB Let Ω be a bounded domain in $\mathbb{R}^n$ and let $K : L^2(\Omega) \to W_0^{1,2}(\Omega)$ be $(-\Delta)^{-1}$ with zero Dirichlet boundary conditions. Define

$$\Phi(v) = \frac{1}{2}\int_\Omega vKv \quad , \quad v \in L^2(\Omega).$$

Let $f \in L^2(\Omega)$ be non-negative and write

$$M = \sup \Phi(R(f)) \qquad (R(f_0) \subset L^2(\Omega))$$
$$m = \inf \Phi(R(f)).$$

If $m \le e \le M$ then there is a $w \in R(f)$, and $\psi = Kw$, such that

$$\Phi(w) = e$$
$$-\Delta\psi = \varphi \circ \psi \quad \text{for some function } \varphi.$$

Proof. K is a compact, self adjoint (in L^2), (strictly) positive linear operator, so Φ is a weakly sequentially continuous (strictly) convex functional on $L^2(\Omega)$, and

$\Phi'(v) = Kv.$

Let Π be a fixed hyperplane and let Π_λ be the hyperplane parallel to Π dividing the volume of Ω in the ratio $\lambda : 1-\lambda$; let the two portions into which Ω is so divided be A_λ and B_λ respectively. Write

$$f_\lambda = f 1_{A_\lambda}$$
$$g_\lambda = f 1_{B_\lambda}$$

so $f_0 = 0$, $g_0 = f$ and $f_1 = f$, $g_1 = 0$. Let

$$F_\lambda = \overline{R}(f_\lambda) \quad (\subset L^2(\Omega))$$
$$G_\lambda = \overline{R}(g_\lambda)$$

Note that the elements of F_λ can live on any part of Ω, and the same for G_λ

and

$$e_\lambda = \sup_{u \in F_\lambda} \inf_{v \in G_\lambda} \Phi(u+v).$$

Then

$$e_0 = \inf \Phi(\overline{R}(f)) \leq m \quad (= m \text{ in fact, by weak density of } R(f) \text{ in } \overline{R}(f)).$$

$$e_1 = \sup \Phi(\overline{R}(f)) \geq M \quad (= M \text{ in fact})$$

It is also easy to show that if λ, μ are close, then the elements of F_λ are close to F_μ, and the elements of G_λ are close to those of G_μ. It follows that e_λ is a continuous function of λ. So λ can be chosen in $[0,1]$ such that $e_\lambda = e$.

<u>Existence of minimax pair $(\tilde{u}, \tilde{v})$</u>

Define

$$J_\lambda(u) = \inf_{v \in G_\lambda} \Phi(u+v), \quad u \in F_\lambda.$$

Let $\{u_k\}_{k=1}^{\infty}$ be a maximising sequence in F_λ for J_λ. Then by weak compactness of G_λ, there is for each k a $v_k \in G_\lambda$ with

$$\Phi(u_k + v_k) = J_\lambda(u_k).$$

Applying weak compactness of F_λ and G_λ, we can, after removing a subsequence suppose $u_k \xrightarrow{w} \tilde{u} \in F_\lambda$ and $v_k \xrightarrow{w} \tilde{v} \in G_\lambda$.

Now
$$\Phi(u_k + v_k) = J_\lambda(u_k) \to e_\lambda$$
hence
$$\Phi(\tilde{u} + \tilde{v}) = e_\lambda .$$
For any $v \in G_\lambda$
$$\Phi(u_k + v) \geq J_\lambda(u_k) = \Phi(u_k + v_k).$$
Letting $k \to \infty$ gives
$$\Phi(\tilde{u} + v) \geq \Phi(\tilde{u} + \tilde{v}).$$
Therefore
$$\Phi(\tilde{u} + \tilde{v}) = J_\lambda(\tilde{u})$$
Thus $\tilde{u}$ maximises J_λ, and $\tilde{v}$ minimises $\Phi(\tilde{u} + v)$ over v.

1st variation condition at minimax pair

Variations in v Let $v \in G_\lambda$. Then for $0 < t \leq 1$, $(1-t)\tilde{v} + tv \in G_\lambda$ by convexity, so by minimality of $\tilde{v}$,
$$\Phi(\tilde{u} + \tilde{v}) \leq \Phi(\tilde{u} + (1-t)\tilde{v} + tv)$$
$$= \Phi(\tilde{u} + \tilde{v}) + t\langle v - \tilde{v}, K(\tilde{u} + \tilde{v})\rangle + t^2 \Phi(v - \tilde{v})$$
whence
$$\langle v - \tilde{v}, K(\tilde{u} + \tilde{v})\rangle + t\,\Phi(v - \tilde{v}) \geq 0.$$
Letting $t \to 0$ we obtain
$$\langle v - \tilde{v}, K(\tilde{u} + \tilde{v})\rangle \geq 0.$$
Thus $\tilde{v}$ minimises $\langle \cdot, K(\tilde{u} + \tilde{v})\rangle$ over G_λ.

Variations in u Let $u \in F_\lambda$. Then choose v such that
$$\Phi(u + v) = J_\lambda(u).$$
We have
$$\Phi(\tilde{u} + \tilde{v}) = J_\lambda(\tilde{u}) \geq J_\lambda(u)$$
$$= \Phi(u + v)$$
$$\geq \Phi(\tilde{u} + \tilde{v}) + \langle u - \tilde{u}, K(\tilde{u} + \tilde{v})\rangle + \langle v - \tilde{v}, K(\tilde{u} + \tilde{v})\rangle$$
by convexity
$$\geq \Phi(\tilde{u} + \tilde{v}) - \langle u - \tilde{u}, K(\tilde{u} + \tilde{v})\rangle$$
by variational condition on v

hence

$$\langle u - \tilde{u}, K(\tilde{u}+\tilde{v}) \rangle \le 0.$$

Thus $\tilde{u}$ maximises $\langle \cdot, K(\tilde{u}+\tilde{v})\rangle$ over F_λ.

Existence of solution. Let $w = \tilde{u}+\tilde{v}$ and $\psi = Kw$. Thus $\psi \in W^{2,2}_{loc}$ by elliptic regularity theory.

Since $\tilde{u}$ maximises $\langle \cdot, \psi\rangle$ on F_λ, and

$$-\Delta\psi = w \ge \tilde{u}$$

we deduce from Lemma 22 that $\tilde{u} \in R(f_\lambda)$, and that $\tilde{u} = \varphi_1 \circ \psi$ for some increasing function φ_1.

Since $\tilde{v}$ maximises $\langle \cdot, -\psi\rangle$ on G_λ, and

$$\Delta(-\psi) = w \ge \tilde{v}$$

we deduce from Lemma 22 that $\tilde{v} \in R(g_\lambda)$ and that $\tilde{v} = \varphi_2 \circ \psi$ for some decreasing function φ_2.

Now $\tilde{u}$ lives on a set of measure $\lambda\mu_1(\Omega)$, and $\tilde{v}$ lives on a set of measure $(1-\lambda)\mu_1(\Omega)$. Since $\tilde{u}$ is an increasing function of ψ, and $\tilde{v}$ is a decreasing function of ψ, it follows that $\tilde{u}$ and $\tilde{v}$ live on essentially disjoint sets. Thus $w = \tilde{u}+\tilde{v}$ is a rearrangement of f, and

$$\begin{aligned} -\Delta\psi = w &= \varphi_1 \circ w + \varphi_2 \circ \psi \\ &= (\varphi_1 + \varphi_2)\circ\psi \\ &= \varphi\circ\psi \quad \text{with } \varphi = \varphi_1 + \varphi_2. \end{aligned}$$

Now w is the desired solution.

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