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Rearrangements of functions, maximization of convex functionals, and vortex rings

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1. Introduction

This paper studies the maximization of a convex functional over the set of rearrangements of a fixed function. Our results ensure the existence of weak solutions of free boundary problems for certain semilinear elliptic equations having unknown monotone nonlinearities, and are especially relevant to Benjamin's theory for vortex rings in an ideal fluid [5]. For the purpose of this Introduction we take $(\Omega, \mathcal{M}, \mu)$ to be a finite, separable, nonatomic (positive) measure space (definitions will be given in Sect. 2), we let $1 \leqq p \leqq \infty$, let q be the conjugate exponent of p, let $f_0 \in L^p(\mu)$ and let $\mathcal{F}$ be the set of *rearrangements* of f_0 on Ω; that is, $f \in \mathcal{F}$ if and only if $\mu(f^{-1}[\beta, \infty)) = \mu(f_0^{-1}[\beta, \infty))$ for every real β. The L^q-*topology* on $L^p(\mu)$ is the weak topology if $1 \leqq p < \infty$, or the weak* topology if $p = \infty$. Our main results are as follows:

Theorem A. *Let Ψ be a real strictly convex functional on $L^p(\mu)$, sequentially continuous in the L^q-topology on $L^p(\mu)$. Then Ψ attains a maximum value relative to $\mathcal{F}$. If f^* is a maximizer and $g \in L^q(\mu)$ is a subgradient of Ψ at f^* (such a g must exist) then $f^* = \varphi \circ g$ almost everywhere, for some increasing function φ.*

Recall that when Θ is a domain in $\mathbb{R}^N$ then $W^m(\Theta)$ denotes the space of locally Lebesgue integrable real functions on Θ whose distributional partial derivatives of orders up to and including m are locally integrable functions. Two measures are said to be *equivalent* if each is absolutely continuous with respect to the other.

Theorem B. *Suppose Ω is a domain in $\mathbb{R}^N$, suppose μ is a finite measure equivalent to Lebesgue measure on Ω, let $1 \leqq p < \infty$, let $m \geqq 1$ and let*

$$\mathcal{L} = \sum_{1 \leqq |\alpha| \leqq m} a_\alpha(x) D^\alpha$$

be an m-th order linear partial differential operator in Ω with measurable coefficients, and having no 0-th order term. Suppose $K: L^p(\mu) \to L^q(\mu)$ is a compact symmetric positive operator such that $Ku \in W^m(\Omega)$ and $\mathcal{L}Ku = u$ almost everywhere in Ω for all $u \in L^p(\mu)$, and suppose $v \in L^q(\mu) \cap W^m(\Omega)$ satisfies $\mathcal{L}v = 0$ almost everywhere in Ω.

Let $f_0 \in L^p(\mu)$ be non-negative, let I be real and suppose there are f_1 and f_2 in $\mathscr{F}$ satisfying

$$\int_\Omega f_1 v d\mu < I < \int_\Omega f_2 v d\mu .$$

Then the functional

$$\Psi(f) = \tfrac{1}{2}\int_\Omega fKfd\mu$$

attains a maximum relative to the set

$$\left\{ f \in \mathscr{F} \,\middle|\, \int_\Omega fvd\mu = I \right\}.$$

If f^* is a maximizer and $u = Kf^*$ then u satisfies

$$\mathscr{L}u = \varphi \circ (u - \lambda v)$$

almost everywhere in Ω for some increasing function φ and some real λ.

These results are proved in Sect. 3, where Theorem A occurs in a slightly more general form as Theorem 7 and Theorem B occurs as Theorem 9.

It transpires that the maximization of linear functionals relative to $\mathscr{F}$ plays a crucial role in the proofs of Theorems A and B. Accordingly we fix $g \in L^q(\mu)$ and study in Sect. 2 the problem of maximizing the functional

$$\langle f, g \rangle = \int_\Omega fgd\mu$$

over $f \in \mathscr{F}$. We show that a maximizer exists, and is unique only if $\varphi \circ g \in \mathscr{F}$ for some increasing function φ; the maximizer is then $\varphi \circ g$. If $1 \leqq p < \infty$ and $f^* = \varphi \circ g \in \mathscr{F}$ for some increasing function φ, we show that every maximizing sequence for $\langle \cdot, g \rangle$ relative to $\mathscr{F}$ converges to f^* in the p-norm; it then follows that f^* is the unique maximizer for $\mathscr{F}$ on $\overline{\text{conv}\,\mathscr{F}}$. If $\bar{\mathscr{F}}$ denotes the closure of $\mathscr{F}$ in the L^q-topology on $L^p(\mu)$, we show that $\bar{\mathscr{F}}$ is convex.

As applications of Theorems A and B, we consider in Sect. 4 two variational problems for vortex rings in an ideal fluid, arising from work of Benjamin [5]. Let W be a bounded axisymmetric domain in $\mathbb{R}^3$ having C^2 boundary, let r, θ, z be cylindrical coordinates with the same axis as W, let Π be the half-plane defined by $\theta = 0$, and let $\Omega = \Pi \cap W$. Fix $1 < p < \infty$, let $p^{-1} + q^{-1} = 1$, let v be the measure on Ω having density $2\pi r$ with respect to Lebesgue measure, and let $\mathscr{L}$ be the linear differential operator

$$\mathscr{L}u = -\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2 u}{\partial z^2}.$$

There is a strictly positive, symmetric, compact linear operator $K: L^p(\Omega, v) \to L^q(\Omega, v)$ satisfying

$$\mathscr{L}Kv = v \quad \text{in } \Omega$$
$$Kv = 0 \quad \text{on } \partial\Omega$$

for $v \in L^p(\Omega, v)$. We fix a non-negative $f_0 \in L^p(\Omega, v)$, we let $\mathscr{F}$ be the set of rearrangements of f_0 on Ω with respect to v, and study the problems

(V1) $$\underset{f \in \mathscr{F}}{\text{maximize}}\ \tfrac{1}{2}\int_\Omega fKfdv - \tfrac{1}{2}\lambda \int_\Omega r^2 fdv$$

(V2) $$\underset{\substack{f \in \mathscr{F} \\ \int_\Omega r^2 fdv = I}}{\text{maximize}}\ \tfrac{1}{2}\int_\Omega fKfdv .$$

For any real λ, Theorem A guarantees the existence of a maximizer f^* for problem (V1), and $u = Kf^*$ satisfies

$$\mathscr{L}u = \varphi(u - \tfrac{1}{2}\lambda r^2) \tag{1}$$

for some increasing function φ. If I is positive and satisfies a certain feasibility condition, then Theorem B guarantees the existence of a maximizer f^* for problem (V2), and $u = Kf^*$ satisfies (1) for some real λ and some increasing function φ.

If u is one of the solutions constructed above, then (1) ensures that $u - \lambda r^2/2$ is the Stokes stream function for a steady axisymmetric flow of an ideal fluid in W. At the boundary of W, this stream function reduces to $-\lambda r^2/2$, which is the stream function for a uniform flow with velocity λ in the negative z-direction; the normal component of velocity therefore matches that due to the uniform flow. The region where $\mathscr{L}u > 0$ is called the *vortex core*; outside it the flow is irrotational.

Benjamin [5] was concerned with flows defined throughout $\mathbb{R}^3$, and posed problems (V1) and (V2), with different boundary conditions, in a rectangular domain Ω. He put forward a strategy for proving the existence of maximizers for (V1) and (V2), and then for constructing a solution defined throughout Π by taking Ω large. We prove our results by a method different from the one envisaged in [5]. It is possible to prove the existence of maximizers for (V1) and (V2) on bounded domains with Benjamin's boundary conditions using Theorems A and B, but we do not give the details in this paper.

Benjamin in fact included in additional constraint

$$\int_\Omega zfdv = 0 \tag{2}$$

in his variational problems, and showed by a symmetrization argument that if maximizers exist unconstrained by (2) then maximizers satisfying (2) must also exist. We make no use of any constraint analogous to (2).

In Sect. 5, we investigate the relationship between problems of the type considered in Theorem A, and certain dual variational problems formulated according to the theory of Toland [14]. In the case of problem (V1), the dual turns out to be a variational principle similar to problems studied by Mossino and Temam [13] in connection with "queer differential equations."

2. Measures, Rearrangements, and Linear Maximization

A *measure space* is defined to be a triple $(\Omega, \mathscr{M}, \mu)$ where Ω is a nonempty set, $\mathscr{M}$ is a σ-algebra on Ω and μ is a positive measure on Ω. That is, $\Omega \in \mathscr{M}$ and $\mathscr{M}$ is closed under complementation with respect to Ω and under countable unions, $0 \leq \mu(A) \leq \infty$ for each $A \in \mathscr{M}$ and μ is additive over countable disjoint subfamilies of $\mathscr{M}$. We say Ω is *finite* if $\mu(\Omega) < \infty$. If $(\Omega, \mathscr{M}, \mu)$ and $(\Omega', \mathscr{M}', \mu')$ are two measure spaces with $\mu(\Omega) = \mu'(\Omega')$ and f and g are real measurable functions on Ω and Ω' respectively, we say that f is a *rearrangement* of g if $\mu(f^{-1}[\beta, \infty)) = \mu'(g^{-1}[\beta, \infty))$ for all real β. Suppose further that Ω and Ω' are finite. Then $\mu(f^{-1}(A)) = \mu'(g^{-1}(A))$ for every Borel set $A \subset \mathbf{R}$. It follows that if $\psi: \mathbf{R} \to \mathbf{R}$ is a Borel measurable function then $\psi \circ f$ is a rearrangement of $\psi \circ g$. If additionally $\psi \circ f \in L^1(\mu)$ then

$$\int_\Omega \psi \circ f d\mu = \int_{\Omega'} \psi \circ g d\mu'$$

which may be proved by applying Fubini's theorem to the ordinate sets of $\psi \circ f$ and $\psi \circ g$. In particular, if $1 \leq p < \infty$ and $f \in L^p(\mu)$ then $g \in L^p(\mu')$ and $\|f\|_p = \|g\|_p$; the case $p = \infty$ follows easily from the definitions. When p and q are conjugate exponents, $g \in L^p(\mu)$ and $h \in L^q(\mu)$ we write

$$\langle g, h \rangle = \int_\Omega gh d\mu.$$

A standard result, presented in Hardy et al. [11] for example, shows that if f_0 and g_0 are non-negative Lebesgue measurable functions on $(0, \infty)$ and f_0^Δ, g_0^Δ denote the rearrangements of f_0 and g_0 as decreasing functions on $(0, \infty)$, then the inequality

$$\int_0^\infty fg \leq \int_0^\infty f_0^\Delta g_0^\Delta \tag{3}$$

holds for all rearrangements f, g of f_0, g_0 on $(0, \infty)$. An analogue of (3) for functions defined on a domain in $\mathbf{R}^N$ can be found in, for example, Bandle [4] and Mossino [12]. Our first theorem is an analogue of (3) for functions on measure spaces. The methods employed in this section are elementary.

Lemma 1. *Let $(\Omega, \mathscr{M}, \mu)$ be a finite measure space and $f: \Omega \to \mathbf{R}$ a measurable function. For real α define*

$$F(\alpha) = \{x \in \Omega \mid f(x) \geq \alpha\},$$
$$\sigma(\alpha) = \mu(F(\alpha)).$$

Define

$$f^\Delta(s) = \max\{\alpha \mid \sigma(\alpha) \geq s\}$$

for $0 < s < \mu(\Omega)$. Then

$$f^\Delta(s) \geq \alpha \Leftrightarrow \sigma(\alpha) \geq s$$

for real α and $0 < s < \mu(\Omega)$. Further f^Δ is the essentially unique decreasing rearrangement of f on $(0, \mu(\Omega))$.

Proof. The function σ is decreasing. If α is real, $0 < s < \mu(\Omega)$ and $\{\alpha_n\}_{n=1}^\infty$ is an increasing sequence converging to α with $\sigma(\alpha_n) \geq s$ for each n, then

$$\sigma(\alpha) = \mu(F(\alpha)) = \mu\left(\bigcap_{n=1}^\infty F(\alpha_n)\right) = \lim_{n \to \infty} \mu(F(\alpha_n)) \geq s$$

so the maximum in the definition of f^Δ is indeed attained. Clearly f^Δ is a decreasing function.

For real α and $0 < s < \mu(\Omega)$ we have

$$\begin{aligned} f^\Delta(s) \geq \alpha &\Leftrightarrow \max\{\beta \mid \sigma(\beta) \geq s\} \geq \alpha \\ &\Leftrightarrow \sigma(\beta) \geq s \quad \text{for some} \quad \beta \geq \alpha \\ &\Leftrightarrow \sigma(\alpha) \geq s. \end{aligned}$$

Thus

$$\{s \in (0, \mu(\Omega)) \mid f^\Delta(s) \geq \alpha\} = (0, \sigma(\alpha)] \backslash \{\mu(\Omega)\}$$

which has measure $\sigma(\alpha) = \mu(F(\alpha))$. Therefore f^Δ is a rearrangement of f.

Finally note that two decreasing rearrangements of f can differ only at points of discontinuity, which are countable.

Lemma 2. *Let $(\Omega, \mathscr{M}, \mu)$ be a finite measure space, let $1 \leq q \leq \infty$, let $g \in L^q(\mu)$, let $U \in \mathscr{M}$ and let $\alpha = \mu(U)$. Then*

$$\int_U g d\mu \leq \int_0^\alpha g^\Delta.$$

Proof. Let m denote one-dimensional Lebesgue measure, and let g_1 and g_2 be the positive and negative parts of g, so $g = g_1 - g_2$ and $g^\Delta = g_1^\Delta + (-g_2)^\Delta$. Write

$$\omega = \mu(\Omega)$$
$$G_1 = \{(x, s) \in \Omega \times \mathbf{R} \mid 0 \leq s \leq g_1(x)\}$$
$$F_1(s) = \{x \in \Omega \mid g_1(x) \geq s\}.$$

Then by Fubini's theorem we have

$$\begin{aligned} \int_U g_1 d\mu &= \int_U \int_0^\infty 1_{G_1}(x, s) ds d\mu(x) = \int_0^\infty \int_U 1_{G_1}(x, s) d\mu(x) ds \\ &= \int_0^\infty \mu(F_1(s) \cap U) ds \leq \int_0^\infty m(\{t \in (0, \alpha) \mid g_1^\Delta(t) \geq s\}) ds \\ &= \int_0^\alpha g_1^\Delta. \end{aligned}$$

The same argument applies with U, g_1 and α replaced by $\Omega \backslash U, g_2$ and $\omega - \alpha$. We obtain

$$-\int_U g_2 = -\int_\Omega g_2 + \int_{\Omega \backslash U} g_2 \leq -\int_0^\omega g_2^\Delta + \int_0^{\omega - \alpha} g_2^\Delta = -\int_{\omega - \alpha}^\omega g_2^\Delta = \int_0^\alpha (-g_2)^\Delta.$$

Hence the result.

Theorem 1. *Let $(\Omega, \mathscr{M}, \mu)$ be a finite measure space, let $1 \leq p \leq \infty$, let q be the conjugate exponent of p, let $\omega = \mu(\Omega)$, let $f_0 \in L^p(\mu)$ and let $g_0 \in L^q(\mu)$. Then for all rearrangements f of f_0 and g of g_0 on Ω we have*

$$\int_\Omega fg d\mu \leq \int_0^\omega f_0^\Delta g_0^\Delta.$$

Proof. Suppose first that f_0 is bounded below. We may assume without loss of generality that $f_0 \geqq 0$. Write

$$G = \{(x,t) \in \Omega \times \mathbf{R} \mid 0 \leqq t \leqq f(x)\}$$

$$F(s) = \{x \in \Omega \mid f(x) \geqq s\}$$

$$\sigma(s) = \mu(F(s))$$

for real s. Then by Fubini's theorem and Lemma 2

$$\int_\Omega fg\,d\mu = \int_\Omega \int_0^\infty 1_G(x,s)g(x)\,ds\,d\mu(x) = \int_0^\infty \int_\Omega 1_G(x,s)g(x)\,d\mu(x)\,ds$$

$$= \int_0^\infty \int_{F(s)} g\,d\mu\,ds \leqq \int_0^\infty \int_0^{\sigma(s)} g_0^\Delta(t)\,dt\,ds = \int_0^\omega f_0^\Delta g_0^\Delta .$$

For a general f_0, write $f_s(x) = \max\{f(x), s\}$ for $x \in \Omega$ and $s < 0$. Then

$$\int_\Omega f_s g\,d\mu \leqq \int_0^\omega f_s^\Delta g_0^\Delta$$

and letting $s \to -\infty$ gives the result.

Lemma 3. *Let $(\Omega, \mathscr{M}, \mu)$ be a finite measure space, let $1 \leqq p \leqq \infty$, let q be the conjugate exponent of p, let $\omega = \mu(\Omega)$, let $f \in L^p(\mu)$ and let $g \in L^q(\mu)$. Suppose f has a rearrangement f^* that satisfies $f^* = \varphi \circ g$ almost everywhere, for some increasing function φ. Then*

$$\int_\Omega f^* g\,d\mu = \int_0^\omega f^\Delta g^\Delta .$$

Proof. We have noted above that $\varphi \circ g^\Delta$ is a rearrangement of $\varphi \circ g$. Thus $f^{*\Delta} = \varphi \circ g^\Delta$. Writing $\psi(s) = s\varphi(s)$ for real s we therefore have

$$\int_\Omega f^* g\,d\mu = \int_\Omega \psi \circ g\,d\mu$$

$$\int_0^\omega f^\Delta g^\Delta = \int_0^\omega \psi \circ g^\Delta$$

and the two right-hand integrals are equal since $\psi \circ g^\Delta$ is a rearrangement of $\psi \circ g$.

Theorem 2. *Let $(\Omega, \mathscr{M}, \mu)$ be a finite measure space, let $1 \leqq p < \infty$, let q be the conjugate exponent of p, let $f_0 \in L^p(\mu)$, let $g \in L^q(\mu)$, and let $\mathscr{F}$ be the set of rearrangements of f_0 on Ω. Suppose there is an increasing function φ such that $f^* = \varphi \circ g \in \mathscr{F}$. If $\{f_n\}_{n=1}^\infty$ is a sequence in $\mathscr{F}$ such that*

$$\langle f_n, g\rangle \to \langle f^*, g\rangle$$

as $n \to \infty$ then $\|f_n - f^*\|_p \to 0$.

Proof. First consider the case when f_0 is bounded. Then we may suppose without loss of generality that $\operatorname{ess\,inf} f_0 = 0$ and $\operatorname{ess\,sup} f_0 = A$, say. Let $f \in \mathscr{F}$ and for real s define

$$F(s) = \{x \in \Omega \mid f(x) \geqq s\}$$

$$F^*(s) = \{x \in \Omega \mid f^*(x) \geqq s\}$$

Observe that $\beta(s)$ is finite for $0 < s < A$, and since φ is increasing, that

$$x \in F^*(s) \Rightarrow g(x) \in \varphi^{-1}[s,\infty) \Rightarrow g(x) \geqq \beta(s)$$

$$x \in \Omega \backslash F^*(s) \Rightarrow g(x) \notin \varphi^{-1}[s,\infty) \Rightarrow g(x) \leqq \beta(s) .$$

Since $\mu(F(s)) = \mu(F^*(s))$ for each s we have

$$\mu(F^*(s) \backslash F(s)) = \mu(F(s) \backslash F^*(s)) .$$

We now have

$$\langle f^* - f, g\rangle = \int_0^A \int_{F^*(s)} g\,d\mu\,ds - \int_0^A \int_{F(s)} g\,d\mu\,ds$$

$$= \int_0^A \left(\int_{F^*(s)\backslash F(s)} (g - \beta(s))\,d\mu + \int_{F(s)\backslash F^*(s)} (\beta(s) - g)\,d\mu \right) ds$$

and the inner integrals of the second line have non-negative integrands.

Fix $\varepsilon > 0$ and choose $M > 0$ such that

$$\mu(\{x \in \Omega \mid |g(x)| \geqq M\}) < \varepsilon .$$

Introduce the notation $I(t,\xi)$ for the open interval $(t - \xi, t + \xi)$. For each $t \in [-M, M]$ we may choose $\xi(t) > 0$ such that

$$\mu(\{x \in \Omega \mid 0 < |g(x) - t| < 2\xi(t)\}) < \varepsilon .$$

By compactness we can choose $t_1, \ldots, t_K \in [-M, M]$ such that

$$[-M, M] \subset \bigcup_{k=1}^K I(t_k, \xi(t_k)) .$$

Write

$$\xi = \min\{\xi(t_k) \mid 1 \leqq k \leqq K\} .$$

Consider $s \in (0, A)$. If $\beta(s) > M$ then $F^*(s) \subset g^{-1}(M, \infty)$ so $\mu(F^*(s)) < \varepsilon$, hence

$$\int_{F^*(s)\backslash F(s)} (g - \beta(s))\,d\mu \geqq 0 > (\mu(F^*(s) \backslash F(s)) - \varepsilon)\xi$$

whereas if $\beta(s) < -M$ then $\Omega \backslash F^*(s) \subset g^{-1}(-\infty, -M)$ so $\mu(\Omega \backslash F^*(s)) < \varepsilon$, hence

$$\int_{F(s)\backslash F^*(s)} (\beta(s) - g)\,d\mu \geqq 0 > (\mu(F(s) \backslash F^*(s)) - \varepsilon)\xi .$$

Now suppose $-M \leqq \beta(s) \leqq M$ so $\beta(s) \in I(t_k, \xi(t_k))$ for some k. Consider the possibility that $\varphi(t_k) \geqq s$, so $t_k \geqq \beta(s)$. For any $x \in F(s) \backslash F^*(s)$ we have $\varphi \circ g(x) < s$ so $g(x) \neq t_k$ and $g(x) \leqq \beta(s)$. Hence if $x \in F(s) \backslash F^*(s)$ and $g(x) > \beta(s) - \xi$ then $t_k - 2\xi(t_k) < g(x) < t_k$. Therefore

$$\mu(\{x \in F(s) \backslash F^*(s) \mid g(x) > \beta(s) - \xi\}) < \varepsilon$$

and consequently

$$\int_{F(s)\backslash F^*(s)} (\beta(s) - g)\,d\mu \geqq (\mu(F(s) \backslash F^*(s)) - \varepsilon)\xi .$$

Consider the other possibility that $\varphi(t_k)<s$ so $t_k \leqq \beta(s)$. For any $x \in F^*(s)\backslash F(s)$ we have $\varphi \circ g(x) \geqq s$ so $g(x) \neq t_k$ and $g(x) \geqq \beta(s)$. Hence if $x \in F^*(s)\backslash F(s)$ and $g(x) < \beta(s)+\xi$ then $t_k < g(x) < t_k + 2\xi(t_k)$. Therefore

$$\mu(\{x \in F^*(s)\backslash F(s) \mid g(x) < \beta(s)+\xi\}) < \varepsilon$$

and consequently

$$\int_{F^*(s)\backslash F(s)} (g-\beta(s))d\mu \geqq (\mu(F^*(s)\backslash F(s))-\varepsilon)\xi .$$

We now have

$$\langle f^*-f, g\rangle \geqq \xi \int_0^A (\mu(F^*(s)\backslash F(s))-\varepsilon)ds$$
$$=2^{-1}\xi\|f^*-f\|_1-\xi A\varepsilon .$$

If $\langle f^*-f, g\rangle < 2^{-1}\xi\varepsilon$ then

$$\|f^*-f\|_p \leqq \|f^*-f\|_1^{1/p}A^{1-1/p} \leqq \varepsilon^{1/p}(2A+1)^{1/p}A^{1-1/p} .$$

It follows that any maximizing sequence for $\langle\cdot, g\rangle$ relative to $\mathscr{I}$ must converge to f^* in the p-norm.

Now consider the general case when f_0 may be unbounded. By Theorem 1 and Lemma 3 we have

$$\langle f^*, g\rangle = \max_{f\in\mathscr{I}} \langle f, g\rangle .$$

Let $\{f_n\}_{n=1}^{\infty}$ be a maximizing sequence for $\langle\cdot, g\rangle$ relative to $\mathscr{I}$. Fix $N>0$ and for real s define

$$\gamma_1(s)=\min\{0, s+N\}$$
$$\gamma_3(s)=\max\{0, s-N\}$$
$$\gamma_2(s)=s-\gamma_1(s)-\gamma_3(s) .$$

Then γ_1, γ_2, and γ_3 are increasing functions. For $1 \leqq i \leqq 3$ notice that $\gamma_i \circ f_n$ and $\gamma_i \circ f^*$ are rearrangements of $\gamma_i \circ f_0$, that $\gamma_i \circ \varphi$ is increasing and that $\gamma_i \circ f^* = (\gamma_i \circ \varphi)\circ g$. We have

$$\langle f^*-f_n, g\rangle = \sum_{i=1}^{3} \langle \gamma_i \circ f^* - \gamma_i \circ f_n, g\rangle$$

and by Theorem 1 and Lemma 3

$$\langle \gamma_i \circ f^* - \gamma_i \circ f_n, g\rangle \geqq 0$$

for $1 \leqq i \leqq 3$. Since

$$\langle f^*-f_n, g\rangle \to 0$$

it follows that

$$\langle \gamma_i \circ f^* - \gamma_i \circ f_n, g\rangle \to 0$$

for $1 \leqq i \leqq 3$.

Let $\varepsilon>0$, and choose N so large that $\|\gamma_1 \circ f_0\|_p < \varepsilon$ and $\|\gamma_3 \circ f_0\|_p < \varepsilon$. Since $\gamma_2 \circ f_0$ is bounded we can apply the first part of the proof to show that

$$\|\gamma_2 \circ f^* - \gamma_2 \circ f_n\|_p \to 0$$

as $n \to \infty$. For all sufficiently large n we now have

$$\|f^*-f_n\|_p \leqq \|\gamma_1 \circ f^*\|_p + \|\gamma_1 \circ f_n\|_p + \|\gamma_2 \circ f^* - \gamma_2 \circ f_n\|_p + \|\gamma_3 \circ f^*\|_p + \|\gamma_3 \circ f_n\|_p < 5\varepsilon$$

hence the result.

Theorem 3. *Let $(\Omega, \mathscr{M}, \mu)$ be a finite measure space, let $1 \leqq p < \infty$, let q be the conjugate exponent of p, let $f_0 \in L^p(\mu)$, let $g \in L^q(\mu)$, and let $\mathscr{I}$ be the set of rearrangements of f_0 on Ω. Suppose there is an increasing function φ such that $f^* = \varphi \circ g \in \mathscr{I}$. Then f^* is the unique maximizer of the functional $\langle\cdot, g\rangle$ relative to $\overline{\mathrm{conv}}\,\mathscr{I}$.*

Proof. It follows from Theorem 1 and Lemma 3 that f^* maximizes the functional $\langle\cdot, g\rangle$ relative to $\overline{\mathrm{conv}}\,\mathscr{I}$. For $0 \leqq s \leqq \infty$ define

$$\psi(s)=\inf\{\langle f^*-f, g\rangle \mid f \in \mathscr{I} \text{ and } \|f^*-f\|_p \geqq s\} .$$

Then ψ is increasing, $\psi(0)=0$, by Theorem 2 we have $\psi(s)>0$ for $s>0$, and we have $\psi(s)=\infty$ for all sufficiently large s, say $s \geqq a$. For all $f \in \mathscr{I}$ we have

$$\psi(\|f^*-f\|_p) \leqq \langle f^*-f, g\rangle .$$

Let $\bar{\psi}$ be the lower semicontinuous convex lower envelope of ψ. Then $\bar{\psi}(0)=0$ and we show that $\bar{\psi}(s)>0$ for $s>0$. Choose σ with $0<\sigma<s$ and $\psi(\sigma)<\infty$ and define the continuous convex function θ by

$$\theta(t)=\begin{cases} 0 & \text{for } 0 \leqq t < \sigma \\ \psi(\sigma)(t-\sigma)/(a-\sigma) & \text{for } t \geqq \sigma . \end{cases}$$

Then $\psi \geqq \theta$ so $\bar{\psi} \geqq \theta$, and $\theta(s)>0$. Therefore $\bar{\psi}(s)>0$. Further $\bar{\psi}$ is increasing. We have

$$\bar{\psi}(\|f^*-f\|_p)-\langle f^*-f, g\rangle \leqq 0 \tag{4}$$

for all $f \in \mathscr{I}$. The left-hand side of (4) is a lower semicontinuous convex functional of f. Hence the set of $f \in L^p(\mu)$ for which (4) holds is a closed convex set containing $\mathscr{I}$. Therefore (4) holds for all $f \in \overline{\mathrm{conv}}\,\mathscr{I}$. Suppose $f \in \overline{\mathrm{conv}}\,\mathscr{I}$ and $\langle f, g\rangle = \langle f^*, g\rangle$. Then $\bar{\psi}(\|f^*-f\|_p)=0$ by (4) so $\|f^*-f\|_p=0$. □

There exist examples of finite measure spaces $(\Omega, \mathscr{M}, \mu)$ with $f_0 \in L^p(\mu)$ and $g \in L^q(\mu)$ such that the supremum of $\langle f, g\rangle$ as f varies over the rearrangements of f_0 is strictly less than

$$\int_0^{\mu(\Omega)} f_0^\Delta g^\Delta .$$

Accordingly we make further assumptions on our measure spaces.

The measure space $(\Omega, \mathscr{M}, \mu)$ is called *nonatomic* if for every $U \in \mathscr{M}$ with $\mu(U)>0$ there exists $V \in \mathscr{M}$ with $V \subset U$ and $0<\mu(V)<\mu(U)$. The measure space $(\Omega, \mathscr{M}, \mu)$ is called *separable* if there is a sequence $\{U_n\}_{n=1}^{\infty}$ of measurable sets such that for every $V \in \mathcal{M}$ and $\varepsilon > 0$ there exists n such that

$$\mu(V \setminus U_n) + \mu(U_n \setminus V) < \varepsilon .$$

It is a standard result that any finite separable nonatomic measure space is isomorphic to an interval of $\mathbf{R}$, in a sense we now describe. Let $(\Omega, \mathcal{M}, \mu)$ and $(\Omega', \mathcal{M}', \mu')$ be two measure spaces, and let their respective families of null sets be $\mathcal{N}$ and $\mathcal{N}'$. Regard two members of $\mathcal{M}$ as equivalent if their symmetric difference lies in $\mathcal{N}$, and write $\mathcal{M}/\mathcal{N}$ for the space of equivalence classes. Similarly define $\mathcal{M}'/\mathcal{N}'$. An *isomorphism* from $(\Omega, \mathcal{M}, \mu)$ to $(\Omega', \mathcal{M}', \mu')$ is a bijection $\Phi : \mathcal{M}/\mathcal{N} \to \mathcal{M}'/\mathcal{N}'$ having the properties

$$\Phi(U \setminus V) = \Phi(U) \setminus \Phi(V)$$

$$\Phi\left(\bigcup_{n=1}^{\infty} V_n\right) = \bigcup_{n=1}^{\infty} \Phi(V_n)$$

$$\mu'(\Phi(U)) = \mu(U)$$

for all $U, V, V_n \in \mathcal{M}$, where we have abused notation by failing to distinguish between measurable sets and their equivalence classes.

Isomorphism Theorem. *Let $(\Omega, \mathcal{M}, \mu)$ be a finite separable nonatomic measure space. Then $(\Omega, \mathcal{M}, \mu)$ is isomorphic to the interval $(0, \mu(\Omega))$ with Lebesgue measure.*

A proof of the Isomorphism Theorem may be found in Halmos [10]. Note that any Lebesgue measurable set in $\mathbf{R}^N$, with any measure that is absolutely continuous with respect to Lebesgue measure, is separable and nonatomic. Further, separability and nonatomicity are hereditary properties.

If $(\Omega, \mathcal{M}, \mu)$ is a measure space, the functions on Ω having the form

$$\sum_{n=1}^{\infty} \alpha_n 1_{A(n)},$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence of real numbers and $\{A(n)\}_{n=1}^{\infty}$ is a partition of Ω into sets from $\mathcal{M}$, are called *simple functions*.

Lemma 4. *Let $(\Omega, \mathcal{M}, \mu)$ and $(\Omega', \mathcal{M}', \mu')$ be isomorphic measure spaces and let Φ be an isomorphism from Ω to Ω'. Let $1 \leq p \leq \infty$, let q be the conjugate exponent of p, and let V and V' be the respective linear subspaces of $L^p(\mu)$ and $L^p(\mu')$ consisting of simple functions. Define $T_p : V \to V'$ by*

$$T_p\left(\sum_{n=1}^{\infty} \alpha_n 1_{A(n)}\right) = \sum_{n=1}^{\infty} \alpha_n 1_{\Phi(A(n))}.$$

Then T_p extends to a unique linear isometry of $L^p(\mu)$ onto $L^p(\mu')$, satisfying:

(i) *If f and $g \in L^p(\mu)$ with $f \leq g$ almost everywhere then $T_p f \leq T_p g$ almost everywhere.*

(ii) *If $f \in L^p(\mu)$ and $A \subset \mathbf{R}$ is a Borel set then*

$$(T_p f)^{-1}(A) = \Phi(f^{-1}(A))$$

apart from a null set.

(iii) *If $f \in L^p(\mu)$ and $g \in L^q(\mu)$ then*

$$\int_{\Omega'} (T_p f)(T_q g) d\mu' = \int_{\Omega} f g d\mu .$$

(iv) *T_p is a homeomorphism of $L^p(\mu)$ and $L^p(\mu')$ with their L^q-topologies.*

Proof. It is routine to verify that T_p is a well-defined linear isometry of V onto V'. The existence and uniqueness of the required extension follow by density. It is readily verified that (i) holds for simple functions, and the general case follows easily by taking limits.

To prove (ii) first consider the case when $A = [\alpha, \infty)$ for some real α. Let $\{f_n\}_{n=1}^{\infty}$ be a decreasing sequence in V converging in $L^p(\mu)$ to f. From the definition of T_p for simple functions it is immediate that

$$(T_p f_n)^{-1}(A) = \Phi(f_n^{-1}(A)).$$

Since $f_n \downarrow f$ and $T_p f_n \downarrow T_p f$ almost everywhere we have

$$f^{-1}(A) = \bigcap_{n=1}^{\infty} f_n^{-1}(A)$$

$$(T_p f)^{-1}(A) = \bigcap_{n=1}^{\infty} (T_p f_n)^{-1}(A)$$

apart from null sets. We now have

$$(T_p f)^{-1}(A) = \Phi(f^{-1}(A)) \tag{5}$$

when $A = [\alpha, \infty)$. But the family of sets A for which (5) holds is closed under complementation and under countable unions. Therefore (5) holds whenever A is a Borel set.

It is easy to prove (iii) in the case when Ω has a countable partition into measurable sets, on each of which f and g are constant. The general case follows by approximating f and g.

When $1 \leq p < \infty$ the fact that T_p and T_p^{-1} are bounded operators gives (iv). When $p = \infty$ we have the formulae $T_\infty = (T_1^{-1})^*$ and $T_\infty^{-1} = T_1^*$ for the adjoints, so T_∞ and T_∞^{-1} are continuous in the weak* topologies.

Theorem 4. *Let $(\Omega, \mathcal{M}, \mu)$ be a finite separable nonatomic measure space, let $1 \leq p \leq \infty$, let q be the conjugate exponent of p, let $f \in L^p(\mu)$ and $g \in L^q(\mu)$. Then there is a rearrangement f^* of f on Ω that satisfies*

$$\int_{\Omega} f^* g d\mu = \int_0^{\mu(\Omega)} f^\Delta g^\Delta .$$

Proof. For real α define

$$\sigma(\alpha) = \mu(\{x \in \Omega \mid f(x) \geq \alpha\})$$

$$G_0(\alpha) = \{x \in \Omega \mid g(x) > \alpha\}$$

$$H(\alpha) = \{x \in \Omega \mid g(x) = \alpha\}$$

$$G(\alpha) = G_0(\alpha) \cup H(\alpha).$$

Let $\alpha_1, \alpha_2, \ldots$ be those numbers for which $H(\alpha_i)$ has positive measure. The Isomorphism Theorem shows there is an isomorphism Φ from $(\Omega, \mathcal{M}, \mu)$ to the interval $(0,\omega)$ with Lebesgue measure, where $\omega = \mu(\Omega)$; we can further assume that $\Phi(H(\alpha_i))$ is an interval for each i. In view of Lemma 4 it is now enough to consider the case when $\Omega = (0,\omega)$, μ is Lebesgue measure, and each $H(\alpha_i)$ is an interval

$$H(\alpha_i) = (\beta_i, \beta_i + \gamma_i).$$

For $0 < s < \omega$ define

$$\Gamma(s) = G(g^{\Delta}(s)) \quad \text{if} \quad g^{\Delta}(s) \neq \alpha_i \quad \text{for all } i,$$

$$\Gamma(s) = G_0(g^{\Delta}(s)) \cup (\beta_i, \beta_i + s - \mu(G_0(\alpha_i))) \quad \text{if} \quad g^{\Delta}(s) = \alpha_i.$$

We have

$$\mu(\Gamma(s)) = s,$$

if $0 < s \leqq t < \omega$ then

$$\Gamma(s) \subset \Gamma(t)$$

and almost every point of Ω belongs to some $\Gamma(s)$.

Write

$$k(x) = \inf\{t \in (0,\omega) \mid x \in \Gamma(t)\}$$

which is defined and satisfies $0 < k(x) < \omega$ for almost all $x \in \Omega$. If τ is real and $x \in \Gamma(\tau)$ then $k(x) \leqq \tau$. Conversely if $k(x) \leqq \tau$ then $x \in \Gamma(t)$ for all $t > \tau$. Thus

$$\Gamma(\tau) \subset \{x \mid k(x) \leqq \tau\} \subset \bigcap_{t > \tau} \Gamma(t).$$

Therefore $\{x \in \Omega \mid k(x) \leqq \tau\}$ has measure τ, and differs from $\Gamma(\tau)$ by a set of zero measure.

Define f^{Δ} as in Lemma 1, and let $f^* = f^{\Delta} \circ k$ which is defined almost everywhere in Ω. Neglecting a set of zero measure in Ω, for real α we have $f^*(x) \geqq \alpha$ if and only if $\sigma(\alpha) \geqq k(x)$, and the set of x for which this inequality holds has measure $\sigma(\alpha)$. Therefore f^* is a rearrangement of f.

For real α write

$$F(\alpha) = \{x \in \Omega \mid f^*(x) \geqq \alpha\}.$$

Then $F(\alpha)$ is essentially $\Gamma(\sigma(\alpha))$, and the decreasing rearrangement of g restricted to $F(\alpha)$ is therefore g^{Δ} restricted to $(0, \sigma(\alpha))$. If $M > 0$ then by Fubini's theorem we have

$$\int_{F(-M)} f^* g + M \int_{F(-M)} g = \int_{F(-M)} (f^* + M) g = \int_0^{\infty} \left(\int_{F(\alpha - M)} g \right) d\alpha = \int_0^{\infty} \left(\int_0^{\sigma(\alpha - M)} g^{\Delta} \right) d\alpha$$
$$= \int_0^{\sigma(-M)} (f^{\Delta} + M) g^{\Delta} = \int_0^{\sigma(-M)} f^{\Delta} g^{\Delta} + M \int_0^{\sigma(-M)} g^{\Delta}$$

and therefore

$$\int_{F(-M)} f^* g = \int_0^{\sigma(-M)} f^{\Delta} g^{\Delta};$$

letting $M \to \infty$ gives the result.

Theorem 5. *Let $(\Omega, \mathcal{M}, \mu)$ be a finite separable nonatomic measure space, let $1 \leqq p \leqq \infty$, let q be the conjugate exponent of p, let $f_0 \in L^p(\mu)$ and $g \in L^q(\mu)$, and let $\mathcal{F}$ be the set of rearrangements of f_0 on Ω. If $\langle \cdot, g \rangle$ has a unique maximizer f^* relative to $\mathcal{F}$ then there is an increasing function φ such that $f^* = \varphi \circ g$ almost everywhere.*

Proof. For $\alpha \in \mathbb{R}$ let

$$G_0(\alpha) = \{x \in \Omega \mid g(x) > \alpha\}$$
$$G(\alpha) = \{x \in \Omega \mid g(x) \geqq \alpha\}$$
$$H(\alpha) = \{x \in \Omega \mid g(x) = \alpha\}$$

and define

$$\varphi(\alpha) = \operatorname{ess\,inf} f^*(G(\alpha)).$$

Then $\varphi: \mathbb{R} \to \overline{\mathbb{R}}$ is an increasing function. We first show that $f^*(x) \leqq \varphi(\alpha)$ for almost all $x \in \Omega \backslash G_0(\alpha)$. Suppose this is false, so for some set $B' \subset \Omega \backslash G_0(\alpha)$ having positive measure and some real β we have $f^*(x) > \beta > \varphi(\alpha)$ for all $x \in B'$. By the definition of $\varphi(\alpha)$ we can choose a set $C' \subset G(\alpha)$ having positive measure such that $\varphi(\alpha) \leqq f^*(x) < \beta$ for all $x \in C'$. Since B' and C' are isomorphic to intervals in $\mathbb{R}$ we can choose subsets $B \subset B'$ and $C \subset C'$ with $\mu(B) = \mu(C) > 0$. There is then an isomorphism Φ from B to C. Let $T_p: L^p(B) \to L^p(C)$ be the linear isometry induced by Φ as in Lemma 4. Clearly $B \cap C = \emptyset$. Define

$$Tf(x) = \begin{cases} T_p(f|B)(x) & \text{if} \quad x \in B \\ T_p^{-1}(f|C)(x) & \text{if} \quad x \in C \\ f(x) & \text{if} \quad x \in \Omega \backslash (B \cup C) \end{cases}$$

for $f \in L^p(\mu)$, which provides a linear isometry of $L^p(\mu)$ onto itself. Observe that Tf is a rearrangement of f by (ii) of Lemma 4. Now

$$\int_{\Omega} (Tf^* - f^*) g\, d\mu = \int_{B \cup C} (Tf^* - f^*)(g - \alpha)\, d\mu$$

since

$$\int_B (Tf^* - f^*)\, d\mu = \int_C (f^* - Tf^*)\, d\mu.$$

We have

$$\int_B Tf^*(g-\alpha)\,d\mu \geqq \beta \int_B (g-\alpha)\,d\mu \quad \text{since} \quad Tf^* < \beta \quad \text{and} \quad g \leqq \alpha \quad \text{on } B,$$
$$\int_B f^*(g-\alpha)\,d\mu \leqq \beta \int_B (g-\alpha)\,d\mu \quad \text{since} \quad f^* > \beta \quad \text{and} \quad g \leqq \alpha \quad \text{on } B,$$
$$\int_C Tf^*(g-\alpha)\,d\mu \geqq \beta \int_C (g-\alpha)\,d\mu \quad \text{since} \quad Tf^* > \beta \quad \text{and} \quad g \geqq \alpha \quad \text{on } C,$$
$$\int_C f^*(g-\alpha)\,d\mu \leqq \beta \int_C (g-\alpha)\,d\mu \quad \text{since} \quad f^* < \beta \quad \text{and} \quad g \geqq \alpha \quad \text{on } C,$$

hence

$$\int_{\Omega} Tf^* g\, d\mu \geqq \int_{\Omega} f^* g\, d\mu.$$

Since $f^*|B > \beta$ and $f^*|C < \beta$ we have $Tf^* \neq f^*$ contradicting the uniqueness of the maximizer. Therefore $f^*(x) \leq \varphi(\alpha)$ for almost all $x \in \Omega \backslash G_0(\alpha)$. In particular we now have $f^*(x) = \varphi(\alpha)$ for almost all x in $H(\alpha)$.

Finally we show $f^*(x) = \varphi(g(x))$ for almost all $x \in \Omega$. Suppose this is false. Then we can choose $\beta \in \mathbf{R}$ and a set $S \subset \Omega$ of positive measure such that $f^*(x) > \beta > \varphi(g(x))$ for all $x \in S$ or $f^*(x) < \beta < \varphi(g(x))$ for all $x \in S$. If there is a real α for which $\mu(H(\alpha) \cap S) > 0$ then we have a contradiction since $f^*(x) = \varphi(g(x))$ for almost all $x \in H(\alpha)$. Hence $\mu(H(\alpha) \cap S) = 0$ for every α. Therefore we can choose α such that $S \cap G(\alpha)$ and $S \backslash G(\alpha)$ both have positive measure. Consider the case $f^*(x) > \beta > \varphi(g(x))$ for all $x \in S$. Since φ is increasing, taking $x \in S \cap G(\alpha)$ we have $\varphi(g(x)) \geq \varphi(\alpha)$. Therefore $\varphi(\alpha) < \beta$. But for almost all $x \in S \backslash G(\alpha)$ we have $f^*(x) \leq \varphi(\alpha)$ giving a contradiction. Now consider the case $f^*(x) < \beta < \varphi(g(x))$ for all $x \in S$. Taking $x \in S \backslash G(\alpha)$ we have $\varphi(g(x)) \leq \varphi(\alpha)$ since φ is increasing, hence $\varphi(\alpha) > \beta$. But for almost all $x \in S \cap G(\alpha)$ we have $f^*(x) \geq \varphi(\alpha)$. From this final contradiction we conclude that $f^*(x) = \varphi(g(x))$ for almost all $x \in \Omega$.

Theorem 6. *Let $(\Omega, \mathscr{M}, \mu)$ be a finite separable nonatomic measure space, let $1 \leq p \leq \infty$, let q be the conjugate exponent of p, let $f \in L^p(\mu)$, let $\mathscr{F}$ be the set of rearrangements of f on Ω and let $\bar{\mathscr{F}}$ denote the closure of $\mathscr{F}$ in the L^q-topology on $L^p(\mu)$. Then $\bar{\mathscr{F}}$ is convex.*

Proof. We first take any $f_1, f_2 \in \mathscr{F}$ and $0 < \lambda < 1$, and show that $\lambda f_1 + (1-\lambda) f_2 \in \bar{\mathscr{F}}$. Let $\varepsilon > 0$ and let $\{F(i) \mid i \in \mathbf{N}\}$ be a partition of Ω into measurable sets on each of which f_1 and f_2 have essential oscillation less than ε. Let α_i and β_i be the respective means of f_1 and f_2 on $F(i)$ and write

$$g_1 = \sum_{i=1}^{\infty} \alpha_i 1_{F(i)}$$

$$g_2 = \sum_{i=1}^{\infty} \beta_i 1_{F(i)}.$$

We show that $\lambda g_1 + (1-\lambda) g_2 \in \bar{\mathscr{F}}$ and then letting ε tend to 0 it follows that $\lambda f_1 + (1-\lambda) f_2 \in \bar{\mathscr{F}}$. There is an isomorphism Φ from $(\Omega, \mathscr{M}, \mu)$ to the interval $I = (0, \mu(\Omega))$. The linear isometry $T_p : L^p(\mu) \to L^p(I)$ provided by Lemma 4 is a homeomorphism of the L^q-topologies. We may therefore suppose that $\Omega = I$ and that μ is Lebesgue measure; further we may suppose $F(i)$ is an interval

$$F(i) = (x_i, x_i + h_i)$$

for each i.

Define functions φ_i and ψ_i on $\mathbf{R}$ satisfying

$$\begin{cases} \varphi_i \mid (0, h_i) \text{ is a rearrangement of } f_1 \mid F(i) \\ \varphi_i \mid (h_i, \lambda^{-1} h_i) = 0 \\ \varphi_i \text{ is } \lambda^{-1} h_i\text{-periodic} \end{cases}$$

$$\begin{cases} \psi_i \mid (0, \lambda(1-\lambda)^{-1} h_i) = 0 \\ \psi_i \mid (\lambda(1-\lambda)^{-1} h_i, (1-\lambda)^{-1} h_i) \text{ is a rearrangement of } f_2 \mid F(i) \\ \psi_i \text{ is } (1-\lambda)^{-1} h_i\text{-periodic}. \end{cases}$$

Write

$$\varphi^n(x) = \sum_{i=1}^{\infty} \varphi_i(n\lambda^{-1}(x - x_i)) 1_{F(i)}(x)$$

$$\psi^n(x) = \sum_{i=1}^{\infty} \psi_i(n(1-\lambda)^{-1}(x - x_i)) 1_{F(i)}(x)$$

for $x \in \Omega$ and n a positive integer. There are disjoint sets A^n and B^n in Ω with $\mu(A^n) = \lambda \mu(\Omega)$ and $\mu(B^n) = (1-\lambda)\mu(\Omega)$ such that φ^n and ψ^n vanish outside A^n and B^n respectively; further $\varphi^n \mid A^n$ and $\psi^n \mid B^n$ are rearrangements of $f(\lambda^{-1} x)$ and $f((1-\lambda)^{-1} x)$ respectively. Hence $\varphi^n + \psi^n \in \mathscr{F}$. As $n \to \infty$ we have $\varphi^n \mid F(i) \to \lambda \alpha_i$ and $\psi^n \mid F(i) \to (1-\lambda)\beta_i$ in the L^q-topology of $L^p(\mu)$. It now follows that $\varphi^n + \psi^n \to \lambda g_1 + (1-\lambda) g_2$ in the L^q-topology on $L^p(\mu)$; this uses the observations that when $1 \leq p < \infty$ the series defining φ^n and ψ^n converge in the p-norm uniformly over n, and that when $p = \infty$ the functions that vanish outside a finite union of $F(i)$'s are dense in $L^1(\mu)$. Thus $\lambda g_1 + (1-\lambda) g_2 \in \bar{\mathscr{F}}$, hence $\lambda f_1 + (1-\lambda) f_2 \in \bar{\mathscr{F}}$ for all f_1 and $f_2 \in \mathscr{F}$. It now follows that $\lambda f_1 + (1-\lambda) f_2 \in \bar{\mathscr{F}}$ for all f_1 and $f_2 \in \bar{\mathscr{F}}$.

Lemma 5. *Let $(\Omega, \mathscr{M}, \mu)$ be a finite measure space, let $1 \leq p \leq \infty$, let $f_0 \in L^p(\mu)$ be non-negative, and let $\mathscr{F}$ denote the set of rearrangements of f_0 on Ω. If $f^* \in \overline{\text{conv}}\,\mathscr{F}$ $(\subset L^p(\mu))$ then*

$$\mu(\{x \in \Omega \mid f^*(x) > 0\}) \geq \mu(\{x \in \Omega \mid f_0(x) > 0\}).$$

Proof. Let $f \in \text{conv}\,\mathscr{F}$, say

$$f = \sum_{n=1}^{N} \lambda_n f_n,$$

where $\lambda_n \geq 0$ and $f_n \in \mathscr{F}$ for $1 \leq n \leq N$ and $\lambda_1 + \ldots + \lambda_n = 1$. Fix $\alpha > 0$ and $0 < \beta < 1$ and write

$$S(n) = \{x \in \Omega \mid f_n(x) \geq \alpha\}$$

$$T = \{x \in \Omega \mid f(x) \geq \alpha\beta\}$$

for $0 \leq n \leq N$. We prove that

$$\mu(T) \geq (\mu(S(0)) - \beta\mu(\Omega))/(1-\beta).$$

We have

$$f \geq \alpha \sum_{n=1}^{N} \lambda_n 1_{S(n)}$$

and therefore

$$\left\{x \in \Omega \,\middle|\, \sum_{n=1}^{N} \lambda_n 1_{S(n)}(x) \geq \beta\right\} \subset T.$$

Hence

$$\int_{\Omega} \sum_{n=1}^{N} \lambda_n 1_{S(n)} d\mu \leq \mu(T) + \beta(\mu(\Omega) - \mu(T)).$$

We also have

$$\int_\Omega \sum_{n=1}^{N} \lambda_n 1_{S(n)} d\mu = \sum_{n=1}^{N} \lambda_n \mu(S(n)) = \mu(S(0))$$

so

$$\mu(S(0)) \leqq \mu(T) + \beta(\mu(\Omega) - \mu(T))$$

and therefore

$$\mu(T) \geqq (\mu(S(0)) - \beta\mu(\Omega))/(1-\beta).$$

This inequality also holds with f^* in place of f, by taking a limit. It now follows that

$$\mu(\{x \in \Omega \mid f^*(x) > 0\}) \geqq \mu(\{x \in \Omega \mid f^*(x) \geqq \alpha\beta\}) \geqq (\mu(S(0)) - \beta\mu(\Omega)) \mid (1-\beta).$$

Letting α and β tend to 0 we obtain the result.

Lemma 6. *Let $(\Omega, \mathcal{M}, \mu)$ be a finite separable measure space, let $1 \leqq p \leqq \infty$, let q be the conjugate exponent of p, let $f_0 \in L^p(\mu)$ and let $\mathcal{F}$ denote the set of rearrangements of f_0 on Ω. Let $\bar{\mathcal{F}}$ denote the closure of $\mathcal{F}$ in the L^q-topology on $L^p(\mu)$. Then $\bar{\mathcal{F}}$ is sequentially compact in the L^q-topology.*

Proof. For all $f \in \mathcal{F}$ we have $\|f\|_p = \|f_0\|_p$, hence $\bar{\mathcal{F}}$ is bounded. The result follows from this when $1 < p < \infty$. The separability of $(\Omega, \mathcal{M}, \mu)$ ensures that $L^1(\mu)$ is separable, so if $p = \infty$ then $\bar{\mathcal{F}}$ is a bounded weak* closed set in the dual of a separable Banach space, hence $\bar{\mathcal{F}}$ is weak* sequentially compact. Consider finally the case $p = 1$. We have

$$\lim_{M \to \infty} \int_{|f(x)| \geqq M} |f| d\mu = 0$$

uniformly over $f \in \mathcal{F}$. The weak sequential compactness of $\bar{\mathcal{F}}$ now follows from the Dunford-Pettis criterion for weak compactness in $L^1(\mu)$, see [6, p. 294].

3. Maximization of Convex Functionals

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space, let $1 \leqq p \leqq \infty$, let q be the conjugate exponent of p, and let $\Psi : L^p(\mu) \to \bar{\mathbb{R}}$ be a convex functional. If $u \in L^p(\mu)$ and $\Psi(u)$ is finite, the *subdifferential* $\partial\Psi(u)$ of Ψ at u is defined by

$$\partial\Psi(u) = \{w \in L^q(\mu) \mid \Psi(v) \geqq \Psi(u) + \langle v-u, w\rangle \ \forall v \in L^p(\mu)\}.$$

If $\partial\Psi(u) \neq \emptyset$ then Ψ is said to be *subdifferentiable* at u, and elements of $\partial\Psi(u)$ are called *subgradients* of Ψ at u. The following is a variant of a standard result:

Lemma 7. *Let $(\Omega, \mathcal{M}, \mu)$ be a separable measurable space, let $1 \leqq p \leqq \infty$, let q be the conjugate exponent of p, let $\Psi : L^p(\mu) \to \mathbb{R}$ be convex, and let $u \in L^p(\mu)$. If Ψ is sequentially continuous at u in the L^q-topology on $L^p(\mu)$ then Ψ is subdifferentiable at u.*

Proof. A fortiori Ψ is norm continuous at u, and an application of the Hahn-Banach theorem shows there exists a bounded linear functional f on $L^p(\mu)$ satisfying

$$\Psi(v) \geqq \Psi(u) + f(v-u) \tag{6}$$

for all $v \in L^p(\mu)$. In case $1 \leqq p < \infty$ this is sufficient to establish the result. In the case $p = \infty$ we must further that f is weak* continuous on $L^p(\mu)$. Inequality (6) together with the weak* sequential continuity of Ψ at u ensures that f is weak* sequentially continuous. The separability of $(\Omega, \mathcal{M}, \mu)$ ensures that $L^1(\mu)$ is separable, so the relative weak* topology on any bounded set in $L^\infty(\mu)$ is metrizable, hence f is continuous in the bounded L^1-topology on $L^\infty(\mu)$. It now follows that f is weak* continuous on $L^\infty(\mu)$ by Theorem 6 on p. 428 of [6]. Thus f can be represented by an L^1 function.

Theorem 7. *Let $(\Omega, \mathcal{M}, \mu)$ be a finite separable nonatomic measure space, let $1 \leqq p \leqq \infty$, let q be the conjugate exponent of p, let $\Psi : L^p(\mu) \to \mathbb{R}$ be convex, let $f_0 \in L^p(\mu)$ and let $\mathcal{F}$ denote the set of rearrangements of f_0 on Ω.*

(i) *Suppose that Ψ is sequentially continuous in the L^q-topology on $L^p(\mu)$. Then Ψ attains a maximum value relative to $\mathcal{F}$.*

(ii) *Suppose Ψ is strictly convex, that f^* is a maximizer for Ψ relative to $\mathcal{F}$ and that $g \in \partial\Psi(f^*)$ $(\subset L^q(\mu))$. Then $f^* = \varphi \circ g$ almost everywhere in Ω for some increasing function φ.*

Proof. To prove (i) let $\bar{\mathcal{F}}$ denote the L^q-closure of $\mathcal{F}$ in $L^p(\mu)$. Then $\bar{\mathcal{F}}$ is L^q-sequentially compact by Lemma 6. Hence Ψ has a maximizer f_1 relative to $\bar{\mathcal{F}}$. It follows from Lemma 7 that Ψ is subdifferentiable at f_1, so choose $h \in \partial\Psi(f_1) \subset L^q(\mu)$. By Theorem 4 there is a maximizer f^* for $\langle \cdot, h\rangle$ relative to $\mathcal{F}$. By L^q-continuity the supremum of $\langle \cdot, h\rangle$ on $\bar{\mathcal{F}}$ is equal to the supremum on $\mathcal{F}$. Hence

$$\langle f_1, h\rangle \leqq \langle f^*, h\rangle.$$

By subdifferentiability we now have

$$\Psi(f^*) \geqq \Psi(f_1) + \langle f^* - f_1, h\rangle \geqq \Psi(f_1)$$

so f^* maximizes Ψ on $\mathcal{F}$.

Now suppose the assumptions of (ii) apply. If $f \in \mathcal{F} \backslash \{f^*\}$ then by strict convexity we have

$$\Psi(f^*) \geqq \Psi(f) > \Psi(f^*) + \langle f - f^*, g\rangle$$

and therefore

$$\langle f, g\rangle < \langle f^*, g\rangle.$$

Thus f^* is the unique maximizer of $\langle \cdot, g\rangle$ on $\mathcal{F}$, so $f^* = \varphi \circ g$ almost everywhere, for some increasing function φ, by Theorem 5.

Corollary 1. *Let $(\Omega, \mathcal{M}, \mu)$ be a finite separable nonatomic measure space, let $1 \leqq p \leqq \infty$, let q be the conjugate exponent of p, let $f_0 \in L^p(\mu)$ and let $\mathcal{F}$ be the set of rearrangements of f_0 on Ω. Let Ψ be a strictly convex Gateaux differentiable* *functional on* $L^p(\mu)$ *and suppose* Ψ *is sequentially continuous in the* L^q*-topology on* $L^p(\mu)$. *Then* Ψ *attains a maximum value relative to* $\mathscr{F}$, *and if* f^* *is any maximizer then* $f^* = \varphi \circ \Psi'(f^*)$ *almost everywhere for some increasing function* φ.

Let $(\Omega, \mathscr{M}, \mu)$ be a measure space, $1 \leqq p \leqq \infty$ and q the conjugate exponent of p. A bounded linear operator $K: L^p(\mu) \to L^q(\mu)$ will be called *symmetric* if

$$\int_\Omega uKw\,d\mu = \int_\Omega wKu\,d\mu$$

for all u and $w \in L^p(\mu)$.

Corollary 2. *Let* $(\Omega, \mathscr{M}, \mu)$ *be a finite separable nonatomic measure space, let* $1 \leqq p < \infty$, *let* q *be the conjugate exponent of* p, *let* $f_0 \in L^p(\mu)$, *let* $v \in L^q(\mu)$ *and let* $K: L^p(\mu) \to L^q(\mu)$ *be a compact strictly positive symmetric linear operator. Define*

$$\Psi(f) = \tfrac{1}{2}\int_\Omega fKf\,d\mu - \int_\Omega vf\,d\mu$$

for $f \in L^p(\mu)$. *Then* Ψ *attains its supremum on the rearrangements of* f_0 *on* Ω, *and if* f^* *is a maximizer then* $f^* = \varphi \circ (Kf^* - v)$ *almost everywhere for some increasing function* φ.

Theorem 8. *Let* $(\Omega, \mathscr{M}, \mu)$ *be a finite separable nonatomic measure space, let* $1 \leqq p < \infty$, *let* q *be the conjugate exponent of* p, *let* $f_0 \in L^p(\mu)$ *and let* $\mathscr{F}$ *be the set of all rearrangements of* f_0 *on* Ω. *Suppose* $v \in L^q(\mu)$, *let* I *be real and suppose there exist* f_1 *and* $f_2 \in \mathscr{F}$ *satisfying*

$$\langle f_1, v\rangle < I < \langle f_2, v\rangle .$$

Let Ψ *be a weakly sequentially continuous convex functional on* $L^p(\mu)$, *such that for each* $f \in \overline{\mathrm{conv}}\,\mathscr{F}$, *each* $g \in \partial\Psi(f)$ *and each real* λ *such that* f *maximizes* $\langle \cdot, g - \lambda v\rangle$ *relative to* $\overline{\mathrm{conv}}\,\mathscr{F}$, *there is an increasing function* ψ *such that* $\psi \circ (g - \lambda v) \in \mathscr{F}$.

Then Ψ *attains a maximum value on*

$$\{f \in \mathscr{F} \mid \langle f, v\rangle = I\};$$

if f^* *is any maximizer and* $g \in \partial\Psi(f^*)$ *then*

$$f^* = \varphi \circ (g - \lambda v)$$

almost everywhere, for some real λ *and some increasing function* φ.

Proof. Write

$$\Lambda = \{u \in L^p(\mu) \mid \langle u, v\rangle = I\}.$$

We claim that $C = \overline{\mathrm{conv}}\,\mathscr{F}$ is weakly compact. When $1 < p < \infty$ it is sufficient to observe that C is closed, bounded, and convex. When $p = 1$, Lemma 6 shows that $\bar{\mathscr{F}}$, the weak closure of $\mathscr{F}$, is weakly compact, and $C = \bar{\mathscr{F}}$ by Theorem 6.

The weak continuity of Ψ now ensures that Ψ attains a maximum relative to $\Lambda \cap C$. Consider any maximizer f^*, and let $g \in \partial\Psi(f^*)$ which is nonempty by weak continuity. For $f \in \Lambda \cap C$ we have

$$\Psi(f^*) \geqq \Psi(f) \geqq \Psi(f^*) + \langle f - f^*, g\rangle$$

and hence

$$\langle f, g\rangle \leqq \langle f^*, g\rangle .$$

Thus f^* maximizes $\langle \cdot, g\rangle$ on $\Lambda \cap C$.

The existence of f_1 and f_2 ensures that f_0 and v are both nonconstant. Further g is not a multiple of v, for otherwise $\psi(0) \in \mathscr{F}$ for some increasing function ψ, which is impossible since f_0 is nonconstant. Let E be the 2-dimensional linear subspace of $L^q(\mu)$ generated by g and v, let $E^\perp \subset L^p(\mu)$ be the annihilator of E, let $J \subset L^p(\mu)$ be a 2-dimensional linear subspace complementary to $E^\perp$ and let $\pi: L^p(\mu) \to J$ be the linear projection map having $\ker \pi = E^\perp$.

Since f^* maximizes the functional $\Gamma = \langle \cdot, g\rangle$ on $\Lambda \cap C$, it follows that πf^* maximizes Γ over $\Lambda \cap \pi C$. The inequalities

$$\langle \pi f_1, v\rangle < \langle \pi f^*, v\rangle = I < \langle \pi f_2, v\rangle$$

ensure that Λ intersects the relative interior of πC, but that πC is not contained in Λ. A Hahn-Banach argument shows that $\Gamma \mid \Lambda \cap J$ has an affine extension Γ_0 to J such that πf^* maximizes Γ_0 on πC. We can represent Γ_0 by

$$\Gamma_0(f) = \langle f, g - \lambda v\rangle + \beta$$

for $f \in J$, where λ and β are some fixed real numbers. Now f^* maximizes $\langle \cdot, g - \lambda v\rangle$ on C. By hypothesis there now exists an increasing function φ such that $\varphi \circ (g - \lambda v) \in \mathscr{F}$. From Theorem 3 it follows that $\varphi \circ (g - \lambda v)$ is the unique maximizer of $\langle \cdot, g - \lambda v\rangle$ on C. Therefore $f^* = \varphi \circ (g - \lambda v)$ almost everywhere, and in particular $f^* \in \mathscr{F}$, so f^* maximizes Ψ on $\Lambda \cap \mathscr{F}$.

Theorem 9. *Let* Ω *be a domain in* $\mathbf{R}^N$, *let* μ *be a finite measure that is equivalent to Lebesgue measure on* Ω, *let* $1 \leqq p < \infty$, *let* q *be the conjugate exponent of* p, *let* $m \geqq 1$ *and let*

$$\mathscr{L} = \sum_{1 \leqq |\alpha| \leqq m} a_\alpha(x) D^\alpha$$

be an m-th order linear partial differential operator in Ω *having measurable coefficients and having no 0-th order term. Suppose* $K: L^p(\mu) \to L^q(\mu)$ *is a compact, symmetric, positive operator such that* $Ku \in W^m(\Omega)$ *and* $\mathscr{L}Ku = u$ *almost everywhere in* Ω *for all* $u \in L^p(\mu)$, *and suppose* $v \in L^q(\mu) \cap W^m(\Omega)$ *satisfies* $\mathscr{L}v = 0$ *almost everywhere in* Ω. *Let* $f_0 \in L^p(\mu)$ *be non-negative, let* $\mathscr{F}$ *be the set of rearrangements of* f_0 *with respect to* μ *on* Ω, *let* I *be real and suppose there are* f_1 *and* f_2 *in* $\mathscr{F}$ *satisfying*

$$\langle f_1, v\rangle < I < \langle f_2, v\rangle .$$

Then the functional

$$\Psi(f) = \tfrac{1}{2}\int_\Omega fKf\,d\mu$$

attains a maximum relative to the set

$$\{f \in \mathscr{F} \mid \langle f, v\rangle = I\}.$$

If f^* *is a maximizer and* $u = Kf^*$ *then* u *satisfies*

$$\mathscr{L}u = \varphi \circ (u - \lambda v)$$

almost everywhere in Ω, *for some increasing function* φ *and some real* λ.

Proof. In order to apply Theorem 8, it remains only to let $f \in \overline{\text{conv}}\,\mathscr{I}$, note that $\partial\Psi(f)=\{Kf\}$, let λ be a real number such that f maximizes $\langle\cdot, Kf-\lambda v\rangle$ on $\overline{\text{conv}}\,\mathscr{I}$, and show that $\varphi\circ(Kf-\lambda f)\in\mathscr{I}$ for some increasing function φ. The existence of f_1 and f_2 shows f is not zero. Write

$$w=Kf-\lambda v$$
$$S=\{x\in\Omega \mid f(x)>0\}$$
$$\gamma=\operatorname{ess\,inf} w(S).$$

We next show that $w(x)\leqq\gamma$ for almost all $x\in\Omega\backslash S$. Suppose this is false. Then we can choose β with $\gamma<\beta<\infty$ such that the sets

$$B'=\{x\in S \mid w(x)<\beta\}$$
$$C'=\{x\in\Omega\backslash S \mid w(x)>\beta\}$$

both have positive measure. Since B' and C' are isomorphic to real intervals, we can choose $B\subset B'$ and $C\subset C'$ such that

$$\mu(B)=\mu(C)>0.$$

Let Φ be an isomorphism from B to C, let $T_p: L^p(B)\to L^p(C)$ be the linear isometry induced by Φ as in Lemma 4 and let

$$Th(x)=\begin{cases} T_p(h\mid B)(x) & \text{for } x\in B\\ T_p^{-1}(h\mid C)(x) & \text{for } x\in C\\ h(x) & \text{for } x\in\Omega\backslash(B\cup C)\end{cases}$$

for $h\in L^p(\Omega)$, which defines a linear isometry of $L^p(\Omega)$ onto itself, such that Th is a rearrangement of h. Then

$$\int_\Omega (Tf)w>\int_\Omega fw.$$

Since $T(\mathscr{I})=\mathscr{I}$ it follows that $T(\overline{\text{conv}}\,\mathscr{I})=\overline{\text{conv}}\,\mathscr{I}$ so $Tf\in\overline{\text{conv}}\,\mathscr{I}$. But f maximizes $\langle\cdot, w\rangle$ over $\overline{\text{conv}}\,\mathscr{I}$. This contradiction shows that $w(x)\leqq\gamma$ for almost all $x\in\Omega\backslash S$.

Now consider $\alpha\in\mathbf{R}$ and let

$$H(\alpha)=\{x\in\Omega \mid w(x)=\alpha\}.$$

We show that $H(\alpha)\cap S$ has zero measure. Since $w\in W^m(\Omega)$ the partial derivatives of w of orders $1,\ldots,m$ vanish almost everywhere in $H(\alpha)$ by, for example, Lemma 7.7 of Gilbarg and Trudinger [9]. Thus $\mathscr{L}w=0$ almost everywhere in $H(\alpha)$. But $\mathscr{L}w=f>0$ almost everywhere in S. Therefore $\mu(H(\alpha)\cap S)=0$.

We now construct φ. Let $\omega=\mu(\Omega)$. For real α define

$$F(\alpha)=\{x\in\Omega \mid f_0(x)\geqq\alpha\}$$
$$G(\alpha)=\{x\in\Omega \mid w(x)\geqq\alpha\}$$
$$\sigma(\alpha)=\mu(F(\alpha))$$
$$\tau(\alpha)=\mu(G(\alpha)).$$

For $0\leqq s\leqq\omega$ define

$$f_0^D(s)=\begin{cases}\operatorname{ess\,sup} f_0 & \text{if } s=0\\ f_0^\Delta(s) & \text{if } 0<s<\omega\\ \operatorname{ess\,inf} f_0 & \text{if } s=\omega.\end{cases}$$

For real t define

$$\varphi(t)=\begin{cases}0 & \text{if } t\leqq\gamma\\ f_0^D(\tau(t)) & \text{if } t>\gamma.\end{cases}$$

Since f_0^D and τ are decreasing and $f_0^D\geqq0$ it follows that φ is increasing. We show that $f_3=\varphi\circ w$ is a rearrangement of f_0. If $\alpha\leqq\operatorname{ess\,inf} f_0$ then

$$\mu(\{x\in\Omega \mid f_3(x)\geqq\alpha\})=\omega=\sigma(\alpha)$$

whereas if $\alpha>\operatorname{ess\,sup} f_0$ then

$$\mu(\{x\in\Omega \mid f_3(x)\geqq\alpha\})=0=\sigma(\alpha),$$

so suppose $\operatorname{ess\,inf} f_0<\alpha\leqq\operatorname{ess\,sup} f_0$. Then $\sigma(\alpha)<\omega$. For all $x\in\Omega$,

$$\begin{aligned} f_3(x)\geqq\alpha &\Leftrightarrow f_0^D(\tau(w(x)))\geqq\alpha\\ &\Leftrightarrow \sigma(\alpha)\geqq\tau(w(x)); \end{aligned}$$

this follows from Lemma 1 if $0<\tau(w(x))<\omega$, and is immediate from the definition of f_0^D if $\tau(w(x))=0$ or ω. We have $\sigma(\alpha)\leqq\mu(S)$ by Lemma 5. Consider first the case $0<\sigma(\alpha)<\mu(S)$. Now $\mu(H(\beta))=0$ for each $\beta>\gamma$ so τ is continuous on (γ,∞); further $\tau(\beta)\to\mu(S)$ as $\beta\to\gamma+$ and $\tau(\beta)\to0$ as $\beta\to\infty$. We can therefore choose $\delta\in(\gamma,\infty)$ with $\tau(\delta)=\sigma(\alpha)$; take δ to be the least possible. Then

$$\begin{aligned}\tau(w(x))\leqq\sigma(\alpha) &\Leftrightarrow w(x)\geqq\delta\\ &\Leftrightarrow x\in G(\delta).\end{aligned}$$

Thus

$$\mu(\{x\in\Omega \mid f_3(x)\geqq\alpha\})=\mu(G(\delta))=\tau(\delta)=\sigma(\alpha).$$

Next consider the case $\sigma(\alpha)=\mu(S)$. If $x\in S\backslash H(\gamma)$ then $\tau(w(x))\leqq\mu(S)$ and since $\mu(S\cap H(\gamma))=0$ we then have

$$\mu(\{x\in\Omega \mid f_3(x)\geqq\alpha\})\geqq\mu(S).$$

Since the sets $G(\beta)$ are totally ordered by inclusion we have

$$\mu(\bigcup\{G(w(x)) \mid \tau(w(x))\leqq\mu(S)\})\leqq\mu(S)$$

and so

$$\mu(\{x\in\Omega \mid f_3(x)\geqq\alpha\})=\mu(S)=\sigma(\alpha)$$

in this case also. The last to consider is when $\sigma(\alpha)=0$. Then $f_0^D(s)\geqq\alpha$ only when $s=0$, so if $f_3(x)\geqq\alpha$ then $\tau(w(x))=0$ which occurs for almost no $x\in\Omega$. This completes the verification that $f_3=\varphi\circ w$ is a rearrangement of f_0.

It now follows from Theorem 8 that Ψ has a maximizer f^* relative to

$$\{f \in \mathscr{F} \mid \langle f, v \rangle = I\}$$

and that $f^* = \varphi \circ (Kf^* - \lambda v)$ almost everywhere for some increasing function φ and some real λ. Then $u = Kf^*$ satisfies

$$\mathscr{L}u = \varphi \circ (u - \lambda v)$$

almost everywhere in Ω.

4. Applications to Vortex Rings

We begin by defining the compact operator K needed to study the problems (V1) and (V2) introduced in Sect. 1. We write

$$\Pi = \{(r, z) \in \mathbf{R}^2 \mid r > 0\} \subset \mathbf{R}^3,$$

we define the measure ν on Π having density $2\pi r$ with respect to Lebesgue measure μ, we take $\Omega \subset \Pi$ to be a bounded domain having the cone property (defined in [1] for example) and let $\mathscr{L}$ be the operator

$$\mathscr{L}u = -\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2 u}{\partial z^2}.$$

Modifying the approach of Amick and Fraenkel [3], we define the Hilbert space H to be the completion of the (Schwartz) test functions on Ω with the scalar product

$$\langle u, v \rangle_H = \int_\Omega \frac{1}{r^2} \nabla u \cdot \nabla v d\nu.$$

Lemma 8. *Let $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$. Then for $v \in L^p(\Omega, \nu)$ there is a unique $Kv \in H$ that is a weak solution of $\mathscr{L}u = v$ in Ω. Further*

(i) *$K : L^p(\Omega, \nu) \to H$ is a bounded linear operator,*
(ii) *$K : L^p(\Omega, \nu) \to L^q(\Omega, \nu)$ is symmetric, strictly positive, and compact,*
(iii) *if $v \in L^p(\Omega, \nu)$ then $Kv \in W^2(\Omega)$.*

Proof. For $v \in L^p(\Omega, \nu)$ define

$$\Phi_v(u) = \tfrac{1}{2}\|u\|_H^2 - \int_\Omega uv d\nu$$

for $u \in H$. The embeddings $H \to W_0^{1,2}(\Omega) \to L^q(\Omega, \mu) \to L^q(\Omega, \nu)$ are bounded. It follows that Φ_v is continuous, coercive, and strictly convex on H. A semicontinuity argument now shows that Φ_v has a minimizer Kv on H, and Kv is the unique stationary point of Φ_v by strict convexity. A function $u \in H$ is a weak solution of $\mathscr{L}u = v$ if and only if

$$\langle u, w \rangle_H = \int_\Omega vw d\nu \quad \forall w \in H \tag{7}$$

which occurs if and only if u is a stationary point of Φ_v, hence the existence and uniqueness of the solution in H.

The linearity of K follows from the linearity of $\mathscr{L}$ and the uniqueness of solutions in H, and from (7) we have

$$\|Kv\|_H^2 \leq \|v\|_p \|Kv\|_q \leq \text{const} \|v\|_p \|Kv\|_H,$$

where the p- and q-norms are calculated with respect to the measure ν, and it follows that $K : L^p(\Omega, \nu) \to H$ is bounded. The compactness of $K : L^p(\Omega, \nu) \to L^q(\Omega, \nu)$ now follows from the compactness of $H \to L^q(\Omega, \nu)$.

If v and $v' \in L^p(\Omega, \nu)$ we may take $u = Kv$ and $w = Kv'$ in (7) to obtain

$$\langle Kv, Kv' \rangle_H = \int_\Omega v Kv' d\nu.$$

from which we deduce that K is symmetric; further taking $v = v' \neq o$ we have $Kv \neq o$ and hence

$$\int_\Omega v Kv d\nu > 0$$

so K is strictly positive.

For $v \in L^p(\Omega, \nu)$ it follows from Theorem 6.1 of Agmon [2] that $Kv \in W^{2,p}_{\text{loc}}(\Omega)$, hence (iii).

Lemma 9. *Suppose Ω is the intersection of Π with a 3-dimensional domain with C^2 boundary, cylindrically symmetric about the z-axis. Let $5 < p < \infty$ and let $v \in L^p(\Omega, \nu)$. Then $Kv \in C^{1,\alpha}(\bar{\Omega})$ for $0 < \alpha < 1 - 5/p$, and $Kv = 0$ on $\partial\Omega$. At any point of $\partial\Omega$ on the z-axis we have $Kv = O(r^2)$, $r^{-1}\partial Kv/\partial z \to 0$ and $r^{-1}\partial Kv/\partial r$ approaches a finite limit.*

Proof. We can identify Π with a half-plane in $\mathbf{R}^5$. Then Ω can be expressed as $\Omega = U \cap \Pi$ where U is a bounded domain in $\mathbf{R}^5$ having a C^2 boundary and having cylindrical symmetry; here and in what follows, cylindrical symmetry is about the z-axis, which forms the edge of Π. We shall use r to denote distances from the z-axis in $\mathbf{R}^5$ also. Modifying the approach of Amick and Fraenkel in Sect. 2 of [3] we define the Hilbert space E to be the completion of the (Schwartz) test functions on U with the scalar product

$$\langle u, v \rangle_E = \int_U \nabla u \cdot \nabla v d\lambda,$$

where the measure λ has density π^{-1} with respect to Lebesgue measure on $\mathbf{R}^5$. Thus E is a renorming of $W_0^{1,2}(U)$. Every function on Ω extends to a cylindrically symmetric function on U, undefined on the z-axis if this intersects U. With this convention, a formal calculation shows that if u and v are test functions on Ω then

$$\langle u, v \rangle_H = \langle r^{-2}u, r^{-2}v \rangle_E. \tag{8}$$

The transformation $u \mapsto r^{-2}u$ thus maps H isometrically to a subspace of E, and (8) holds for all u and v in H.

Suppose φ is a test function on U that vanishes near the z-axis. We can write $\varphi = \varphi(x, t)$ with $x \in \Omega$ and $t \in S$, where $S \subset \mathbf{R}^5$ is the set of unit vectors perpendicular to the z-axis. Then for any test function u on Ω we can calculate formally that

$$\langle u, r^2\varphi(\cdot, t) \rangle_H = \int_\Omega \nabla(r^{-2}u) \cdot \nabla\varphi(x, t) r^2 d\nu(x)$$

for each $t \in S$, where ∇ is the 5-dimensional gradient operator. Integrating over t with respect to the Lebesgue measure ω on S, suitably normalised, yields

$$\int_S \langle u, r^2\varphi(\cdot, t)\rangle_H d\omega(t) = \langle r^{-2}u, \varphi\rangle_E. \tag{9}$$

The integrand of the left-hand side is continuous over $(u, t) \in H \times S$, hence (9) holds for all $u \in H$.

Consider $v \in L^p(\Omega, \nu)$ and φ as above. Then

$$\int_S \int_\Omega v(x) r^2 \varphi(x, t) d\nu(x) d\omega(t) = \int_U v\varphi d\lambda.$$

From this equation and (9) we deduce that

$$\langle r^{-2}Kv, \varphi\rangle_E = \int_U v\varphi d\lambda. \tag{10}$$

The test functions on U that vanish near the z-axis are dense in E; this is proved by the device employed in Lemma 2.3 of [3] as follows. Let ψ be an increasing C^∞ function on $\mathbf{R}$ that satisfies $\psi(s) = 0$ when $s \leq 1$ and $\psi(s) = 1$ when $s \geq 2$. If φ is a test function on U then $\psi(nr)\varphi$ defines a test function vanishing near the z-axis, that converges in E to φ as $n \to \infty$. We deduce that (10) holds for all $\varphi \in E$. In particular $r^{-2}Kv$ is a weak solution of $-\Delta u = v$ in U.

For any y and w in E such that $\Delta y \in L^2(U)$ we have

$$\langle y, w\rangle_E = \int_U (-\Delta y) w d\lambda. \tag{11}$$

Let $v \in L^p(\Omega, \nu)$ and $u = r^{-2}Kv \in E$. Then from (10) and (11) we have

$$\int_U u(-\Delta w) d\lambda = \int_U vw d\lambda \tag{12}$$

if $w \in E$ and $\Delta w \in L^2(U)$. Since U has a C^2 boundary, E contains

$$C_0^2(\bar{U}) = \{w \in C^2(\bar{U}) \mid w = 0 \text{ on } \partial U\}.$$

Therefore (12) holds for all $w \in C_0^2(\bar{U})$. Thus u satisfies the hypotheses of Theorem 8.1 of Agmon [2], and it follows that $u \in W^{2,p}(U)$. Since $p > 5$ we now have $u \in C^{1,\alpha}(\bar{U})$ for $0 < \alpha < 1 - 5/p$, and since u is a limit in E of test functions it follows that $u = 0$ on ∂U. Hence $Kv \in C^{1,\alpha}(\bar{\Omega})$ for $0 < \alpha < 1 - 5/p$, and $Kv = 0$ on $\Pi \cap \partial\Omega$. Since u is bounded we have $Kv = O(r^2)$ as $r \to 0$ if $\partial\Omega$ intersects the z-axis, hence $Kv = 0$ on $\partial\Omega$. Since ∇u is bounded it follows that $r^{-1}\partial Kv/\partial z \to 0$ and $r^{-1}\partial Kv/\partial r$ has a limit at any point of $\partial\Omega$ on the z-axis.

The following result is an immediate consequence of Theorem 7, Theorem 9, and Lemma 8.

Theorem 10. *Let $1 < p < \infty$, let $\Omega \subset \Pi$ be a bounded domain having the cone property, let K be as in Lemma 8, let $f_0 \in L^p(\Omega, \nu)$ and let $\mathscr{F}$ be the set of rearrangements of f_0 on Ω with respect to ν.*

(i) *If λ is any real number then the functional*

$$\tfrac{1}{2}\int_\Omega fKf d\nu - \tfrac{1}{2}\lambda \int_\Omega r^2 f d\nu$$

attains a maximum over $f \in \mathscr{F}$. If f^* is any maximizer and $u = Kf^*$ then u satisfies

$$\mathscr{L}u = \varphi(u - \lambda r^2/2) \tag{13}$$

almost everywhere in Ω for some increasing function φ.

(ii) *Suppose $f_0 \geq 0$, suppose I is a positive number, and suppose there exist f_1 and f_2 in $\mathscr{F}$ satisfying*

$$\int_\Omega r^2 f_1 d\nu < I < \int_\Omega r^2 f_2 d\nu.$$

Then the functional

$$\tfrac{1}{2}\int_\Omega fKf d\nu$$

attains a maximum over the set

$$\left\{f \in \mathscr{F} \,\middle|\, \int_\Omega r^2 f d\nu = I\right\}.$$

If f^* is any maximizer and $u = Kf^*$ then u satisfies (13) almost everywhere in Ω for some increasing function φ and some real λ.

Let us make some remarks on the physical interpretation of the solutions u found in Theorem 10. The stream function $u - \lambda r^2/2$ gives rise to an axisymmetric velocity field $\mathbf{v}$ whose components in cylindrical coordinates r, θ, z are given by

$$\left(-\frac{1}{r}\frac{\partial u}{\partial z}, 0, \frac{1}{r}\frac{\partial u}{\partial r} - \lambda\right).$$

If the stronger assumptions on Ω of Lemma 9 are made, and $p > 5$, then the tangential derivative of u at $\partial\Omega$ vanishes, so the normal component of $\mathbf{v}$ at points of $\Pi \cap \partial\Omega$ matches that due to the uniform stream $(0, 0, -\lambda)$. At points of $\partial\Omega$ lying on the z-axis, $\mathbf{v}$ is axial. The vorticity $\mathbf{curl}\,\mathbf{v}$ has the form

$$\mathbf{curl}\,\mathbf{v} = (0, \omega, 0)$$

$$\omega = r\mathscr{L}u.$$

Benjamin's approach to vortex rings seeks a solution for which ω/r is a rearrangement of a prescribed function f_0, and for which a value is prescribed for either the speed λ of the uniform flow matched at the boundary, or the *impulse* which is given for a fluid of unit density by

$$I = \int_{\mathbf{R}^3} r\omega.$$

One of the virtues claimed by Benjamin for his approach is that I and the measures of the sets $\{\omega/r \geq \alpha\}$ are preserved in all axisymmetric (including unsteady) motions of an ideal fluid in $\mathbf{R}^3$, so the quantities I and f_0 are physically meaningful.

Although this was not assumed in Theorem 10(i), the assumption $f_0 \geq 0$ is appropriate to the physical problem in hand, and, at least for flows in the whole of $\mathbf{R}^3$, one would also expect $\lambda > 0$. Problems for vortex rings with our boundary conditions have been studied by Fraenkel and Berger [8] as approximations to flows defined throughout $\mathbb{R}^3$, and by Amick and Fraenkel [3] who proved a uniqueness theorem for the known explicit examples of flows in a ball with ω/r constant in the vortex core.

5. Duality Theory

Let $(\Omega, \mathcal{M}, \mu)$ be a finite separable nonatomic measure space, let $1 \leqq p \leqq \infty$, let q be the conjugate exponent of p, let $f_0 \in L^p(\mu)$, let $\mathcal{F}$ be the set of rearrangements of f_0 on Ω, let $w \in L^q(\mu)$ and let χ be the *characteristic function* of $\mathcal{F}$, that is

$$\chi(u) = \begin{cases} 0 & \text{if} \quad u \in \mathcal{F} \\ \infty & \text{if} \quad u \in L^p(\mu) \setminus \mathcal{F} . \end{cases}$$

Define

$$\chi_w(u) = \langle u, w \rangle + \chi(u) .$$

For any functional $\Psi : L^p(\mu) \to \overline{\mathbb{R}}$ the *conjugate convex functional* $\Psi^* : L^q(\mu) \to \overline{\mathbb{R}}$ of Ψ is defined by

$$\Psi^*(v) = \sup_{u \in L^p(\mu)} \langle u, v \rangle - \Psi(u)$$

for $v \in L^q(\mu)$; this definition as a supremum of linear functionals ensures that Ψ^* is convex and lower semicontinuous in the L^p-topology on $L^q(\mu)$.

If Ψ is finite-valued, the problem of maximizing $\Psi - \langle \cdot , w \rangle$ over $\mathcal{F}$ is the same as that of maximizing $\Psi - \chi_w$ over $L^p(\mu)$. In the duality theory of Toland [14] the dual functional of $\Psi - \chi_w$ is $\chi_w^* - \Psi^*$. By Theorems 1 and 4 we have the formula

$$\chi^*(v) = \int_0^\omega v^\Delta f_0^\Delta ,$$

where $\omega = \mu(\Omega)$, and therefore

$$\chi_w^*(v) = \chi^*(v - w) = \int_0^\omega (v-w)^\Delta f_0^\Delta$$

for all $v \in L^q(\mu)$.

Lemma 10. *With the above assumptions, for each $v \in L^q(\mu)$ the set $\mathcal{F} \cap \partial \chi_w^*(v)$ consists of all maximizers of $\langle \cdot , v - w \rangle$ relative to $\mathcal{F}$, and is therefore nonempty.*

Proof. For $f \in L^p(\mu)$ we have

$$f \in \partial \chi_w^*(v) \Leftrightarrow v \in \partial \chi_w^{**}(f)$$

by, for example, Proposition 5.2 on p. 22 of [7]. It follows from Proposition 4.1 on p. 18 of [7] that χ_w^{**} is the largest convex function that minorizes χ_w and is lower semicontinuous in the L^q-topology on $L^p(\mu)$. Therefore

$$\chi_w^{**}(f) = \begin{cases} 0 & \text{if} \quad f \in C \\ \infty & \text{if} \quad f \in L^p(\mu) \setminus C , \end{cases}$$

where C is the L^q-topology closed convex hull of $\mathcal{F}$. It follows that $v \in \partial \chi_w^{**}(f)$ if and only if f maximizes $\langle \cdot , v - w \rangle$ on C. Now

$$\sup_{f \in C} \langle f, v - w \rangle = \sup_{f \in \mathcal{F}} \langle f, v - w \rangle .$$

The result follows from this and Theorem 4.

Theorem 11. *Let $\Psi : L^p(\mu) \to \mathbb{R}$ be convex and everywhere subdifferentiable. Then under the above assumptions we have*

$$\sup_{v \in L^q(\mu)} \chi_w^*(v) - \Psi^*(v) = \sup_{u \in L^p(\mu)} \Psi(u) - \chi_w(u) .$$

If u is a maximizer of $\Psi - \chi_w$ and $v \in \partial \Psi(u)$ then $v \in \partial \chi_w(u)$ and v is a maximizer for $\chi_w^* - \Psi^*$. Conversely if v is a maximizer for $\chi_w^* - \Psi^*$ and $u \in \mathcal{F} \cap \partial \chi_w^*(v)$ then $u \in \partial \Psi^*(v)$ and u is a maximizer for $\Psi - \chi_w$.

Proof. The subdifferentiability of Ψ ensures that Ψ is lower semicontinuous in the L^q-topology on $L^p(\mu)$; the equality of $\sup \chi_w^* - \Psi^*$ and $\sup \Psi - \chi_w$ is now immediate from Theorem 2.2 of Toland [14]. Suppose $\Psi - \chi_w$ has a maximizer u and let $v \in \partial \Psi(u)$. Then v is a maximizer of $\chi_w^* - \Psi^*$ by Theorem 2.3 of [14]; during the proof of that result it is shown that $v \in \partial \chi_w(u)$.

Now suppose v is a maximizer of $\chi_w^* - \Psi^*$. Then by Lemma 10 there exists $u \in \mathcal{F} \cap \partial \chi_w^*(v)$, so we have

$$\chi_w(u) = \chi_w^{**}(u) = \langle u, v \rangle - \chi_w^*(v) .$$

From the definition of Ψ^* we have

$$\Psi(u) \geqq \langle u, v \rangle - \Psi^*(v)$$

so

$$\Psi(u) - \chi_w(u) \geqq \chi_w^*(v) - \Psi^*(v)$$

which shows that u is a maximizer for $\Psi - \chi_w$. The proof of Theorem 2.3 of [14] shows that $u \in \partial \Psi^*(v)$. □

We now apply the above duality theory to one of the variational problems for vortex rings in a bounded domain, discussed during Sect. 4. Let $1 < p < \infty$, let q be the conjugate exponent of p, and let Π, Ω, ν, $\mathcal{L}$, K, and H be defined as at the beginning of Sect. 4 and in Lemma 8. For $u \in L^p(\Omega, \nu)$ recall from the proof of Lemma 8 that Ku is the minimizer of the functional

$$\tfrac{1}{2} \|v\|_H^2 - \int_\Omega u v d\nu$$

over $v \in H$. Define

$$\Psi(u) = \tfrac{1}{2} \int_\Omega u K u d\nu \qquad \text{if} \quad u \in L^p(\Omega, \nu)$$

$$\Phi(v) = \begin{cases} \tfrac{1}{2} \int_\Omega \dfrac{1}{r^2} |\nabla v|^2 d\nu & \text{if} \quad v \in H \\ \infty & \text{if} \quad v \in L^q(\Omega, \nu) \setminus H . \end{cases}$$

Lemma 11. *In the usual duality between $L^p(\Omega, \nu)$ and $L^q(\Omega, \nu)$ we have $\Phi^* = \Psi$ and $\Psi^* = \Phi$.*

Proof. Let $u \in L^p(\Omega, \nu)$. It is immediate from the definition of Ku as a minimizer that writing $v = Ku$ we have

$$\begin{aligned}\Phi^*(u) &= \int_\Omega uv d\nu - \tfrac{1}{2}\int_\Omega \frac{1}{r^2}|\nabla v|^2 d\nu \\ &= \int_\Omega (uv - \tfrac{1}{2}v\mathscr{L}v) d\nu \\ &= \tfrac{1}{2}\int_\Omega uv d\nu = \Psi(u).\end{aligned}$$

Using the compactness of the embedding $H \to L^q(\Omega, \nu)$ it is easily shown that Φ is strongly lower semicontinuous, and since Φ is convex it follows that Φ is weakly lower semicontinuous, hence $\Phi^{**} = \Phi$. Since $\Psi^* = \Phi^{**}$ the proof is complete.

Fix a real number λ and $u_0 \in L^p(\Omega, \nu)$, and let $\mathscr{F}$ be the set of rearrangements of u_0 in $L^p(\Omega, \nu)$. Then using Theorem 11 and Lemma 11 we have

$$\sup_{u \in \mathscr{F}} \tfrac{1}{2}\int_\Omega uKu d\nu - \lambda \int_\Omega \tfrac{1}{2}r^2 u d\nu = \sup_{v \in H} \int_0^\omega u_0^\Delta(v - \tfrac{1}{2}\lambda r^2)^\Delta - \tfrac{1}{2}\int_\Omega \frac{1}{r^2}|\nabla v|^2 d\nu, \qquad (14)$$

where $\omega = \nu(\Omega)$. Maximizers u of the left hand side and v of the right-hand side occur in dual pairs connected by the relations

$$\left.\begin{array}{c} v = Ku \\ u \text{ maximizes } \int_\Omega (v - \tfrac{1}{2}\lambda r^2) u d\nu \text{ over } \mathscr{F}. \end{array}\right\}$$

The variational functional in the right-hand side of (14) is of the form arising in the study of "queer differential equations," and similar functionals have been studied by Mossino and Temam [13].

We have not found a duality theory appropriate to the variational problems of Theorem 9.

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