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Vortex rings in a cylinder and rearrangements

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Vortex Rings in a Cylinder and Rearrangements

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1. INTRODUCTION

In this paper we prove the existence of a weak solution to the boundary value problem for a steady vortex ring in an ideal fluid flowing along an infinite pipe of circular cross section. This result is proved by means of a variational principle proposed by Benjamin [3], involving the maximization of a convex functional over the set of rearrangements of a fixed function. If ω denotes the scalar field of vorticity strength, and r denotes the distance from the axis of the pipe, this approach yields a solution for which ω/r is a rearrangement of a prescribed non-negative function f_0 in L^p $(p > 5)$ having bounded support.

The vortex core of our solution, that is, the region where $\omega > 0$, is cylindrically symmetric, bounded and bounded away from both the axis and the boundary of the pipe. At infinity the fluid velocity approaches a uniform stream of speed λ relative to the vortex core, and for a given f_0 a solution exists for all sufficiently small λ, whereas for large λ the method fails. We have not been able to prove that the vortex core is connected, nor that the solution is unique.

In view of the cylindrical symmetry we work in a plane infinite strip representing the intersection of the inside of the pipe with a half-plane bounded by its axis. Thus we define

$$\Omega = \{(r, z) \in \mathbb{R}^2 \mid 0 < r < R\},$$

where $R > 0$ is fixed, and endow Ω with the measure ν having density $2\pi r$ with respect to plane Lebesgue measure. A differential operator $\mathscr{L}$ is defined in Ω by

$$\mathscr{L}u = -\frac{1}{r}\left(\frac{1}{r}u_r\right)_r - \frac{1}{r^2}u_{zz}.$$

THEOREM. *Let $p > 5$ and let $f_0 \in L^p(\Omega) \setminus \{0\}$ be non-negative and have bounded support. Then there is a $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$ there exists a positive function u on Ω satisfying*

(i) *$u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ and satisfying*

$$\mathscr{L}u = \varphi(u - \lambda r^2/2) \tag{1}$$

almost everywhere in Ω, for some increasing function φ.

(ii) *$u \in C^{1,\alpha}(\bar{\Omega})$ for all $0 < \alpha < 1 - 5/p$,*

(iii) *$f = \mathscr{L}u$ is a rearrangement of f_0 with respect to ν, and the support of f is bounded away from infinity, from $r = 0$ and from $r = R$,*

(iv) *u and f are symmetric decreasing in z,*

(v) (a) *$u(r, z) \to 0$ uniformly as $z \to \pm \infty$,*

(b) *$u(r, z) = 0$ when $r = 0$ and when $r = R$,*

(c) *$u(r, z) = O(r^2)$ uniformly,*

(d) *$r^{-1}u_z(r, z) \to 0$ as $r \to 0$,*

(e) *$r^{-1}\nabla u(r, z) \to 0$ as $z \to \pm \infty$,*

(f) *$u_z(r, z) = 0$ when $r = R$.*

We believe the restriction $\lambda < \lambda_0$ is not just a technical one, and given f_0, for all sufficiently large λ there is no function u satisfying $\mathscr{L}u = \varphi(u - \lambda r^2/2)$ for φ increasing and for which $\mathscr{L}u$ is a rearrangement of f_0. Solutions for arbitrary λ, and for which $\mathscr{L}u$ is a rearrangement of αf_0 where $\alpha > 0$ is a priori unknown, can of course be obtained by rescaling.

The function $u(r, z) - \lambda r^2/2$ represents the stream function for the flow and gives rise to a velocity field

$$\mathbf{v} = (-r^{-1}u_z, 0, r^{-1}u_r - \lambda)$$

in cylindrical coordinates r, θ, z. From (vd) the velocity is parallel to the z axis when $r = 0$, and from (ve) the velocity at infinity approaches a uniform flow of magnitude λ in the negative z direction. From (vf) the velocity is parallel to the boundary when $r = R$. The vorticity ω is given by

$$\operatorname{curl} \mathbf{v} = (0, \omega, 0)$$

and consequently $\omega = r\mathscr{L}u$. Hence ω/r is a rearrangement of f_0.

It should be noted that we do not establish any smoothness properties of the function φ in (1), so we cannot assert that $\mathbf{v}$ satisfies the Euler equations of hydrodynamics.

2. DESCRIPTION OF THE METHOD

We now describe Benjamin's variational principle for vortex rings, as it applies to the present problem. For $\xi > 0$ let

$$\Omega(\xi) = \{(r, z) \in \Omega \mid |z| < \xi\}.$$

Suppose $p > 5$ and $\xi_0 > 0$, and let $f_0 \in L^p(\Omega)$ be a nontrivial non-negative function vanishing outside $\Omega(\xi_0)$. Let $\mathscr{F}$ denote the set of all rearrangements of f_0 with respect to ν, that is, the set of all real ν-measurable functions f on Ω that satisfy

$$\nu(f^{-1}[\beta, \infty)) = \nu(f_0^{-1}[\beta, \infty))$$

for every real β. For $\xi \geq \xi_0$ let $\mathscr{F}(\xi)$ comprise those functions in $\mathscr{F}$ that vanish outside $\Omega(\xi)$. We will define an inverse K for $\mathscr{L}$ satisfying suitable boundary conditions, and when $\lambda > 0$, a variational functional will be defined by

$$\Phi_\lambda(v) = \tfrac{1}{2}\int_\Omega vKv \, d\nu - \tfrac{1}{2}\lambda \int_\Omega r^2 v \, d\nu$$

for $v \in L^p(\Omega)$.

The first step of the proof is to show for $\xi \geq \xi_0$ that Φ_λ attains a maximum value relative to $\mathscr{F}(\xi)$, and that if f^ξ is a maximizer and $u^\xi = Kf^\xi$ then u^ξ satisfies (1) in $\Omega(\xi)$, for some increasing function φ. This is a routine application of the following result, which is Theorem A of Burton [4]:

THEOREM A. *Let $(\Theta, \mathscr{M}, \mu)$ be a finite, separable, non-atomic, positive measure space, let $1 \leq p \leq \infty$, let q be the conjugate exponent of p, let $g_0 \in L^p(\Theta)$, and let $\mathscr{G}$ be the set of all rearrangements of g_0 relative to μ. Let Φ be a strictly convex real functional on $L^p(\Theta)$ that is weakly (weak* if $p = \infty$) sequentially continuous. Then Φ attains a maximum value relative to $\mathscr{G}$, and if g is a maximizer and $h \in \partial\Phi(g)$ $(\subset L^q(\Theta))$ then $g = \psi \circ h$ almost everywhere in Θ, for some increasing function ψ.*

The second step is to show that the Steiner symmetrization of f^ξ with respect to the line $z = 0$ is also a maximizer, so f^ξ can be assumed to be symmetrically decreasing in z, that is,

$$f^\xi(r, -z) = f^\xi(r, z)$$

$$f^\xi(r, z) \leq f^\xi(r, z')$$

The third step is to show that for some $\lambda_0>0$ and some $\zeta \geqslant \zeta_0$, if $0<\lambda<\lambda_0$ and $\zeta > \zeta_0$ then f^ζ vanishes outside $\Omega(\zeta)$. It is then deduced that u satisfies (1) throughout Ω.

In Section 5 we show this approach fails for large λ.

3. Inversion of $\mathcal{L}$, Steiner Symmetrization, and Estimates

3.1. *Definition of Operators K and $\mathcal{K}$*

In choosing the spaces appropriate to the study of $\mathcal{L}$ we have been guided by Amick and Fraenkel [2, Sect. 2.2]. Let U be the cylinder in $\mathbb{R}^5$ comprising all points whose distances from the z axis are less than R. We shall regard Ω as the intersection of U with a half-plane bounded by the z axis, and we shall use r to denote distances from the z axis. Cylindrical symmetry in $\mathbb{R}^5$ is understood to be relative to the z axis. Then functions defined almost everywhere on Ω can be identified with cylindrically symmetric functions defined almost everywhere on U. With this convention we formally have

$$\mathcal{L}(r^2u) = -\Delta_5 u$$

for functions on Ω, where Δ_5 is the 5-dimensional Laplacian.

Define H to be the completion of the (Schwartz) test functions on Ω with the scalar product

$$\langle u, v\rangle_H = \int_\Omega r^{-2}\,\nabla u\cdot\nabla v\,dv.$$

Since Ω is a strip it follows that H is embedded in $W_0^{1,2}(\Omega)$; let us emphasise that v is the measure used to define $L^2(\Omega)$, but Lebesgue measure is used to define Sobolev spaces. For each $v\in L^2(\Omega)$ there is a unique element Kv of H that is a weak solution of the equation $\mathcal{L}u=v$, in the sense that

$$\int_\Omega u\,\mathcal{L}\varphi\,dv = \int_\Omega v\varphi\,dv$$

for every test function φ on Ω. Agmon [1, Theorem 6.1] shows that $u\in W^{2,2}_{loc}(\Omega)$ so $\mathcal{L}u=v$ almost everywhere in Ω. We can characterize Kv as the unique minimizer of the convex functional defined by

$$\Psi_H^v(u) = \tfrac{1}{2}\,\|u\|_H^2 - \int_\Omega uv\,dv$$

for $u\in H$. Then $K: L^2(\Omega)\to H$ is a bounded linear operator and $K: L^2(\Omega)\to L^2(\Omega)$ is self-adjoint and strictly positive.

Define E to be the completion of the test functions on U with the scalar product

$$\langle u, v\rangle_E = \int_U \nabla u\cdot\nabla v\,d\mu,$$

where the measure μ has density $1/\pi$ with respect to 5-dimensional Lebesgue measure. Since U is a cylinder it follows that E is a renorming of $W_0^{1,2}(U)$. For each $w\in L^2(U)$ there is a unique element $\mathcal{K}w$ of E that is a weak solution of the equation $-\Delta_5 u = w$. We can characterize $\mathcal{K}w$ as the unique minimizer of the convex functional defined by

$$\Psi_E^w(u) = \tfrac{1}{2}\,\|u\|_E^2 - \int_U uw\,d\mu$$

for $u\in E$. Then $\mathcal{K}: L^2(U)\to E$ is a bounded linear operator, and $\mathcal{K}: L^2(U)\to L^2(U)$ is self-adjoint and strictly positive.

Lemma 1. *Let $v\in L^2(\Omega)$. Then $\mathcal{K}v = r^{-2}Kv$.*

Proof. The argument has been given on bounded domains in proving [4, Lemma 9], but we repeat it here for completeness. A direct calculation shows that

$$\langle u, w\rangle_H = \langle r^{-2}u, r^{-2}w\rangle_E \tag{2}$$

if u and w are test functions on Ω. It now follows that if u and w belong to H then $r^{-2}u$ and $r^{-2}w$ belong to E, and (2) holds.

Suppose φ is a test function on U that vanishes near the z axis. We can write $\varphi=\varphi(x,t)$ with $x\in\Omega$ and $t\in S$, where S is the set of unit vectors in $\mathbb{R}^5$ perpendicular to the z axis. If $u\in H$ then (2) yields

$$\langle u, r^2\varphi(\cdot,t)\rangle_H = \int_\Omega \nabla(r^{-2}u)\cdot\nabla\varphi(x,t)\,r^2\,dv(x)$$

for each $t\in S$, where ∇ is the 5-dimensional gradient operator. Integrating over t, with respect to the suitably normalized Lebesgue measure σ on S, yields

$$\int_S \langle u, r^2\varphi(\cdot,t)\rangle_H\,d\sigma(t) = \langle r^{-2}u, \varphi\rangle_E. \tag{3}$$

Since Kv is a critical point of Ψ^v_H we have

$$\langle Kv, w\rangle_H = \int_\Omega vw \, d\nu \tag{4}$$

for every $w \in H$. From (3) and (4) we now have

$$\langle r^{-2}Kv, \varphi\rangle_E = \int_S \int_\Omega v r^2 \varphi(x, t) \, d\nu(x) \, d\sigma(t)$$

$$= \int_U v\varphi \, d\mu \tag{5}$$

for every test function φ on U that vanishes near the z axis. We claim that the test functions on U that vanish near the z axis are dense in E. Suppose φ is a test function on U. Choose an increasing C^∞ function ψ on $\mathbb{R}$ that satisfies $\psi(s) = 0$ for $s \leq 1$ and $\psi(s) = 1$ for $s \geq 2$. Then $\{\psi(nr)\varphi\}_{n=1}^\infty$ is a sequence of test functions vanishing near the z axis and converging to φ in E. Our claim follows from this. Hence (5) holds for all $\varphi \in E$. Thus $r^{-2}Kv$ is a critical point of Ψ^v_E, so $r^{-2}Kv = \mathcal{K}v$.

3.2. *Steiner Symmetrization*

If $v \in L^2(\Omega)$ is non-negative, we define the Steiner symmetrization v^* of v with respect to the line $z = 0$ to be the essentially unique non-negative function in $L^2(\Omega)$ such that for each $\alpha > 0$ and almost every $r \in (0, R)$ the set

$$\{z \mid v^*(r, z) \geq \alpha\}$$

is an interval with centre 0, whose length equals the linear measure of the set

$$\{z \mid v(r, z) \geq \alpha\}.$$

Then v^* is a rearrangement of v with respect to ν. For any rearrangement v' of v we have

$$\int_\Omega v'^s \, d\nu = \int_\Omega v^s \, d\nu \tag{6}$$

for all real s; for v^* we additionally have

for all real s and t. We require some results on Steiner symmetrization from Appendix 1 of Fraenkel and Berger [6], where v^* was defined as above for non-negative continuous functions v with compact supports, and then defined for a general non-negative L^2-function v by approximation in the 2-norm. It is easily verified that the two definitions are equivalent. Fraenkel and Berger studied functions defined on a half-plane, but their results are equally applicable to functions on the strip Ω. We therefore have the inequalities

$$\int_\Omega uv \, d\nu \leq \int_\Omega u^* v^* \, d\nu \tag{8}$$

$$\| u^* - v^* \|_{\Omega, 2} \leq \| u - v \|_{\Omega, 2} \tag{9}$$

for all non-negative $v \in L^2(\Omega)$; further, if $u \in H$ is non-negative then $u^* \in H$ and

$$\| u^* \|_H \leq \| u \|_H. \tag{10}$$

For $v \in L^2(U)$ the Steiner symmetrization v^* of v with respect to the hyperplane $z = 0$ is similarly defined by rearranging the restriction of v to each line parallel to the z axis as a symmetrically decreasing function. The inequalities analogous to (8), (9), and (10), with U, E, and μ in place of Ω, H, and ν, are also valid.

Lemma 2. *Let $v \in L^2(\Omega)$ be non-negative. Then $Kv \geq 0$ and*

$$\int_\Omega v^* K v^* \, d\nu \geq \int_\Omega v K v \, d\nu.$$

Further if $v^* = v$ then $(Kv)^* = Kv$. The same conclusions are true with $\mathcal{K}$, U, and μ in place of K, Ω, and ν.

Proof. The argument is similar to the proof of [5, Lemma 4], but we include it for completeness. For all $w \in L^2(\Omega)$ we have

$$\int_\Omega wKw \, d\nu = \| Kw \|_H^2$$

and consequently

$$-\tfrac{1}{2} \int_\Omega wKw \, d\nu = \inf_{y \in H} \tfrac{1}{2} \| y \|_H^2 - \int_\Omega yw \, d\nu \tag{11}$$

from the characterization of Kw as the minimizer of Ψ^w

Let $u = Kv$. Then $u \geqslant 0$ by the maximum principle applied to $\Delta_5(r^{-2}u) = v$. We take $y = u$ and $w = v$ in (11) to obtain

$$\tfrac{1}{2}\|u\|_H^2 + \tfrac{1}{2}\int_\Omega vKv\,dv = \int_\Omega uv\,dv.$$

Then we take $y = u^*$ and $w = v^*$ in (11) to obtain

$$\tfrac{1}{2}\|u^*\|_H^2 + \tfrac{1}{2}\int_\Omega v^*Kv^*\,dv \geqslant \int_\Omega u^*v^*\,dv.$$

Then

$$\tfrac{1}{2}\int_\Omega v^*Kv^*\,dv - \tfrac{1}{2}\int_\Omega vKv\,dv$$

$$\geqslant \int_\Omega u^*v^*\,dv - \int_\Omega uv\,dv + \tfrac{1}{2}\|u\|_H^2 - \tfrac{1}{2}\|u^*\|_H^2 \geqslant 0$$

by (8) and (10).

Suppose additionally that $v = v^*$. Then from (8) and (10) we have

$$\tfrac{1}{2}\|u^*\|_H^2 - \int_\Omega u^*v\,dv \leqslant \tfrac{1}{2}\|u\|_H^2 - \int_\Omega uv\,dv.$$

Since u is the unique minimizer for Ψ_H^v we now have $u^* = u$.

The analogous results for $\mathscr{K}$ follow by the same arguments, using the appropriate analogues of (8) and (10).

3.3. L^p Estimates

For $\xi > 0$ we write $U(\xi)$ for the set of points in U that satisfy $|z| < \xi$. If G is a measurable subset of Ω or U and $1 \leqslant p \leqslant \infty$, we take $L^p(G)$ to consist of those functions in $L^p(\Omega)$ or $L^p(U)$ that vanish outside G, and we denote the norm on $L^p(G)$ by $\|\ \|_{G,p}$; recall that the measure used in defining $L^p(\Omega)$ is v. If G is a domain in $\mathbb{R}^n$ and $m \geqslant 1$, the Sobolev space $W^{m,p}(G)$ is defined in the usual way using Lebesgue measure and its norm is denoted $\|\ \|_{G,m,p}$. If, for example, u is a function in $L^p(\Omega)$ then $\|u\|_{\Omega(\xi),p}$ and $\|u\|_{\Omega(\xi),m,p}$ are to be interpreted as norms of the restriction of u to $\Omega(\xi)$.

Lemma 3. (a) *Suppose $p \geqslant 2$, $0 < \xi < \xi'$, $v \in L^2(U)$, and $\|v\|_{U(\xi'),p} < \infty$. Then*

$$\|\mathscr{K}v\|_{U(\xi),2,p} \leqslant \text{const}(\|\mathscr{K}v\|_{U(\xi'),p} + \|v\|_{U(\xi'),p}).$$

(b) *Suppose $p \geqslant 2$ and $v \in L^2(U) \cap L^p(U)$. Then*

$$\|\mathscr{K}v\|_{U,2,p} \leqslant \text{const}(\|v\|_{U,2} + \|v\|_{U,p}).$$

Proof. We are going to use the L^p estimates up to the boundary given in Agmon [1, Sect. 8]. We have first to show that $\mathscr{K}v$ is a solution of $-\Delta_5 u = v$ in the slightly stronger sense required by [1]. Observe that if $u, w \in E$ and $\Delta_5 u \in L^2(U)$ then

$$\int_U \nabla u \cdot \nabla w\,d\mu = -\int_U (\Delta_5 u)\,w\,d\mu.$$

Hence if $v \in L^2(U)$ and $\varphi \in E$ with $\Delta_5\varphi \in L^2(U)$ then

$$\int_U v\varphi\,d\mu = -\int_U (\Delta_5\mathscr{K}v)\,\varphi\,d\mu$$

$$= \int_U (\nabla\mathscr{K}v)\cdot\nabla\varphi\,d\mu = \int_U (\mathscr{K}v)(-\Delta_5\varphi)\,d\mu.$$

In particular if $v \in L^2(U)$, and $\varphi \in C^2(\bar{U})$ vanishes on ∂U and has bounded support then

$$\int_U (\mathscr{K}v)(-\Delta_5\varphi)\,d\mu = \int_U v\varphi\,d\mu.$$

Thus $\mathscr{K}v$ is a solution in the required sense.

To prove (a) let $v \in L^2(U)$ with $\|v\|_{U(\xi'),p} < \infty$. A trivial modification of the proof of Agmon [1, Theorem 8.1] shows that $\|\mathscr{K}v\|_{U(\xi),2,p} < \infty$ and moreover

$$\|\mathscr{K}v\|_{U(\xi),2,p} \leqslant \text{const}(\|\mathscr{K}v\|_{U(\xi'),p} + \|v\|_{U(\xi'),p}).$$

Now consider $v \in L^2(U) \cap L^p(U)$ as in (b). By covering U with translates of $U(\xi)$, we conclude from (a) that, provided $\mathscr{K}v \in L^p(U)$, we have $\mathscr{K}v \in W^{2,p}(U)$ and

$$\|\mathscr{K}v\|_{U,2,p} \leqslant \text{const}(\|\mathscr{K}v\|_{U,p} + \|v\|_{U,p}).$$

If $2 \leqslant p \leqslant \frac{10}{3}$ then $E \to L^p(U)$ is bounded; since $\mathscr{K}: L^2(U) \to E$ is bounded we have $\mathscr{K}v \in L^p(U)$ and

$$\|\mathscr{K}v\|_{U,2,p} \leqslant \text{const}(\|v\|_{U,2} + \|v\|_{U,p}). \tag{12}$$

If $p > \frac{10}{3}$ then $v \in L^{5/2}(U)$ and we have

$$\|\mathscr{K}v\|_{U,p} \leqslant \text{const}\ \|\mathscr{K}v\|_{U,2,5/2}$$

$$\leqslant \text{const}\ (\|v\|_{U,2} + \|v\|_{U,5/2})$$

$$\leqslant \text{const}\ (\|v\|_{U,2} + \|v\|_{U,p}).$$

LEMMA 4. *Suppose* $p > \frac{5}{2}$. *Then it is possible to choose* $h(z) = \text{const} \min\{1, |z|^{-1/p}\}$ *such that*

$$\mathscr{K}v(x_1, ..., x_4, z) \leqslant h(z)(\|v\|_{U,2} + \|v\|_{U,p})$$

for all non-negative functions $v \in L^2(U) \cap L^p(U)$ *that satisfy* $v^* = v$.

Proof. Consider $z \geqslant 1$ and let

$$A = (0, ..., 0, z) + U(1)$$
$$B = (0, ..., 0, z) + U(\tfrac{1}{2}).$$

Since $v \geqslant 0$ and $v^* = v$, we have $\mathscr{K}v \geqslant 0$ and $(\mathscr{K}v)^* = \mathscr{K}v$ by Lemma 2. Hence

$$\|v\|_{A,p} \leqslant z^{-1/p} \|v\|_{U(z),p} \leqslant z^{-1/p} \|v\|_{U,p}$$
$$\|\mathscr{K}v\|_{A,p} \leqslant z^{-1/p} \|\mathscr{K}v\|_{U(z),p} \leqslant z^{-1/p} \|\mathscr{K}v\|_{U,p}.$$

By Lemma 3(a) we now have

$$\|\mathscr{K}v\|_{B,2,p} \leqslant \text{const } z^{-1/p}(\|v\|_{U,p} + \|\mathscr{K}v\|_{U,p}),$$

where the constant is independent of z and v. By Lemma 3b and the embedding of $W^{2,p}(U(\frac{1}{2}))$ in the space $C_B(U(\frac{1}{2}))$ of bounded continuous functions on $U(\frac{1}{2})$ we now have

$$\mathscr{K}v(x_1, ..., x_4, z) \leqslant \text{const } |z|^{-1/p} (\|v\|_{U,2} + \|v\|_{U,p})$$

for $|z| \geqslant 1$ and

$$\|\mathscr{K}v\|_{\sup} \leqslant \text{const } (\|v\|_{U,2} + \|v\|_{U,p}),$$

hence the result.

LEMMA 5. *Let* $p > \frac{5}{2}$, $a > 0$, $c > 0$, *let* F *be a bounded open subset of* U, *let* G *be a compact subset of* U, *and suppose that*

$$D = \left\{v \in L^2(U) \cap L^p(U) \,|\, v \geqslant 0,\ \|v\|_{U,p} \leqslant c,\ \int_F v\, d\mu \geqslant a\right\}$$

is non-empty. Then

$$d = \inf\{r^2 \mathscr{K}v(x) \,|\, x \in G,\ v \in D\} > 0.$$

Proof. Define

$$D' = \left\{v \in L^p(U) \,|\, v = 0 \text{ in } U \setminus F,\ v \geqslant 0 \text{ in } F,\ \|v\|_{\ldots} \leqslant c,\ \int \ldots\, d\mu \geqslant a\right\}$$

Then $D' \subset D$. If $v \in D$ and v' denotes the restriction of v to F, then $v' \in D'$ and $\mathscr{K}v' \leqslant \mathscr{K}v$ by the maximum principle; hence

$$\inf\{r^2 \mathscr{K}v(x) \,|\, v \in D',\ x \in G\} = d.$$

Now $\mathscr{K}: L^p(F) \to W^{2,p}(U)$ is bounded by Lemma 3b. Since D' is weakly compact in $L^p(F)$ it follows that $\mathscr{K}D'$ is weakly compact in $W^{2,p}(U)$, so $\mathscr{K}D'$ is norm compact in $C(G)$. We can therefore choose $v_1 \in D'$ with

$$\inf r^2 \mathscr{K}v_1(G) = d.$$

Since $v_1 \neq 0$ and $v_1 \geqslant 0$ we have $\mathscr{K}v_1 > 0$ in U by the maximum principle, hence $d > 0$.

LEMMA 6. *Let* $p > \frac{5}{2}$, $\alpha > 0$, $\beta > 0$ *and suppose that*

$$D = \{v \in L^2(\Omega) \cap L^p(\Omega) \,|\, v \geqslant 0,\ v^* = v,\ \|v\|_{\Omega,2} + \|v\|_{\Omega,p} \leqslant \alpha,\ \|Kv\|_{\sup} \geqslant \beta\}$$

is nonempty. Then for some $\eta > 0$ *we have*

$$\inf\{\|v\|_{U(\eta),p} \,|\, v \in D\} > 0.$$

Proof. Write

$$S = \{x \in \mathbb{R}^4 \,|\, |x| < R\}.$$

Consider $v \in D$, and let $\eta > 0$. Write

$$v_0 = 1_{U(\eta)} v$$
$$v_1(x, z) = v(x, |z - \eta|)$$
$$v_2(x, z) = v(x, |z + \eta|)$$

for $x \in S$ and real z, so

$$v \leqslant v_0 + v_1 + v_2.$$

Since $(\mathscr{K}v)^* = \mathscr{K}v$ we may choose $x \in S$ such that

$$\mathscr{K}v(x, 0) = \|\mathscr{K}v\|_{\sup}.$$

Then Lemma 4 is applicable to v_0, v_1, and v_2, and in conjunction with the maximum principle this yields

$$\begin{aligned}\beta R^{-2} \leqslant \mathscr{K}v(x, 0) &\leqslant h(0)(\|v_0\|_{U,2} + \|v_0\|_{U,p}) + h(\eta)(\|v_1\|_{U,2} + \|v_1\|_{U,p}) \\ &\quad + h(-\eta)(\|v_2\|_{U,2} + \|v_2\|_{U,p}) \\ &\leqslant h(0)(\|v_0\|_{U,2} + \|v_0\|_{U,p}) + 20h(\eta)(\|v\|_{\Omega,2} + \|v\|_{\Omega,p})\end{aligned}$$

where

$$\theta = \max\{R, R^{2-p}\}.$$

Therefore

$$\|v_0\|_{U,2} + \|v_0\|_{U,p} \geqslant (\beta R^{-2} - 2\theta\alpha b(\eta))/b(0).$$

Now fix $\eta > 0$ with

$$4\theta\alpha b(\eta) < \beta R^{-2}.$$

We have

$$\|v_0\|_{U,2} \leqslant \gamma \|v_0\|_{U,p},$$

where

$$\gamma = \|1\|_{U(\eta),2p/(p-2)}.$$

Hence

$$\|v\|_{U(\eta),p} = \|v_0\|_{U,p} \geqslant \frac{\beta R^{-2}}{2(1+\gamma)\, b(0)}.$$

4. Proof of the Theorem

Let us adopt the notation of Sections 2 and 3, let $\xi \geqslant \xi_0$, and let $\lambda > 0$. The operator $K: L^2(\Omega) \to H$ is bounded, the embedding $H \to W_0^{1,2}(\Omega)$ is bounded, and the embedding $W^{1,2}(\Omega(\xi)) \to L^2(\Omega(\xi))$ is compact. Hence the functional

$$\Phi_\lambda(v) = \tfrac{1}{2}\int_\Omega vKv\, dv - \tfrac{1}{2}\lambda \int_\Omega r^2 v\, dv$$

is weakly sequentially continuous on $L^2(\Omega(\xi))$, and the strict positivity of K ensures that Φ_λ is strictly convex. Theorem A in Section 2 now shows that Φ_λ attains a maximum value relative to $\mathscr{F}(\xi)$, and that if f^ξ is any maximizer then

$$f^\xi = \varphi^\xi(Kf^\xi - \lambda r^2/2)$$

almost everywhere in $\Omega(\xi)$, for some increasing function φ^ξ. Thus $u^\xi \equiv Kf^\xi$ satisfies

$$\mathscr{L}u^\xi = \varphi^\xi(u^\xi - \lambda r^2/2)$$

By Lemma 2 and (7) we have

$$\int_\Omega v^* K v^*\, dv \geqslant \int_\Omega vKv\, dv$$

$$\int_\Omega r^2 v^*\, dv = \int_\Omega r^2 v\, dv$$

for all non-negative $v \in L^2(\Omega(\xi))$; hence

$$\Phi_\lambda(v^*) \geqslant \Phi_\lambda(v).$$

Henceforth we shall assume

$$f^{\xi *} = f^\xi.$$

By Lemma 2 we then have

$$u^{\xi *} = u^\xi.$$

Let $\Lambda > 0$ be chosen so small that

$$g \equiv \Phi_\Lambda(f_0) > 0.$$

Henceforth we shall assume $\lambda < \Lambda$. Then

$$\tfrac{1}{2}\int_\Omega f^\xi K f^\xi\, dv \geqslant g$$

and with (6) this yields

$$\|Kf^\xi\|_{\sup} \geqslant 2g/\|f^\xi\|_{\Omega,1} = 2g/\|f_0\|_{\Omega,1}.$$

It now follows from Lemma 6 that there exists $m > 0$ and $\eta > 0$, independent of ξ and λ, such that

$$\|f^\xi\|_{U(\eta),p} \geqslant m.$$

If we choose $M > 0$ such that

$$\int_{f_0 > M} f_0^p\, d\mu \leqslant \frac{m^p}{2}$$

then

$$\int_{U(\eta)} f_0^p d\mu \geqslant \frac{m^p}{2}$$

and consequently

$$\int_{U(\eta)} f^{\xi}\, d\mu \geq \frac{m^{p}}{2M^{p-1}}.$$

Define

$$A = \nu(\{x \in \Omega \mid f_0(x) > 0\})$$

and choose a compact set $G \subset \Omega \subset U$ such that

$$\nu(G) > A.$$

By Lemma 5 there is a number $d > 0$, independent of ξ and λ, with

$$\mathscr{K} f^{\xi}(x) \geq d \qquad \forall x \in G.$$

Let ρ be the least value of r on G and let $\varepsilon = d\rho^2/2$.

Suppose henceforth that $\lambda < d$ and $G \subset \Omega(\xi)$. Defining

$$J(\xi) = \{(r, z) \in \Omega \mid u^{\xi}(r, z) - \lambda r^2/2 \geq \varepsilon\}$$

we now have $G \subset J(\xi)$. Let

$$S(\xi) = \{(r, z) \in \Omega \mid f^{\xi}(r, z) > 0\}.$$

Then

$$\nu(S(\xi)) = A < \nu(G) \leq \nu(J(\xi) \cap \Omega(\xi)).$$

Since f^{ξ} is essentially an increasing function of $u^{\xi} - \lambda r^2/2$ on $\Omega(\xi)$ it follows that, apart from a set of measure zero,

$$S(\xi) \subset J(\xi).$$

By Lemma 4 we have

$$J(\xi) \subset \{(r, z) \in \Omega \mid r^2 h(z) \geq \varepsilon\}$$
$$\subset \{(r, z) \mid \delta < r < R, |z| < \zeta\}$$

for some positive δ and ζ, independent of ξ. Since u^{ξ} is continuous and vanishes on $r = R$, it follows further that $J(\xi)$ is bounded away from $r = R$.

Let φ be extended to an increasing function φ defined on an interval containing $(-\infty, \varepsilon)$ and satisfying $\varphi(s) = 0$ for $s < \varepsilon$. Since $u^{\xi} - \lambda r^2/2 < \varepsilon$ outside $\Omega(\xi)$ we now have

almost everywhere in Ω. By Agmon [1, Theorem 6.1] we have $u^{\xi} \in W^{2,p}_{\text{loc}}(\Omega)$.

It remains only to verify the boundary and asymptotic conditions of (v). Henceforth we write $f = f^{\xi}$ and $u = u^{\xi}$. Since $\mathscr{K} f \in W^{2,p}(U) \subset C^{1,\alpha}(\bar{U})$ for $0 < \alpha < 1 - 5/p$, it follows that $u \in C^{1,\alpha}(\bar{\Omega})$. Lemma 4 yields (a). Since $\mathscr{K} f \in W^{1,2}_0(U) \cap C^{1,\alpha}(\bar{U})$ it follows that $\mathscr{K} f$ vanishes on ∂U so u vanishes when $r = R$, and

$$u(r, z) \leq r^2 \|\mathscr{K} f\|_{\sup},$$

hence (b) and (c). Since $\nabla \mathscr{K} f$ is bounded we have (d). For $\xi > 0$ write

$$V(\xi) = \{(x, z) \in U \mid \xi - 1 < z < \xi + 1\}.$$

Since $\mathscr{K} f \in W^{2,p}(U)$ we have

$$\|\mathscr{K} f\|_{V(\xi), 2, p} \to 0$$

as $\xi \to \pm\infty$; hence

$$\|\mathscr{K} f\|_{C^{1,\alpha}(\bar{V}(\xi))} \to 0$$

from which (e) follows. Since $u \in C^{1,\alpha}(\bar{\Omega})$ and $u = 0$ when $r = R$ we have (f).

5. Failure of the Method for Large λ

Fix $R > 0$, $p > 5$, and a non-negative $f_0 \in L^p(\Omega)$ having bounded support; say f_0 vanishes outside $\Omega(\xi_0)$. Define

$$k = 2 \sup_{f \in \mathscr{F}} \|r^{-2} K f\|_{\sup}$$

which is finite by Lemma 4b and the embedding $W^{2,p}(U) \to C_B(U)$. Fix $\lambda > k$. Consider $\xi \geq \xi_0$ and let f^{ξ} be a maximizer for Φ_λ relative to $\mathscr{F}(\xi)$. Let

$$\alpha = \lambda A/(4\pi\xi),$$

where A is the ν-measure of the set where $f_0 > 0$, and let

$$S = \{(r, z) \mid 0 < r < R, |z| < \xi, K f^{\xi} - \lambda r^2/2 > -\alpha\}.$$

Since $K f^{\xi} > 0$ the rectangle $|z| < \xi$, $r^2 < 2\alpha/\lambda$ is contained in S, so

Since f^{ξ} is an increasing function of $Kf^{\xi} - \lambda r^2/2$ in $\Omega(\xi)$, it follows that f^{ξ} vanishes outside S. But by choice of k, all points (r, z) of S satisfy

$$r^2(k-\lambda)/2 > -\alpha,$$

so

$$r^2 < r_{\xi}^2 = 2\alpha/(\lambda - k) = \lambda A/(2\pi(\lambda - k)\xi).$$

Thus f^{ξ} vanishes for $r > r_{\xi}$, and since $r_{\xi} \to 0$ as $\xi \to \infty$ it follows that no uniform bound exists on the support of f^{ξ}. Indeed $f^{\xi} \to 0$ weakly in $L^2(\Omega)$ as $\xi \to \infty$.

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Elliptic Equations with Nearly Critical Growth

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1. Introduction

Consider the problem

$$(\mathrm{I}) \begin{cases} -\Delta u = u^p, & u > 0 \text{ in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

in which Ω is a bounded, star-shaped domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. It is well known that if $N > 2$ the character of this problem changes when the exponent p passes through the critical Sobolev exponent

$$p = \frac{N+2}{N-2}.$$

If $p < (N+2)/(N-2)$, then Problem I always has a solution, whatever the domain Ω [11, 14], whilst if $p \geq (N+2)/(N-2)$ it has no solution for any (star-shaped) domain [13].

Recently, considerable interest has grown around problems like (I) in which the right hand side u^p is replaced by a perturbation $f(u)$ of the pure power, such as

$$f(u) = \lambda u^q + u^p,$$

where $\lambda \in \mathbb{R}$ and $0 < q < p$. The dichotomy above at $p = (N+2)/(N-2)$ may then be resolved by means of the additional parameters λ and q [2, 7, 12]. For a review of recent results we refer to [5].