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Some results on the existence of geodesics in Lorentz manifolds with nonsmooth boundary

D. Fortunato
Dipartimento di Matematica
Università di Bari
Italy
&
F. Giannoni
Istituto di Matematiche Applicate
Facoltà di Ingegneria
Università di Pisa
Italy

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MAIN BUILDING STRADA COSTIERA, 11 TEL. 22[illegible] TELEFAX 224163 TELEX 460392 ADRIATICO GUEST HOUSE VIA GRIGNANO, 9 TEL. 224241 TELEFAX 224531 TELEX 460449
MICROPROCESSOR LAB. VIA BEIRUT, 31 TEL. 224[illegible] TELEFAX 224163 TELEX 460392 GALILEO GUEST HOUSE VIA BEIRUT, 7 TEL. 22401 TELEFAX 224559 TELEX 460392

SOME RESULTS ON THE EXISTENCE OF GEODESICS IN LORENTZ MANIFOLDS WITH NONSMOOTH BOUNDARY(1)

V.Benci
Ist.Mat.Appl.
Fac.Ingegneria
Univ. di Pisa
Pisa, Italy

D.Fortunato
Dip.Matematica
Univ. di Bari
Bari, Italy

F.Giannoni
Ist.Mat.Appl.
Fac.Ingegneria
Univ. di Pisa
Pisa, Italy

ABSTRACT. *In this note we deal with the problem of the existence of geodesics joining two given points and not touching the boundary of certain Lorentz manifolds, of which the Schwartzchild spacetime is the simplest example.*

§1. INTRODUCTION.

In this note we state some existence and multiplicity results about geodesics joining two given points, in Lorentz manifolds having nonsmooth boundary. These geodesics are required to not touch the boundary.

We recall that a Lorentz manifold $\mathcal{L}$ is a smooth manifold equipped with a metric tensor g of the second order having index 1 (i.e. every matrix representation of it has exactly one negative eigenvalue, and no null eigenvalue). (see e.g. [10]). We write also $\langle , \rangle_L$ instead of $g(,)$.

A 4-dimensional Lorentz manifold is called *spacetime*.

We consider Lorentz manifolds with assumptions that are satisfied for example by the Schwartzchild spacetime and the Reissner-Nordström spacetime (see e.g. [8]).

The Schwartzchild metric is the unique solution (up to isometric change of variables) of the Einstein equations in the empty space, when the

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curvature of the spacetime is produced by a single, static, spherically symmetric massive body.

Using polar coordinates, the metric can be given in the form

$$ds^2 = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta \cdot d\varphi^2\right) - c^2\left(1 - \frac{2m}{r}\right)dt^2, \qquad (1.1)$$

where $2m=2GM/c^2$, G is the universal gravitation constant, M is the mass of the body and c is the speed of the light.

The Schwartzchild spacetime is a relativistic model which can be applied to regions around any astronomical object which is approximately static and spherically symmetric. For example in the case of the sun it gives a model for the solar system even better than the highly accurate Newtonian model.

Since (1.1) is a solution of the Einstein equations in the empty space, it is physically meaningful to equipped all $\{r > 2m\} \times \mathbb{R}$ with the metric (1.1), only if the radius r_M of the body is less than 2m. This spacetime is an example of universe with a black hole.

The name is justified by the fact that a light ray can not leave the region $\{r_M \leq 2m\}$. If an astronaut "falls" in the black hole, he spends a finite "proper" time, but an observer far from the black hole does not see the astronaut falling in it in a finite time.

The metric (1.1) is singular when r=2m and r=0. However, the singularity r=2m is not a physical singularity but it is the result of a "bad choice" of the coordinates. In fact, using changes of coordinates which are singular when r=2m, it is possible to show that the spacetime

$$\mathcal{S} = \{r > 2m\} \times \mathbb{R} \textit{ with metric } (1.1), \qquad (1.2)$$

is isometric to an open subset of a Lorentz manifold whose metric extends (1.1) (see e.g. [8, pp. 150–152]).

Notice that using the coordinates introduced by Kruskal in 1960 (see e.g. [8, pp. 153–155]), it is possible to construct the maximal analytical extension of the Schwartzchild spacetime. In this coordinates we see that the topological boundary $\partial\mathcal{S}$ of $\mathcal{S}$ is not a smooth manifold.

The previous considerations on the Schwartzchild spacetime leads to the following

Definition (1.3). *Let $\mathcal{M}$ be an open subset of a Lorentz manifold $(\mathcal{L},\langle,\rangle_L)$. $\mathcal{M}$ is said to be* static *Lorentz manifold if there exists a Riemann manifold $(\mathcal{M}_0,\langle,\rangle_R)$ of class C^2 (where $\langle,\rangle_R$ denotes the Riemann structure on $\mathcal{M}_0$) such that $\mathcal{M}$ is isometric to $\mathcal{M}_0 \times \mathbb{R}$*[2]*, and, in the coordinate system* (x,t) *(with $x\in\mathcal{M}_0$ and $t\in\mathbb{R}$),*

$$\langle\cdot,\cdot\rangle_L = \langle\cdot,\cdot\rangle_R - \beta(x)dt^2 \tag{1.4}$$

where $\beta\in C^2(\mathcal{M}_0,\mathbb{R}^+\setminus\{0\})$.

The spacetime $\mathcal{S}$ defined in (1.2) is a static Lorentz manifold. Notice that we lose the static structure on $\partial\mathcal{S}$ where the coordinate systems (x,t) does not have sense. However in order to look for geodesics on a Lorentz manifold it can be useful to consider its static structure even if the completeness is lost.

Indeed geodesics joining two given events $z_0=(x_0,t_0)$ and $z_1=(x_1,t_1)$ in $\mathcal{M}$ are the critical points of the functional

$$f(\gamma) = \int_0^1 \langle\dot\gamma(s),\dot\gamma(s)\rangle_L ds = \int_0^1 \left(\langle\dot x(s),\dot x(s)\rangle_R - \beta(x(s))(\dot t(s))^2\right) ds^{(3)} \tag{1.5}$$

on the space of the smooth curves $\gamma(s)=(x(s),t(s))$ on $\mathcal{M}$ such that $\gamma(0)=z_0$, $\gamma(1)=z_1$. The coordinate function t is called universal time and its existence means that there is a way to synchronize all the watches in $\mathcal{M}_0$. The parameter s is proportional to the proper time which is the time

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(2) i.e. there is a diffeomorphism between $\mathcal{M}$ and $\mathcal{M}_0 \times \mathbb{R}$ preserving the Lorentz metric.

(3) here $\dot\gamma(s)$, $\dot x(s)$ and $\dot t(s)$ denote the derivatives of $\gamma(s)$, $x(s)$ and $t(s)$ respectively.

measured by an observer moving along a geodesic.

The functional (1.5) is indefinite, i.e. *sup* $f=+\infty$ and *inf* $f=-\infty$. This fact creates technical difficulties for the research of critical points of f. But in the static case we can reduce our problem to the study of a functional bounded from below (at least if the function β is bounded from above).

In fact, as proved in [5] by a simple calculation, $\gamma(s)=(x(s),t(s))$ is a critical points for (1.5) with $\gamma(0)=z_0$ and $\gamma(1)=z_1$ if and only if $x(s)$ is a critical point for the functional

$$J(x) = \int_0^1 \langle \dot{x}(s),\dot{x}(s)\rangle_R ds - (t_1-t_0)^2 \left[\int_0^1 \frac{1}{\beta(x(s))} ds\right]^{-1} \tag{1.6}$$

with $x(0)=x_0$ and $x(1)=x_1$, and $t(s)$ solves the Cauchy problem

$$\begin{cases} \dot{t}(s) = (t_1-t_0)\left[\int_0^1 \frac{1}{\beta(x(\tau))} d\tau\right]^{-1} \frac{1}{\beta(x(s))} \\ t(0) = t_0 \end{cases} . \tag{1.7}$$

Moreover for the critical points γ and x we have

$$f(\gamma) = J(x). \tag{1.8}$$

Finally we recall that if γ is a geodesic in $\mathcal{M}$ there exists $E_\gamma \in \mathbb{R}$ such that, for every s,

$$\langle\dot{\gamma}(s),\dot{\gamma}(s)\rangle_L = E_\gamma. \tag{1.9}$$

γ is said *time-like, light-like* or *space-like* geodesic if $E_\gamma<0$, $E_\gamma=0$, or $E_\gamma>0$ respectively. A time-like geodesic is physically interpreted as the world line of a material particle under the action of a gravitational field, while a light-like geodesic is a world line of a light ray. Space-like geodesics have less physical relevance, however they can be useful in the study of the global properties of a Lorentz manifold.

§2. Static universes.

Motivated by the considerations in section 1 we give the following

Definition (2.1). *Let U be an open connected subset of a Lorentz manifold*

$(\mathcal{L},g)$ *and let* ∂U *be its topological boundary.* U *is said to be* static universe (with a black hole) *if*

(*i*) $U = \mathcal{M}_0 \times \mathbb{R}$ *is a static Lorentz manifold (see* (1.3));

(*ii*) $\sup_U \beta < +\infty$, *where* β *is the function in* (1.4);

(*iii*) *there exists* $\phi \in C^2(U, \mathbb{R}^+ \setminus \{0\})$ *such that*

$$\lim_{(x,t)\to z\in\partial U} \phi(x,t) = 0 \text{ and } \phi(x,t) = \phi(x,0) \equiv \varphi(x) \quad \forall (x,t)\in U;$$

(*iv*) $U\cup\partial U$ *is complete (i.e. every geodesic* $\gamma:]a,b[\longrightarrow U$ *has a continuous extension* $\bar{\gamma}:[a,b]\longrightarrow U\cup\partial U$);

(*v*) *let* $\gamma(s)=(x(s),t(s))$ *be a time-like geodesic in* U *with* $\lim_{s\to s_0^-}\gamma(s) \in \partial U$, *then* $\lim_{s\to s_0^-} t(s) = \pm\infty$.

REMARK (2.2). *Condition iv) means that the lack of completeness is due only to the the boundary* ∂U.

Condition v) says that if a material particle reaches the topological boundary of U, an observer far from the boundary does not see this event in a finite time, since its proper time is a reparametrization of the universal time. This condition justifies the name of the structure defined in (2.1).

A straightforward calculation shows that the spacetime $\mathcal{S}$ defined in (1.2) is a static universe.

The same computations show that, when $m^2 > e^2$, (m represents the gravitational mass and e is the electric charge of the body), also the Reissner–Nordström spacetime is a static universe for $r > m + \sqrt{m^2-e^2}$, (see [6]).

Assume that U is a static universe (with a black hole). We have the following results about existence of time-like geodesics joining two given events.

Theorem (2.3). *Let* $z_0=(x_0,t_0)$ *and* $z_1=(x_1,t_1)$ *be events in* U. *There exists a time-like geodesic* γ *in* U *such that* $\gamma(0)=z_0$ *and* $\gamma(1)=z_1$ *if and only if*

$$\exists\ x \in C^1([0,1],\mathcal{M}_0):\ x(0)=x_0,\ x(1)=x_1 \text{ and}$$
$$\left[\int_0^1 \frac{1}{\beta(x(s))}\,ds\right]\cdot\left[\int_0^1 \langle\dot{x}(s),\dot{x}(s)\rangle_R ds\right] < (t_1-t_0)^2. \tag{2.4}$$

Remark (2.5). *Notice that condition* (2.4) *is certainly satisfied if* $|t_1-t_0|$ *is large enough.*

Let $N(x_0,x_1,|t_1-t_0|)$ be the number of time-like geodesics in U joining $z_0=(x_0,t_0)$ and $z_1=(x_1,t_1)$.(4) If U has a non trivial topology we get the following multiplicity results of time-like geodesics joining z_0 and z_1.

Theorem (2.6). *Assume* $\mathcal{M}_0$ *to be not contractible in itself and of class* C^3. *Then*

$$\lim_{|t_1-t_0|\to+\infty} N(x_0,x_1,|t_1-t_0|) = +\infty.$$

Remark (2.7). *Condition* (*ii*) *of Definition* (2.1) *is essential to obtain our geodesics existence results. For example Anti-de Sitter space* (*see e.g.*[8,12]) *furnishes counterexamples.*

However if $\beta(x)$ *goes to* $+\infty$ *with a mild rate as* x *goes to* ∞, *Theorems* (2.3) *and* (2.6) *still hold.*

Also the condition (*iv*) *can be weakened. In fact since we shall look for critical points of the functional* (1.6) *it is sufficient to require that the sets* $\{x\in\mathcal{M}_0:\phi(x)\geq\delta\}$ *are complete with respect to the Riemann structure of* $\mathcal{M}_0$, *for every* $\delta>0$.

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(4) It depends only by x_0,x_1 and $|t_1-t_0|$ because the metric tensor g is independent of t.

Theorems (2.3) and (2.6) are related to some results of [1,14,16]. In these papers is always assumed that the Lorentz manifolds are globally hyperbolic (but not necessarily static). However the assumption of global hyperbolicity is not always easy to verify.

§3. LORENTZIAN MANIFOLDS WITH CONVEX BOUNDARY.

Now we consider the problem of the geodesical connectivity for Lorentz manifolds.

A Lorentz manifold is said to be geodetically connected if for every z_0, z_1 in $\mathcal{M}$ there exist a geodesic γ in $\mathcal{M}$ such that $\gamma(0)=z_0$ and $\gamma(1)=z_1$.

Clearly for studying the geodesical connectivity, it is necessary to consider also space like geodesics which are more difficult to deal with. The geodesical connectivity has not been treated in the previous works on this topic which deal only with time-like and light-like geodesics.

This problem has been faced for the first time in [3,4] for stationary Lorentz manifolds without boundary. Here we consider the case of static Lorentz manifolds with nonsmooth boundary.

For the study of the geodesical connectivity the condition of static universe is not appropriated (see example (3.9)). For this reason we introduce the following geometrical condition:

DEFINITION (3.1). *Let $\mathcal{M}$ be an open connected subset of a Lorentz manifold $(\mathcal{L},g)$ and $\partial\mathcal{M}$ its topological boundary. $\mathcal{M}$ is said to be* static Lorentz manifold with convex boundary *if*

(i)–(iv) of Definition (2.1) *hold,*

while v) of (2.1) *is replaced by*

v′) there exists a neighbourhood $\mathcal{N}$ of $\partial\mathcal{M}$ and there exist $\nu, N, M \in \mathbb{R}^+\setminus\{0\}$ *such that in $(\mathcal{N}\cap\mathcal{M})\setminus\partial\mathcal{M}$ the function ϕ of* (2.1)*(iii) satisfies:*

$$N \geq \langle grad\ \phi(z), grad\ \phi(z)\rangle_L \geq \nu^{(5)}, \tag{3.2}$$

$$H^{\phi}_{L}(z)[v,v] \leq M\cdot|\langle v,v\rangle_L|\cdot\phi(z) \qquad \forall v\in T_z(\mathcal{M})^{(6)}. \tag{3.3}$$

Remark (3.4). *Schwartzchild spacetime* (1.2) *satisfies* v') *with the function*

$$\phi(r,\vartheta,\varphi,t) = \sqrt{1 - \frac{2m}{r}},$$

(*see* [6]), *so* (1.2) *is a static Lorentz manifold with convex boundary.*

Also the Reissner–Nordström spacetime (*for* $r > m + \sqrt{m^2-e^2}$) *is a static Lorentz manifold with convex boundary provided that* $m^2 > \frac{9}{5}\cdot e^2$, *as we can verify* (*see* [6]) *using the function*

$$\phi(r,\vartheta,\varphi,t) = \sqrt{1 - \frac{2m}{r} + \frac{e^2}{r^2}}.$$

Theorem (3.5). *A static Lorentz manifold with convex boundary is geodesically connected.*

Remark (3.6). *Now we give a geometric interpretation of assumption* (3.3).

A geometric notion of convexity of a smooth boundary $\partial\mathcal{M}$ *of a Lorentz manifold* $\mathcal{M}$ *can be the following one*:

$\partial\mathcal{M}$ *is said to be convex if for every geodesic* $\gamma:[a,b] \longrightarrow \mathcal{M}\cup\partial\mathcal{M}$

$$\gamma(a)\in\partial\mathcal{M},\ \dot\gamma(a)\in T_{\gamma(a)}(\partial\mathcal{M}),\ \Rightarrow \gamma([a,b]) \cap \mathcal{M} = \emptyset. \tag{g.1}$$

It is easy to check that condition (g.1) *implies that*

$$H^{\phi}_{L}(z)[v,v] \leq 0 \text{ for all } z\in\partial\mathcal{M}, \text{ for all } v\in T_z(\partial\mathcal{M}). \tag{g.2}$$

In our case, since the boundary is not smooth, the assumptions (g.1) *and* (g.2) *do not make any sense. However condition* (g.2) *can be generalized in the following way*:

() *grad* $\phi(z)$ denotes the gradient of the function ϕ with respect to the Lorentz structure, i.e it is the unique vector $G \in T_z(\mathcal{M})$ (the tangent plane of $\mathcal{M}$ at z) such that $\langle G,v\rangle_L = d\phi(z)v\ \forall v\in T_z(\mathcal{M})$.

() $H^{\phi}_{L}(z)[v,v]$ denotes the Hessian of the function ϕ at z in the direction v, i.e. $\frac{d^2}{ds^2}\left(\phi(\gamma(s)\right)_{|s=0}$ where γ is a geodesic such that $\gamma(0)$=z and $\dot\gamma(0)$=v.

$$\limsup_{z\to z_0\in\partial\mathcal{M}} H^r_L(z)[v,v] \le 0$$

for all v *such that* $|\langle v,v\rangle_L| \le 1$ *and* $\langle grad\ \phi(z),v\rangle_L = 0.$ (g.3)

It can be proved that (g.3) *is sufficient to guarantee the geodesical connectivity of* $\mathcal{M}\cup\partial\mathcal{M}$ *but not of* $\mathcal{M}$.

In order to get the geodesical connectivity of $\mathcal{M}$ *it seems we need a control of the rate for which the limit in* (g.3) *is achieved.*

The assumptions (3.3) *provides this control.*

When the topology of $\mathcal{M}$ is not trivial we have the following multiplicity results about space-like geodesics.

THEOREM (3.7). *Let* $\mathcal{M}$ *be a static Lorentz manifold with convex boundary. Assume* $\mathcal{M}_0$ *to be not contractible in itself and of class* C^3.

Then for every $z_0,z_1\in\mathcal{M}$ *there exists a sequence* $\{\gamma_n\}_{n\in\mathbb{N}}$ *of geodesics in* $\mathcal{M}$ *joining* z_0 *and* z_1 *such that*

$$\lim_{n\to+\infty} E_{\gamma_n} = +\infty.$$

Before to conclude this section we wish to point out that the notion of static universe and the notion of static Lorentz manifold with convex boundary are independent as the following examples show. Consider

$$\{x\in\mathbb{R} : x > 1\} \times \mathbb{R} \text{ with metric } ds^2 = dx^2 - \beta(x)dt^2,$$

$$\text{where } \beta \text{ is bounded and } \beta(x) = x-1 \text{ if } x \le 2; \tag{3.8}$$

$$\{(x,y) \in \mathbb{R}^2: x^2 + y^2 > 1\} \times \mathbb{R} \text{ with metric } ds^2 = dx^2 + dy^2 - \beta(x,y)dt^2,$$

$$\text{where } \beta \text{ is bounded and } \beta(x,y) = \left(\sqrt{x^2+y^2}-1\right)^2 \text{ if } \sqrt{x^2+y^2} \le 2. \tag{3.9}$$

Simple calculations show that (3.8) is a static Lorentz manifold with convex boundary (take $\phi(x) = x-1$) and it is not a static universe. On the other hand (3.9) is a static universe and it is not a static Lorentz manifold with convex boundary (the events of the type (x_1,x_2,t_0) and $(-x_1,-x_2,t_0)$ can not be joining by geodesics lying in the spacetime (3.9)).

Remark (3.10). *Theorems* (2.3) *and* (2.5) *hold even for a static Lorentz manifold with convex boundary, while theorems* (3.5) *and* (3.7) *in general do not hold for a static universe, as we can see using the spacetime* (3.9).

§4. Sketch of the proofs.

The complete proofs of the above results are in [6]; here we give a sketch of such proofs.

Denote by $AC(0,1;\mathcal{M}_0)$ and $AC(0,1;T\mathcal{M}_0)$ the set of the absolutely continuous curves from [0,1] to the Riemann manifold $\mathcal{M}_0$ and to the tangent bundle $T\mathcal{M}_0$ respectively.

We put

$$\Omega^1 := \Omega^1(x_0,x_1) = \left\{x\in AC(0,1;\mathcal{M}_0) : \int_0^1 \langle\dot{x}(s),\dot{x}(s)\rangle_R ds < +\infty, \right.$$
$$\left. \textit{and } x(0)=x_0,\ x(1)=x_1\right\}. \tag{4.1}$$

Ω^1 is a Riemann manifold with tangent plane at $x\in\Omega^1$ given by

$$T_x(\Omega^1)=\left\{\xi\in AC(0,1;T\mathcal{M}_0):\ \xi(s)\in T_{x(s)}(\mathcal{M}_0)\ \forall s\in[0,1],\right.$$
$$\left.\int_0^1 \langle D_s\xi(s),D_s\xi(s)\rangle_R ds < +\infty, \text{ and } \xi(0)=\xi(1)=0\right\},$$

where D_s denotes the covariant derivative with respect to the Riemann structure of $\mathcal{M}_0$.

Consider the function φ defined in (*iii*) of (2.1). Using a partition of the unity we can modify φ in order to have the existence of $\delta>0$ and $N>0$ such that

$$\varphi(x)\le\delta \Rightarrow \langle grad\ \varphi(x),grad\ \varphi(x)\rangle_R \le N. \tag{4.2}$$

By a standard computation (cf. e.g. [2]), we get the following

Lemma (4.3). *Let* $\varphi:\mathcal{M}_0\to\mathbb{R}^+\setminus\{0\}$ *satisfying* (4.2) *and* $\{x_n\}_{n\in\mathbb{N}}$ *a sequence of* *curves in* Ω^1 *such that* $\int_0^1 \langle \dot{x}_n(s), \dot{x}_n(s)\rangle_R ds$ *is bounded and* $\lim_{n\to+\infty} \varphi(x_n(s_n))=0$ *for some* $s_n \in]0,1[$. *Then*

$$\lim_{n\to+\infty}\int_0^1 \frac{1}{\varphi^2(x_n(s))}\, ds = +\infty.$$

SKETCH OF THE PROOF OF THEOREM (2.3).

By virtue of (1.6) and (1.8) we see immediately that the condition (2.4) is necessary to guarantee the existence of a time-like geodesics joining z_0 and z_1.

The prove of the sufficiency is divided in various steps.

First step. Since $\mathcal{M}_0$ is non-complete, Ω^1 is non-complete.

To overcome this lack of compactness we introduce, for every $\varepsilon > 0$, the penalized functional $J_\varepsilon:\Omega^1 \to \mathbb{R}$ such that

$$J_\varepsilon(x) = J(x) + \varepsilon\int_0^1 \frac{1}{\varphi^2(x(s))}\, ds \tag{4.4}$$

where J is the functional (1.6) and φ is as in Lemma (4.3).

By virtue of lemma (4.3), for every $a \in \mathbb{R}$ the sublevels $J_\varepsilon^a=\{x\in\Omega^1: J_\varepsilon(x)\le a\}$ are complete in Ω^1. Then, since J_ε is bounded from below, it is not difficult to see that J_ε possesses a point of minimum $x_\varepsilon \in \Omega^1$.

Moreover, by assumption (2.4), there exists $\lambda>0$ such that, for ε small enough, we have

$$J_\varepsilon(x_\varepsilon) \le -\lambda. \tag{4.5}$$

Second step. The above points of minimum x_ε satisfy

$$\inf_{\varepsilon,s} \phi(\gamma_\varepsilon(s)) > 0. \tag{4.6}$$

To prove (4.6) we argue by contradiction and assume that

$$\inf_{\varepsilon,s} \phi(\gamma_\varepsilon(s)) = 0. \tag{4.7}$$

Let t_ε be the solution of (1.7) with x replaced by x_ε. Clearly (if $t_0 < t_1$) we have

$$t_0 \le t_\varepsilon(s) \le t_1 \quad \textit{for all } s \in [0,1].$$

Using the approximated solution $\gamma_\varepsilon=(x_\varepsilon,t_\varepsilon)$ and by (4.7) it is possible to construct, following a diagonal procedure, a time-like geodesic $\gamma(s)=(x(s),t(s))$ ($s\in[0,s_0[$) in U, such that

$$\forall s\in[0,s_0[,\quad t_0\le t(s)\le t_1 \text{ and } \lim_{s\to s_0}\gamma(s)\in\partial U.$$

This clearly contradicts the assumption that U is a static universe with a black hole.

<u>Third</u> <u>step.</u> Clearly we have

$$J'_\varepsilon(x_\varepsilon) = 0, \tag{4.8}$$

where J'_ε denotes the Frechet derivative of the functional J_ε.

By (4.6) we deduce the existence of an infinitesimal sequence ε_n such that

$$x_\varepsilon \longrightarrow x \in \Omega^1 \text{ weakly in } \Omega^1,$$

while (4.5) and (4.8) we get

$$J(x) \le -\lambda,\ J'(x) = 0.$$

Therefore the conclusion follows by using the variational principle stated in section 1 (see (1.6)-(1.8)). ∎

SKETCH OF THE PROOF OF THEOREM (3.5).

The proof of (3.5) can be carried out by the same penalization argument used in Theorem (2.3). In order to prove that the approximating solution γ_ε does not approach $\partial\mathcal{M}$ we use condition v'). By Gromwall Lemma we see that if $\inf_s \phi(\gamma_\varepsilon(s))\xrightarrow[\varepsilon]{}0$, then $\phi(\gamma_\varepsilon(s))\xrightarrow[\varepsilon]{}0$ uniformly in $s\in[0,1]$ and this contradicts the boundary conditions. ∎

SKETCH OF THE PROOF OF THEOREMS (2.6) AND (3.7).

To carry out the proofs of (2.6) and (3.7) we use the Liusternik-Schnirelmann category. It is the smallest number of closed contractible subsets of a topological space X covering a subset A of X, and it is denoted by $cat_X(A)$ (see e.g. [13]).

Using a well known theorem of Serre (see [15]), in [7] it has been proved that, if $\mathcal{M}_0$ is not contractible in itself, then there exists a sequence K_m of compact subsets of Ω^1 such that

$$\lim_{m\to+\infty} cat_{\Omega^1}(K_m) = +\infty. \tag{4.9}$$

Using the Nash immersion theorem (see [9]) and for instance the methods developed in [4] we get that J_ε satisfies the Palais-Smale compactness conditions, so the classical Deformation Lemma holds (see [11]). Hence if $\Gamma_k = \{A\subset\Omega^1 : cat_{\Omega^1}(A)\geq k\}$ we have that

$$c_k^\varepsilon = \inf_{A\in\Gamma_k} \sup_{x\in A} J_\varepsilon(x) \tag{4.10}$$

are critical level of J_ε. Moreover (see e.g.[13]) if two of these critical level are equal for different values of k, there exist infinitely many critical points having such a critical level.

Moreover for every $a\in\mathbb{R}$, $cat_{\Omega^1}(J_\varepsilon^a)$ is bounded independently of ε, and this fact permit us to get The proof of Theorem (3.7).

In order to get Theorem (2.6) we cannot take the limit for $\varepsilon\to 0$, since in this case two different critical levels may collaps and we lose multiplicity informations. This difficulty can be overcome using a suitable penalization for which

$$J_\varepsilon(x)=J(x) \text{ when } \varepsilon \text{ is small and } x\in\Omega^1.$$

∎

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