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Subharmonic oscillations of forced pendulum-type equations
Semicoercive variational problems at resonance: an abstract approach
Subharmonic solutions for second order differential equations

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Subharmonic Oscillations of Forced Pendulum-Type Equations

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1. INTRODUCTION

In this paper we are concerned with the existence of subharmonic solutions of second order differential equations of the form

$$\ddot{x} + g(x) = f(t),$$

where f is periodic with minimal period T and mean value zero. We have in mind as a particular case the pendulum equation, where $g(x) = A \sin x$.

First results on the existence of subharmonic orbits in a neighborhood of a given periodic motion were obtained by Birkhoff and Lewis (cf. [3] and [14]) by perturbation-type techniques. Rabinowitz [15] was able to prove the existence of subharmonic solutions for Hamiltonian systems by the use of variational methods. His approach is not of local type like the one in [3], and enables one to obtain a sequence of solutions whose minimal period tends toward infinity in the case when the Hamiltonian function has subquadratic or superquadratic growth. These results have been extended in various directions, cf. [2, 5, 6, 8, 13, 16–18]. Local results on subharmonics for the forced pendulum equation can be found in [19].

Hamiltonian systems with periodic nonlinearity were studied by Conley and Zehnder [6]. They proved the existence of subharmonic solutions under some assumptions on the nondegenerateness of the solutions, by the use of Morse–Conley theory.

In this paper we will prove the existence of subharmonic oscillations of a pendulum-type equation by the use of classical Morse theory together with an iteration formula for the index due to Bott [4] and developed in [7] and [1].

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2. The Main Result

Let T be a fixed positive number and $k \geq 2$ an integer. Assume $f: \mathbb{R} \to \mathbb{R}$ to be a continuous periodic function, with minimal period T, and such that

$$\int_0^T f(t)\, dt = 0. \tag{1}$$

We consider the equation

$$\ddot{x}(t) + g(x(t)) = f(t), \tag{2}$$

where $g: \mathbb{R} \to \mathbb{R}$ is a continuous function such that, setting

$$G(x) = \int_0^x g(s)\, ds,$$

the function G is 2π-periodic.

We want to prove the existence of subharmonic solutions; i.e., we look for periodic solutions of (2) having kT as minimal period. The kT-periodic solutions of (2) correspond to the critical points of the functional ϕ_k, defined on the Hilbert space $H^1_{kT} = \{x \in H^1([0, kT]): x(0) = x(kT)\}$ as follows:

$$\phi_k(x) = \int_0^{kT} [\tfrac{1}{2}(\dot{x}(t))^2 - G(x(t)) + f(t)\, x(t)]\, dt. \tag{3}$$

However, the critical points of ϕ_k do not necessarily correspond to periodic solutions of (2) with *minimal* period kT, as can be seen from the case $g \equiv 0$. In fact, in this case the kT-periodic solutions of (2) are of the form

$$x(t) = C_0 - t\frac{1}{kT}\int_0^{kT}\left(\int_0^s f(u)\, du\right) ds + \int_0^t \left(\int_0^s f(u)\, du\right) ds, \tag{4}$$

where $C_0 = x(0)$ can be chosen arbitrarily in $\mathbb{R}$. Because of (1),

$$\frac{1}{kT}\int_0^{kT}\left(\int_0^s f(u)\, du\right) ds = \frac{1}{T}\int_0^T \left(\int_0^s f(u)\, du\right) ds,$$

and then any $x(t)$, as in (4), has in fact period T.

It can be shown, cf. [10–12], that the functional ϕ_k is bounded from below and satisfies the Palais–Smale condition. So ϕ_k always has a minimum. If $g \equiv 0$, the minimum points of ϕ_k are as in (4), where C_0 is an arbitrary real number. In particular, they are not isolated.

Let x_0 be a T-periodic solution of Eq. (2). Define, for λ and t in $\mathbb{R}$, the matrix

$$A_\lambda(t) = \begin{pmatrix} 0 & -1 \\ \lambda + g'(x_0(t)) & 0 \end{pmatrix}$$

and consider the fundamental solution $X_\lambda(t)$ which satisfies

$$\dot{X}_\lambda(t) = A_\lambda(t)\, X_\lambda(t)$$
$$X_\lambda(0) = Id.$$

It is well known (see e.g. [9]) that the eigenvalues $\sigma'_{\lambda,T}$ and $\sigma''_{\lambda,T}$ of $X_\lambda(T)$ have the following properties:

(i) either both $\sigma'_{\lambda,T}$ and $\sigma''_{\lambda,T}$ are in $\mathbb{R}$, or $\sigma'_{\lambda,T} = \bar{\sigma}''_{\lambda,T}$;

(ii) $\sigma'_{\lambda,T} \cdot \sigma''_{\lambda,T} = 1$;

(iii) there exists $\lambda_0 < \lambda_1$ such that the maps $\lambda \mapsto \sigma'_{\lambda,T}$ and $\lambda \mapsto \sigma''_{\lambda,T}$ are continuous and one to one if $\lambda_0 \leq \lambda \leq \lambda_1$. Moreover,

$$0 < \sigma'_{\lambda,T} < 1 < \sigma''_{\lambda,T} \qquad (\lambda < \lambda_0),$$
$$\sigma'_{\lambda,T} = \bar{\sigma}''_{\lambda,T} \in S^1 \qquad (\lambda_0 \leq \lambda \leq \lambda_1).$$

The T-periodic solution x_0 is said to be nondegenerate if $1 \notin \{\sigma'_{0,T}, \sigma''_{0,T}\}$.

Given $\sigma \in S^1$, we define $J(x_0, T, \sigma)$ to be the number of negative λ's for which $\sigma \in \{\sigma'_{\lambda,T}, \sigma''_{\lambda,T}\}$. The number $J(x_0, T, 1)$ is then the Morse index of the T-periodic solution x_0.

We are now able to formulate our main result.

THEOREM 1. *Assume the following conditions:*

(a) *the T-periodic solutions of Eq.* (2) *are isolated;*

(b) *every T-periodic solution of* (2) *having Morse index equal to zero is nondegenerate.*

Then there exists a $k_0 \geq 2$ such that, for every prime integer $k \geq k_0$, there is a periodic solution of (2) *with minimal period kT.*

Remarks. (1) We have seen above that in the case $g \equiv 0$ there are no subharmonic solutions of (2), and the T-periodic solutions are not isolated, and therefore degenerate. So neither (a) nor (b) is verified in this case.

(2) In [6], Conley and Zehnder proved the existence of subharmonic solutions for a system with Hamiltonian function periodic in each of its variables. They showed that when all the T-periodic solutions, together with their iterates, are nondegenerate, then there exists a periodic solution with minimal period kT if k is a sufficiently large prime number.

We do not need to assume, as in [6], that also the iterates of the T-periodic solutions of (2) are nondegenerate. Since for a T-periodic solution x_0 one has $\sigma'_{\lambda,kT}=(\sigma'_{\lambda,T})^k$ and $\sigma''_{\lambda,kT}=(\sigma''_{\lambda,T})^k$, it could then happen in principle that $1\in\{\sigma'_{\lambda,kT},\sigma''_{\lambda,kT}\}$ even if $1\notin\{\sigma'_{\lambda,T},\sigma''_{\lambda,T}\}$.

Proof of Theorem 1. Let us introduce the Hilbert space

$$\tilde{H}_k=\left\{\tilde{x}\in H^1_{kT}:\int_0^{kT}\tilde{x}(t)\,dt=0\right\}.$$

By (1) and the 2π-periodicity of G, we have that

$$\phi_k(x+2\pi)=\phi_k(x)$$

for every $x\in H^1_{kT}$. Set $S^1=\mathbb{R}/(2\pi\mathbb{Z})$. It is then equivalent to consider the functional ψ_k defined on $S^1\times\tilde{H}_k$ by

$$\psi_k(x)=\phi_k(\bar{x}+\tilde{x})$$

for every $x=(\bar{x},\tilde{x})\in S^1\times\tilde{H}_k$. The functionals ψ_k are bounded from below and satisfy the Palais Smale condition (cf. [10 12]). By assumption (a), the functional ψ_1 has only a finite number of critical points $x_0,\ldots,x_n$. It is clear that the functions x_i $(0\leqslant i\leqslant n)$, extended by T-periodicity on $[0,kT]$, are also critical points of ψ_k for $k\geqslant 2$.

We now assert the following.

Claim. There exists an integer k_0 such that, for $k\geqslant k_0$ and $0\leqslant i\leqslant n$, either $J(x_i,kT,1)=0$ and x_i is nondegenerate, or $J(x_i,kT,1)\geqslant 2$.

Assume for the moment that the above Claim holds true. In case $k\geqslant k_0$ is a prime number, since f has minimal period T, the critical points of ψ_k have as minimal period either T or kT. Assume by contradiction that $x_0,\ldots,x_n$ are the oly critical points of ψ_k. Since the Poincaré polynomial of $S^1\times\tilde{H}_k$ is $(1+t)$, we have

$$\sum_{i=0}^{n}P_k(t,x_i)=(1+t)[1+Q(t)],\qquad (5)$$

where $Q(t)$ is a polynomial with nonnegative integer coefficients and $P_k(t,x_i)=\sum_j\dim C_j(\psi_k,x_i)t^j$ is the usual Morse polynomial of x_i (see e.g. [12]). By the Claim, if $J(x_i,kT,1)=0$, then $P_k(t,x_i)=1$. Otherwise, if $J(x_i,kT,1)\geqslant 2$, then $\dim C_j(\psi_k,x_i)=0$ for $j=0,1$. This implies that Eq. (5) can never be satisfied, and we have a contradiction.

To conclude the proof of the theorem we need then to prove the above Claim. In order to do so, let x_i be a critical point of ψ_1 and let $\lambda_0<\lambda_1$ be as in property (iii). First of all, we claim that $\lambda_0\neq 0$. Indeed, if on the contrary $\lambda_0=0$, we would have, for every negative λ, $0<\sigma'_{\lambda,T}<1<\sigma''_{\lambda,T}$, which implies $J(x_i,T,1)=0$. On the other hand, by (iii), $\sigma'_{0,T}=1=\sigma''_{0,T}$, so that x_i would be a degenerate T-periodic solution with Morse index equal to zero, in contradiction with assumption (b).

Suppose $\lambda_0>0$. Then, for every $\lambda\leqslant 0$, we have $0<\sigma'_{\lambda,T}<1<\sigma''_{\lambda,T}$ and hence $J(x_i,T,\sigma)=0$ for every $\sigma\in S^1$. By [4, Theorem 1] we have

$$J(x_i,kT,1)=\sum_{\sigma^k=1}J(x_i,T,\sigma)=0.$$

Moreover x_i, as a critical point of ψ_k, is also nondegenerate, since

$$0<\sigma'_{0,kT}=(\sigma'_{0,T})^k<1<\sigma''_{0,kT}=(\sigma''_{0,T})^k.$$

Suppose now $\lambda_0<0$. Then for every $\lambda\in\,]\lambda_0,\lambda_0+\varepsilon[$, for $\varepsilon>0$ small enough, we have $\sigma'_{\lambda,T}=\bar{\sigma}''_{\lambda,T}\in S^1$ and

$$J(x_i,T,\sigma'_{\lambda,T})=J(x_i,T,\sigma''_{\lambda,T})>0.$$

Hence, for k large enough, we have

$$J(x_i,kT,1)=\sum_{\sigma^k=1}J(x_i,T,\sigma)\geqslant 2.$$

This proves the Claim, and completes the proof of Theorem 1.

Under a stronger assumption, in the following theorem we will obtain the existence of *two* subharmonic oscillations.

THEOREM 2. *Suppose that the kT-periodic solutions of* (2) *are nondegenerate for $k=1$ and for every prime integer k. Then there exists $k_0\geqslant 3$ such that, for every prime integer $k\geqslant k_0$, there are two geometrically distinct periodic solutions of* (2) *with minimal period kT.*

Proof. As a consequence of the assumption, for every prime number k, the number n_k of critical points of ψ_k is finite. Since the Poincaré polynomial of $S^1\times\tilde{H}_k$ is $(1+t)$, n_k must be even. It follows from Theorem 1 that, for $k\geqslant k_0$, $n_k\geqslant n_1+k$. Then $n_k\geqslant n_1+k$, and the proof is complete.

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SEMICOERCIVE VARIATIONAL PROBLEMS AT RESONANCE: AN ABSTRACT APPROACH*

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Abstract. We study the coercivity of functionals of the form $a+b$ where a is semicoercive with respect to a subspace and b is coercive on the complementary subspace. Applications are given to the existence of solutions for a semilinear Dirichlet problem.

1. Introduction. This paper is concerned with the existence of solutions to elliptic boundary value problems at resonance with the first eigenvalue. We consider the Dirichlet problem

$$-\Delta u - \lambda_1 u + g(x,u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{P}$$

where Ω is a bounded open subset of $\mathbf{R}^N$, and λ_1 the first eigenvalue of $(-\Delta)$ on $\mathrm{H}_0^1(\Omega)$. The Caratheodory function $g : \Omega \times \mathbf{R} \to \mathbf{R}$ is supposed to satisfy the usual growth condition

$$|g(x,u)| \le a|u|^{q-1} + b(x)$$

where $q < \infty$ if $N = 2$, $q < 2^* = 2N/(N-2)$ if $N \ge 3$, and where $b(x) \in L^{q'}(\Omega)$, with q' the Hölder conjugate exponent of q; if $N = 1$, it suffices to assume that for any $r > 0$,

$$\sup_{|u|\le r} |g(x,u)| \in L^1(\Omega).$$

Under this condition, the associated functional

$$f(u) = \tfrac{1}{2}\int_\Omega \left[|\nabla u|^2 - \lambda_1 |u|^2\right] + \int_\Omega G(x,u(x))\,dx,$$

where $G(x,u) = \int_0^u g(x,s)\,ds$, is a weakly lower semicontinuous $\mathbf{C}^1$ functional on H_0^1 whose critical points are the weak solutions of (P). It follows that if f is coercive (i.e., $f(u) \to +\infty$

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as $\|u\| \to \infty$ in H_0^1), then f has a minimum and consequently (P) has a solution. We are mainly interested in this paper in the conditions which guarantee the coercivity of f.

The functional $f(u)$ can be written as the sum of a quadratic term

$$a(u) = \tfrac{1}{2}\int_\Omega \left[|\nabla u|^2 - \lambda_1 |u|^2\right]$$

coming from the linear part of the equation, and a term

$$b(u) = \int_\Omega G(x,u(x))\,dx$$

coming from the nonlinear part. Denoting by $\overline{\mathrm{H}}$ the space spanned by the first (positive) eigenfunction ϕ_1 of $(-\Delta)$ on H_0^1, we see that a necessary condition for f to be coercive on H_0^1 is that f be coercive on $\overline{\mathrm{H}}$; i.e., b be coercive on $\overline{\mathrm{H}}$. A condition of this type was first considered by Ahmad, Lazer and Paul [1] in a slightly different situation. One can ask whether this condition is also sufficient for the coercivity of f, or at least for the existence of a solution to (P). The answer to this question is negative, as is shown by the following.

Example 1. Consider the one-dimensional linear problem

$$u'' + u = s(x)u + h(x) \quad \text{on }]0,\pi[, \quad u(0) = 0 = u(\pi). \tag{Q}$$

The associated functional is

$$f(u) = \int_0^\pi \left[\tfrac{1}{2}(u')^2 - \tfrac{1}{2}(1 - s(x))u^2 + h(x)u\right].$$

The coefficient $s(x)$ in (Q) is defined as follows: first choose $C > 0$ in such a way that the graphs of the two functions

$$u_1(x) = C(e^{2x} - e^{-2x}), \quad u_2(x) = -\sin(2x)$$

be tangent one to the other at a certain point $\bar{x} \in]\frac{\pi}{2}, \pi[$; then take

$$s(x) = \begin{cases} 5 & \text{if } x \in [0,\bar{x}[\\ -3 & \text{if } x \in [\bar{x},\pi]. \end{cases}$$

It follows that the corresponding homogeneous problem

$$u'' + u = s(x)u, \quad u(0) = 0 = u(\pi)$$

has a nontrivial solution given by

$$u_0(x) = \begin{cases} u_1(x) & \text{if } x \in [0,\bar{x}] \\ u_2(x) & \text{if } x \in [\bar{x},\pi]. \end{cases}$$

This implies that if $\int_0^\pi h u_0 \ne 0$ then problem (Q) has no solution. For such h's the functional f can not be coercive; actually it is easily seen directly that f is not coercive on the line $\mathbf{R}u_0$. Nevertheless, f is coercive on $\overline{\mathrm{H}}$. Indeed

$$f(r\sin x) = (r^2/2)\int_0^\pi s(x)\sin^2 x\,dx + r\int_0^\pi h(x)\sin x\,dx$$

and since

$$\int_0^\pi s(x)\sin^2 x\,dx = \int_0^{\bar{x}} 5\sin^2 x\,dx + \int_{\bar{x}}^\pi -3\sin^2 x\,dx > 0,$$

one has $f(r\sin x) \to +\infty$ as $|r| \to \infty$.

In order to try to understand what can go wrong with the coercivity of f when b is coercive on $\overline{\mathbf{H}}$, let us write the orthogonal decomposition $\mathbf{H}_0^1 = \overline{\mathbf{H}} \oplus \widetilde{\mathbf{H}}$, where $\widetilde{\mathbf{H}}$ is generated by the higher eigenfunctions of $(-\Delta)$ on $\mathbf{H}_0^1$, and for every $u \in \mathbf{H}_0^1$, let $u = \bar{u} + \tilde{u}$, with $\bar{u} \in \overline{\mathbf{H}}$, $\tilde{u} \in \widetilde{\mathbf{H}}$. Taking Fourier's expansions, it is easily seen that the functional a is coercive on $\widetilde{\mathbf{H}}$; more precisely, there exists $\delta > 0$ such that

$$a(u) \geq \delta\|\tilde{u}\|^2$$

for every $u \in \mathbf{H}_0^1$. One reason why this semi-coercivity of a on $\mathbf{H}_0^1$ together with a coercivity assumption of b on $\overline{\mathbf{H}}$ do not necessarily imply the coercivity of f on $\mathbf{H}_0^1$ is that b may decrease too rapidly outside $\overline{\mathbf{H}}$. This phenomenon is apparent in the above example where one has $b(ru_0) \to -\infty$ with speed r^2 as $|r| \to \infty$ (since $b(ru_0) = -a(ru_0) + r\int_0^\pi hu_0$).

This suggests to impose some control on the decreasing speed of b outside $\overline{\mathbf{H}}$. One can also reinforce the coercivity of b on $\overline{\mathbf{H}}$ and look for a compromise with the decreasing speed of b outside $\overline{\mathbf{H}}$. These ideas are the content of condition (iii) of our abstract theorem in section 2. A particular case of this theorem deals with the situation where b is Lipschitz continuous (see Remark 2). We provide in this way an abstract explanation for the use in [8] of a boundedness assumption on the nonlinearity $g(x,u)$. this particular case can also be used to recover a theorem of Mawhin [9] relative to the situation where b is convex (see Corollary 1).

Beginning with Hammerstein [7], several papers have been concerned with the study of problem (P) under conditions on the asymptotic behaviour of the quotient $2G(x,u)/u^2$. In [10], Mawhin, Ward and Willem proved an existence result by assuming

$$\liminf_{|u|\to\infty} 2G(x,u)/u^2 \geq 0 \tag{1}$$

for almost every $x \in \Omega$, the inequality being strict on a set of positive measure. In [5], de Figueiredo and Gossez considered the case in which equality holds in (1) for almost every $x \in \Omega$ and proposed a so called density condition in order to obtain existence. Conditions on the quotient $G(x,u)/|u|^p$ for $1 \leq p < 2$ were also condidered by Anane in [2] (see also [3]).

In section 3, we show how these results can all be recovered and sometimes improved or generalized by using our abstract theorem. Beside unification, the present approach provides a new insight to the role of some assumptions in the above mentioned results; this is particularly apparent for the linear growth restriction imposed in [5] on the nonlinearity $g(x,u)$. We also consider the limiting situation where $p = 0$ in the quotient $G(x,u)/|u|^p$, in which case the functional may not be coercive. Our result here is related to some recent work of Ramos and Sanchez [11] and provides an improvement of a theorem of Berger (cf. [4], [8]).

We finally observe that most of our results can be adapted to systems of ordinary differential equations or to other types of boundary conditions, like the Neumann conditions, or, for ODE's, the periodic conditions.

2. An abstract theorem.

In this section we study the coercivity of functionals of the form $f = a + b$, where a is semicoercive with respect to a subspace and b is coercive on a complementary subspace.

Theorem 1. *Let* $\mathbf{H}$ *be a normed space,* $\mathbf{H} = \overline{\mathbf{H}} \oplus \widetilde{\mathbf{H}}$, *and for any* $u \in \mathbf{H}$, *write* $u = \bar{u} + \tilde{u}$, *with* $\bar{u} \in \overline{\mathbf{H}}$, $\tilde{u} \in \widetilde{\mathbf{H}}$. *Let* $a, b : \mathbf{H} \to \mathbf{R}$ *be two functionals satisfying the following properties:*

(i) *there exists* $\delta > 0$ *such that* $a(u) \geq \delta\|\tilde{u}\|^2$ *for every* $u \in \mathbf{H}$;

(ii) $\liminf_{\|u\|\to\infty} \frac{b(u)}{\|u\|^2} \geq 0$;

(iii) *there exist a functional* $\hat{b} : \mathbf{H} \to \mathbf{R}$, $\hat{b} \leq b$, *and* $\beta \geq 1$ *such that*

$$|\hat{b}(u) - \hat{b}(w)| \leq \|u - w\|[A(\|u\| + \|w\|)^{\beta-1} + B] \tag{2}$$

for every $u, w \in \mathbf{H}$, *and*

$$\lim_{\substack{\|\bar{u}\|\to\infty \\ \bar{u}\in\overline{\mathbf{H}}}} \frac{\hat{b}(\bar{u})}{\|\bar{u}\|^{2(\beta-1)}} = +\infty \tag{3}$$

if $1 \leq \beta < 2$, *or*

$$\liminf_{\substack{\|\bar{u}\|\to\infty \\ \bar{u}\in\overline{\mathbf{H}}}} \frac{\hat{b}(\bar{u})}{\|\bar{u}\|^{\beta}} > 0 \tag{4}$$

if $\beta \geq 2$.

Then the functional $f = a + b$ *is coercive on* $\mathbf{H}$.

Remark 1. When $1 \leq \beta < 2$, property (ii) is a consequence of (2); moreover one has $2(\beta - 1) < \beta$ and so (3) is a less restrictive requirement than (4). On the contrary when $\beta \geq 2$, (4) becomes less restrictive than (3). If $\hat{b}$ is differentiable, it is easily seen that (2) is equivalent to the following growth condition on $\hat{b}'$:

$$\|\hat{b}'(u)\| \leq A'\|u\|^{\beta-1} + B'. \tag{2'}$$

Remark 2. In the case $\beta = 1$, condition (iii) becomes:

(iii′) there exists $\hat{b} : \mathbf{H} \to \mathbf{R}$ such that $\hat{b} \leq b$ on $\mathbf{H}$, $\hat{b}$ is coercive on $\overline{\mathbf{H}}$ and Lipschitz continuous on $\mathbf{H}$.

Proof of Theorem 1: Assume by contradiction that there exists a sequence (u_n) in $\mathbf{H}$ and a real constant C such that $\|u_n\| \to \infty$ and $f(u_n) \leq C$ for every n. Then, by (i),

$$\delta\frac{\|\tilde{u}_n\|^2}{\|u_n\|^2} + \frac{b(u_n)}{\|u_n\|^2} \leq \frac{C}{\|u_n\|^2},$$

which by (ii) implies that $\|\tilde{u}_n\|/\|u_n\| \to 0$. As a consequence,

$$\liminf \|\bar{u}_n\|/\|u_n\| > 0, \tag{5}$$

and, in particular, $\|\bar{u}_n\| \to \infty$.

Assume that (3) holds. By (iii),

$$\begin{aligned} f(u_n) &\geq \delta\|\tilde{u}_n\|^2 + \hat{b}(u_n) - \hat{b}(u_n) + \hat{b}(\bar{u}_n) \\ &\geq \delta\|\tilde{u}_n\|^2 - \|\tilde{u}_n\|[A(\|u_n\| + \|\bar{u}_n\|)^{\beta-1} + B] + \hat{b}(\bar{u}_n) \\ &\geq \delta\|\tilde{u}_n\|^2 - B\|\tilde{u}_n\| - \frac{\delta}{2}\|\tilde{u}_n\|^2 - \frac{A^2}{2\delta}(\|u_n\| + \|\bar{u}_n\|)^{2(\beta-1)} + \hat{b}(\bar{u}_n) \\ &\geq \left[\frac{\delta}{2}\|\tilde{u}_n\|^2 - B\|\tilde{u}_n\|\right] + \left[\frac{\hat{b}(\bar{u}_n)}{\|\bar{u}_n\|^{2(\beta-1)}} - \frac{A^2}{2\delta}\left(\frac{\|u_n\|}{\|\bar{u}_n\|} + 1\right)^{2(\beta-1)}\right]\|\bar{u}_n\|^{2(\beta-1)}, \end{aligned} \tag{6}$$

which by (5) and (3) goes to $+\infty$ as $n \to \infty$, in contradiction with our assumption.

Assume now that (4) holds. By (iii),

$$\left|\frac{\hat{b}(u_n)}{\|\tilde{u}_n\|^\beta} - \frac{\hat{b}(\tilde{u}_n)}{\|\tilde{u}_n\|^\beta}\right| \le \frac{\|\bar{u}_n\|}{\|\tilde{u}_n\|}\left[A\left(\frac{\|u_n\|}{\|\tilde{u}_n\|}+1\right)^{\beta-1} + \frac{B}{\|\tilde{u}_n\|^{\beta-1}}\right],$$

which goes to 0, by (5). Consequently, by (4),

$$\liminf \frac{\hat{b}(u_n)}{\|\tilde{u}_n\|^\beta} > 0,$$

which implies $\hat{b}(u_n) \to +\infty$ and so also $f(u_n) \to +\infty$, a contradiction. The theorem is thus proved.

Remark 3. When $\beta = 1$, condition (i) of Theorem 1 can be weakened to the following:

(i)′ there exists $\alpha : \mathbf{R}_+ \to \mathbf{R}$, bounded below, with $\alpha(t) \to +\infty$ as $t \to +\infty$, such that

$$a(u) \ge \alpha(\|\tilde{u}\|)\|\tilde{u}\|$$

for every $u \in \mathbf{H}$.

Indeed assume by contradiction that there exists a sequence (u_n) in $\mathbf{H}$ and a constant C such that $\|u_n\| \to \infty$ and $f(u_n) \le C$ for every n. Then

$$\frac{\alpha(\|\tilde{u}_n\|)\|\tilde{u}_n\|}{\alpha(\|u_n\|)\|u_n\|} + \frac{\hat{b}(u_n)}{\alpha(\|u_n\|)\|u_n\|} \le \frac{C}{\alpha(\|u_n\|)\|u_n\|}.$$

Since $\hat{b}$ is Lipschitz continuous, this implies

$$\frac{\alpha(\|\tilde{u}_n\|)\|\tilde{u}_n\|}{\alpha(\|u_n\|)\|u_n\|} \to 0. \tag{7}$$

It is not restrictive to assume α increasing and concave (cf. [6]). As a consequence, $\|\tilde{u}_n\| \to \infty$. Indeed, if this is not true, there exists a subsequence, still denoted (u_n), such that $\|\bar{u}_n\|$ is bounded. Then

$$\liminf \frac{\|\tilde{u}_n\|}{\|u_n\|} \ge 1 \tag{8}$$

and so, for n large enough,

$$\liminf \frac{\alpha(\|\tilde{u}_n\|)}{\alpha(\|u_n\|)} \ge \liminf \frac{\alpha(\frac{1}{2}\|u_n\|)}{\alpha(\|u_n\|)} > 0 \tag{9}$$

since $\alpha(2t) \le K\alpha(t)$ for a certain $K > 0$ and all t sufficiently large. A contradiction with (7) then follows from (8) and (9). Now, if ℓ is the Lipschitz constant of $\hat{b}$, one has

$$\begin{aligned} f(u_n) &\ge \alpha(\|\tilde{u}_n\|)\|\tilde{u}_n\| + \hat{b}(u_n) - \hat{b}(\tilde{u}_n) + \hat{b}(\tilde{u}_n) \\ &\ge [\alpha(\|\tilde{u}_n\|) - \ell]\|\tilde{u}_n\| + \hat{b}(\tilde{u}_n). \end{aligned}$$

Since the first term in the above sum is bounded below and $\hat{b}(\tilde{u}_n) \to +\infty$, we have that $f(u_n) \to +\infty$, a contradiction.

We deduce from Theorem 1 with $\beta = 1$ (actually from Remark 3) the following result which contains a theorem of Mawhin [9] obtained by convex analysis methods.

Corollary 1. *Let* $\mathbf{H}$ *be as in Theorem 1,* $a : \mathbf{H} \to \mathbf{R}$ *a functional satisfying property (i)′ and suppose that* $b : \mathbf{H} \to \mathbf{R}$ *satisfies* $b \ge \hat{b}$, *with* $\hat{b} : \mathbf{H} \to \mathbf{R}$ *a convex and continuous functional which is coercive on* $\overline{\mathbf{H}}$. *Then* $f = a + b$ *is coercive on* $\mathbf{H}$.

Proof: By Remark 3, it suffices to show that there exists $\hat{\hat{b}} : \mathbf{H} \to \mathbf{R}$ with $\hat{\hat{b}} \le \hat{b}$ on H, $\hat{\hat{b}}$ coercive on $\overline{\mathbf{H}}$ and Lipschitz continuous on $\mathbf{H}$. Since $\hat{b}$ is convex continuous and coercive on $\overline{\mathbf{H}}$, it is easily seen that there exist constants $C_1 > 0$ and C_2 such that

$$\hat{b}(\bar{u}) \ge C_1\|\bar{u}\| - C_2$$

for every $\bar{u} \in \overline{\mathbf{H}}$. Moreover, adding a constant to $\hat{b}$, we can always assume $C_2 = 0$. For every $\bar{u} \in \overline{\mathbf{H}}$, define a linear functional $L_{\bar{u}}$ on the one-dimensional space $\mathbf{R}\bar{u}$ as follows: $L_{\bar{u}}(t\bar{u}) = C_1 t\|\bar{u}\|$. By the Hahn-Banach theorem and the continuity of $\hat{b}$, $L_{\bar{u}}$ can be extended on the whole space $\mathbf{H}$ into a continuous linear functional $L'_{\bar{u}}$ such that $L'_{\bar{u}}(w) \le \hat{b}(w)$ for every $w \in \mathbf{H}$. Define

$$\hat{\hat{b}}(w) = \sup_{\bar{u} \in \overline{\mathbf{H}}} L'_{\bar{u}}(w).$$

Clearly $\hat{\hat{b}} \le \hat{b}$ on $\mathbf{H}$; moreover, $\hat{\hat{b}}(\bar{u}) \ge C_1\|\bar{u}\|$ on $\overline{\mathbf{H}}$, and so $\hat{\hat{b}}$ is coercive on $\overline{\mathbf{H}}$. It remains to see that $\hat{\hat{b}}$ is Lipschitz continuous on $\mathbf{H}$. The continuity of $\hat{b}$ implies that the linear functionals $L'_{\bar{u}}$, $\bar{u} \in \overline{\mathbf{H}}$, remain bounded, say $\|L'_{\bar{u}}\| \le M$. Given $v, w \in \mathbf{H}$ and $\epsilon > 0$, take $\bar{u} \in \overline{\mathbf{H}}$ such that $\hat{\hat{b}}(v) - \epsilon \le L'_{\bar{u}}(v)$. Adding and substracting $L'_{\bar{u}}(w)$ and letting $\epsilon \to 0$, one gets $\hat{\hat{b}}(v) - \hat{\hat{b}}(w) \le M\|v - w\|$, which yields the conclusion.

Remark 4. If

$$|b(u) - \hat{b}(u)| \le C$$

for some constant C and all $u \in \overline{\mathbf{H}}$, then the coercivity requirement on $\hat{b}$ in the above results is equivalent to an analogous requirement on b.

3. Applications to problem (P).

As an immediate consequence of Remark 2, Corollary 1 and Remark 4, we have the following result (which could also be deduced from the arguments in [8]).

Theorem 2. *Assume that*

$$\lim_{|r| \to \infty} \int_\Omega G(x, r\phi_1(x))\,dx = +\infty. \tag{10}$$

If moreover there exists a Caratheodory function $\widehat{G}(x,u)$ *which is either convex in* u *or Lipschitzian in* u *with a Lipschitz constant independent of* x, *and which satisfies*

$$|G(x,u) - \widehat{G}(x,u)| \le c(x)$$

for some function $c(x) \in \mathbf{L}^1(\Omega)$, *then problem (P) has a solution.*

Theorem 2 is an application of Theorem 1 with the choice $\beta = 1$. The following result deals with a case where $\beta \in [1, 2[$.

Theorem 3. *Let $0 < p < 2$ and $c(x) \in \mathbf{L}^s(\Omega)$ (with $s = N/2$ if $N \geq 3$, $s > 1$ if $N = 2$ and $s = 1$ if $N = 1$) be such that*

$$\liminf_{|u| \to \infty} G(x,u)/|u|^p \geq c(x) \tag{11}$$

uniformly for almost every $x \in \Omega$. If

$$\int_\Omega c(x)\phi_1(x)^p\,dx > 0, \tag{12}$$

then problem (P) has a solution.

Proof: Fix $\epsilon > 0$ as follows:

$$\epsilon < \int_\Omega c(x)\phi_1(x)^p\,dx \Big(\int_\Omega \phi_1(x)^p\,dx \Big)^{-1}.$$

There exists $k_\epsilon \in \mathbf{L}(\Omega)$ such that

$$G(x,u) \geq (c(x) - \epsilon)|u|^p - k_\epsilon(x).$$

Suppose first $1 < p < 2$. Set in this case

$$\hat{b}(u) = \int_\Omega [(c(x) - \epsilon)|u(x)|^p - k_\epsilon(x)]\,dx.$$

Then

$$|\hat{b}'(u)v| \leq \int_\Omega |c(x) - \epsilon|p|u(x)|^{p-1}|v(x)|\,dx \leq C\|u\|^{p-1}\|v\|,$$

for a certain $C > 0$ and every $v \in \mathbf{H}_0^1$, so that $\|\hat{b}'(u)\| \leq C\|u\|^{p-1}$. This implies (2) with $\beta = p$, as was observed in Remark 1. Moreover

$$\hat{b}(r\phi_1) \geq |r|^p \int_\Omega (c(x) - \epsilon)\phi_1(x)^p\,dx - \|k_\epsilon\|_{\mathbf{L}^1}.$$

By the choice of ϵ, $\lim_{|r| \to \infty} \hat{b}(r\phi_1)/|r|^p > 0$, and consequently (4) is verified. So Theorem 1 with $\beta = p$ implies the result in this case.

Suppose now $0 < p \leq 1$. Define $\widehat{G}(x,u)$ as follows:

$$\widehat{G}(x,u) = \begin{cases} (c(x) - \epsilon)|u|^p - |c(x) - \epsilon| - k_\epsilon(x) & \text{if } |u| \geq 1 \\ (c(x) - \epsilon) - |c(x) - \epsilon| - k_\epsilon(x) & \text{if } |u| \leq 1. \end{cases}$$

Then $\widehat{G}(x,u)$ is a Caratheodory function, which is Lipschitz continuous in u, with a Lipschitz constant equal to $|c(x) - \epsilon|$. Moreover, $\widehat{G} \leq G$, and defining $\hat{b}(u) = \int_\Omega \widehat{G}(x,u(x))\,dx$, one has $\hat{b}$ is Lipschitz continuous and

$$\begin{aligned} \hat{b}(r\phi_1) &\geq \int_\Omega [(c(x) - \epsilon)|r\phi_1(x)|^p - 2|c(x) - \epsilon| - k_\epsilon(x)]\,dx \\ &\geq |r|^p \int_\Omega (c(x) - \epsilon)\phi_1(x)^p\,dx - 2\|c(\cdot) - \epsilon\|_{\mathbf{L}^1} - \|k_\epsilon\|_{\mathbf{L}^1}. \end{aligned}$$

So, $\hat{b}(r\phi_1) \to +\infty$ as $|r| \to \infty$, and the conclusion follows from Theorem 1 with $\beta = 1$. ∎

A result analogous to Theorem 3 for $1 \leq p < 2$ was obtained in [2, 3]. The limiting case $p = 0$ will be considered at the end of this section.

As can be seen from Example 1 given in the Introduction, Theorem 3 is not true when $p = 2$. In this case one has to reinforce condition (12). This is done in the following theorem. (The fact that the condition on $c(x)$ below implies (12) with $p = 2$ follows easily from the variational characterization of the first eigenvalue of an elliptic operator.)

Theorem 4. *Let $c(x)$ be such that*

$$\liminf_{|u| \to \infty} 2G(x,u)/u^2 \geq c(x) \tag{13}$$

uniformly for almost every $x \in \Omega$, with $c \in \mathbf{L}^{N/2}(\Omega)$ if $N \geq 3$, $c \in \mathbf{L}^r(\Omega)$ for some $r > 1$ if $N = 2$, $c \in \mathbf{L}^1(\Omega)$ if $N = 1$. Suppose that the first eigenvalue of the operator $Lu = -\Delta u - \lambda_1 u + c(.)u$, with Dirichlet boundary conditions, is positive. Then problem (P) has a solution.

Proof: Let μ_1 be the first eigenvalue of L. Problem (P) can then be written as

$$-Lu + \mu_1 u = m(x,u),$$

where $m(x,u) = g(x,u) + (\mu_1 - c(x))u$. Denoting by M the primitive of m with respect to u, we have

$$\liminf_{|u| \to \infty} 2M(x,u)/u^2 \geq \mu_1$$

uniformly for almost every $x \in \Omega$. So, there is a $k \in \mathbf{L}^1(\Omega)$ such that

$$M(x,u) \geq (\mu_1/4)u^2 - k(x). \tag{14}$$

Let us now decompose $\mathbf{H}_0^1$ according to the eigenfunctions of L (instead of those of $-\Delta$). We set $\overline{\mathbf{H}}$ to be the space spanned by the first eigenfunction of L and we define

$$a(u) = \tfrac{1}{2}[(Lu,u)_{\mathbf{L}^2} - \mu_1|u|^2_{\mathbf{L}^2}], \quad b(u) = \int_\Omega M(x,u(x))\,dx.$$

The result then follows from Corollary 1 since, by (14), b is minorized by a convex continuous functional which is coercive on $\overline{\mathbf{H}}$. ∎

As a consequence of Theorem 4, we have the following

Corollary 2. (Mawhin, Ward and Willem [10]). *Let $c \in \mathbf{L}^\infty(\Omega)$ be such that (13) holds uniformly for almost every $x \in \Omega$. If $c(x) \geq 0$ for almost every $x \in \Omega$, with strict inequality on a set of positive measure, then problem (P) has a solution.*

Proof: Let L be the operator defined in Theorem 4 and let μ_1 and Ψ_1 be the first eigenvalue and the corresponding normalized eigenfunction of L. By the variational characterization of the first eigenvalue of an elliptic operator and Poincaré's inequality, we have:

$$\mu_1 = \int_\Omega \big[|\nabla\Psi_1|^2 - (\lambda_1 - c(x))|\Psi_1|^2\big] > \int_\Omega \big[|\nabla\Psi_1|^2 - \lambda_1|\Psi_1|^2\big] \geq 0.$$

Hence we are in the situation of Theorem 4. ∎

The following example, which is a variation of Example 1, shows that a function $c(x)$ may satisfy the assumption of Theorem 4 without being nonnegative almost everywhere. Theorem 4 applies to problem (P) with $g(x,u) = c(x)u + h(x)$, while Corollary 2 does not.

Example 2. Choose $C > 0$ such that the graphs of the two functions

$$u_1(x) = C(e^{\sqrt{3}x} - e^{-\sqrt{3}x}), \quad u_2(x) = -\sin(\sqrt{5}(x - \pi))$$

be tangent one to the other at a certain point $\bar{x} \in [0,\pi]$. We define

$$c(x) = \begin{cases} 5 & \text{if } x \in [0,\bar{x}[\\ -3 & \text{if } x \in [\bar{x},\pi]. \end{cases}$$

Then the first eigenvalue of the operator $Lu = -u'' - u + c(x)u$ on $H_0^1(0,\pi)$ is 1 and the corresponding (positive) eigenfunction is

$$u_0(x) = \begin{cases} u_1(x) & \text{if } x \in [0,\bar{x}], \\ u_2(x) & \text{if } x \in [\bar{x},\pi]. \end{cases}$$

We will now apply Theorem 1 with β possibly greater than 2 to a situation where $c(x)$ in (13) may be identically equal to zero. Let us first recall the following

Definition 1. *Let E be a measurable subset of $\mathbf{R}$ and $v \in [0,1[$. We say that E has a positive v-density at $+\infty$ if*

$$\liminf_{r\to+\infty} \frac{m_1(E \cap [vr,r])}{m_1([vr,r])} > 0,$$

where m_1 is the Lebesgue measure on $\mathbf{R}$. An analogous definition can be given at $-\infty$.

The following two theorems provide some improvement to results in [5]. For $\beta = 2$ they could also be derived from the arguments in [5].

Theorem 5. *Assume that there exist $\beta \geq 1$, with $\beta \leq 2^*$ if $N \geq 3$, and a Caratheodory function $\hat{g} : \Omega \times \mathbf{R} \to \mathbf{R}$ such that, if $\widehat{G}(x,u) = \int_0^u \hat{g}(x,s)\,ds$, one has $\widehat{G} \leq G$ and*

(a) $\liminf_{|u|\to\infty} \frac{\widehat{G}(x,u)}{|u|^\sigma} \geq 0$ *uniformly for almost every $x \in \Omega$, where $\sigma = \min\{\beta,2\}$;*

(b) *theere exist a constant A and a function $B(x) \in L^{q'}(\Omega)$ such that*

$$|\hat{g}(x,u)| \leq A|u|^{\beta-1} + B(x) \tag{15}$$

for almost every $x \in \Omega$ and all $u \in \mathbf{R}$ (here q' is the Hölder conjugate exponent of the exponent q considered in the introduction; if $N = 1$, then $q' = 1$);

(c) *there exists a full subset $\Omega' \subset \Omega$ (i.e., $\Omega \setminus \Omega'$ of zero mesure) and $\eta > 0$ such that the set*

$$E_\eta = \bigcap_{x\in\Omega'} \{u \in \mathbf{R} : \widehat{G}(x,u) \geq \eta|u|^\beta\}$$

has a positive 0-density at both $+\infty$ and $-\infty$.

Then problem (P) has a solution.

Theorem 6. *The conclusion of Theorem 6 remains true if condition (c) is replaced by the following:*

(c′) *there exist ω_+, ω_- open subsets of Ω and corresponding full subsets $\omega'_+ \subset \omega_+$ and $\omega'_- \subset \omega_-$ with the following property: for every $v \in [0,1[$ there exists $\eta > 0$ such that the set*

$$E_\eta^+ = \bigcap_{x\in\omega'_+} \{u \in \mathbf{R} : \widehat{G}(x,u) \geq \eta|u|^\beta\}$$

has a positive v-density at $+\infty$, and the set

$$E_\eta^- = \bigcap_{x\in\omega'_-} \{u \in \mathbf{R} : \widehat{G}(x,u) \geq \eta|u|^\beta\}$$

has a positive v-density at $-\infty$.

The proofs of these two theorems are based, as in [5], on the following lemma.

Lemma 1. (cf. [5]). *Let Ω be an open subset of $\mathbf{R}^N$ with finite Lebesgue measure $m_N(\Omega)$. There exists a constant $c > 0$ such that*

$$m_N(u^{-1}(B)) \geq c\frac{m_1(B)^N}{\ell(u)^N}$$

for any nonconstant Lipschitz continuous function u, with Lipschitz constant $\ell(u)$, which vanishes on $\partial\Omega$ and any Borelian set B in the range of u. The same inequality holds for functions u which do not necessarily vanish on $\partial\Omega$ when Ω is an open parallelepiped.

Proof of Theorem 5: We will apply Theorem 1 with β as above. Condition (ii) follows from (a). Set $\hat{b}(u) = \int_\Omega \widehat{G}(x,u(x))\,dx$. From (15) one can easily deduce (2) (see Remark 1). In order to prove (4), suppose by contradiction that there exists a sequence (r_n) such that $|r_n| \to \infty$ and

$$\lim |r_n|^{-\beta} \int_\Omega \widehat{G}(x,r_n\phi_1(x))\,dx \leq 0. \tag{16}$$

Taking a subsequence if necessary, we have either $r_n \to +\infty$ or $r_n \to -\infty$. Let us consider the first case (the second case is treated similarly). By assumption (c), there exists a positive number γ such that, for n sufficiently large,

$$m_1(E \cap [0,r_n \max\phi_1]) \geq \gamma r_n \max\phi_1.$$

Setting $u = r_n\phi_1$ and $B = E_\eta \cap [0,r_n\max\phi_1]$, we have $u^{-1}(B) \cap \Omega' \subset \{x \in \Omega : \widehat{G}(x,r_n\phi_1(x)) \geq \eta r_n^\beta \phi_1(x)^\beta\} := F_n$, and consequently, by Lemma 1,

$$m_N(F_n) \geq c\left(\frac{\gamma r_n \max\phi_1}{r_n L}\right)^N := k > 0$$

for n sufficiently large, where L is the Lipschitz constant of ϕ_1. Now, by (16),

$$\lim\left(\int_{F_n} \frac{\widehat{G}(x,r_n\phi_1(x))}{r_n^\beta\phi_1(x)^\beta}\phi_1(x)^\beta\,dx + \int_{\Omega\setminus F_n} \frac{\widehat{G}(x,r_n\phi_1(x))}{r_n^\beta\phi_1(x)^\beta}\phi_1(x)^\beta\,dx\right) \leq 0,$$

and consequently, by Fatou's Lemma together with (a) and the definition of F_n,

$$\eta \liminf \int_{F_n} \phi_1(x)^\beta\,dx \leq 0.$$

This leads to a contradiction since $m_N(F_n) \geq k$ for every n.

Proof of Theorem 6: The proof goes as that of Theorem 5, except for showing that there exists $k > 0$ for which $m_N(F_n) \geq k$ for every n. Assuming without loss of generality that ω_+ is an open parallelepiped and setting

$$u = r_n\phi_1, \quad B = E_\eta^+ \cap [r_n \inf_{\omega_+}\phi_1, r_n\sup_{\omega_+}\phi_1],$$

we get here from Lemma 1 and Definition 1 with $v = \inf_{\omega_+}\phi_1/\sup_{\omega_+}\phi_1$,

$$m_N(F_n) \geq c\left(\frac{\gamma^+ r_n(\sup_{\omega_+}\phi_1 - \inf_{\omega_+}\phi_1)}{r_n L}\right)^N := k > 0$$

for a certain positive constant γ^+ and n sufficiently large.

Remark 5. Reading carefully the above proofs, one easily sees that if $G(x,u) = G_1(x,u) + h(x)u$ with, say, $h(x) \in L^2(\Omega)$, and if G_1 satisfies the assumptions of Theorem 5 or 6 with $\beta > 1$ (i.e., there exists $\widehat{G}_1 \leq G_1 \ldots$), then (P) is solvable.

Theorems 5 and 6 exhibit some kind of compromise between the growth condition on the nonlinearity and the density condition. This goes in the line of the main idea behind our abstract Theorem 1. The linear growth restriction on the nonlinearity imposed in [5] can be partially relaxed in this way, as illustrated by the following

Example 3. Take $\beta > 1$, with $\beta < 2^*$ if $N \geq 3$, and define

$$G(x,u) = |u|^\beta(1 - \sin\log(1+|u|)) + h(x)u$$

where, say, $h(x) \in L^2(\Omega)$. Direct computation shows that Theorem 5 applies (with $\widehat{G} = G$).

To conclude, we show that theorem 3 still holds when $p = 0$. In this case the functional f need not be coercive.

Theorem 7. *Assume that there exists an integrable function $c(x)$ such that*

$$\liminf_{|u|\to\infty} G(x,u) \geq c(x) \tag{17}$$

uniformly for almost every $x \in \Omega$, and

$$\int_\Omega c(x)\,dx > 0.$$

Then problem (P) has a solution.

While this paper was being completed, we learned of a slightly more general result by Ramos and Sanchez [11]. Their proof is based on the verification of the Palais-Smale condition at the level of the infimum of f. The proof below gives some insight of the geometry of the functional. It is based on the following simple lemma.

Lemma 2. *Let $\mathbf{H}$ be a reflexive Banach space and $f : \mathbf{H} \to \mathbf{R}$ be a weakly lower semicontinuous and differentiable functional. Assume that there exists $R > 0$ such that for every u with $\|u\| = R$, one has $f(u) > f(0)$. Then f has a critical point.*

Proof: The restriction of f to $B_R = \{u : \|u\| \leq R\}$ attains its minimum at some point $\bar{u} \in B_R$. By assumption, $\bar{u}$ must be in the interior of B_R, and is thus a local minimum for f on $\mathbf{H}$, hence a critical point.

Proof of Theorem 7: Suppose by contradiction that f has no critical point. Then by Lemma 2 there exists a sequence (u_n) in $\mathbf{H}_0^1$ such that $\|u_n\| = n$ and $f(u_n) \leq f(0)$. By (17), there exists $d \in L^1(\Omega)$ such that $G(x,u) \geq d(x)$ for almost every x and all u. Since

$$f(0) \geq f(u_n) \geq \delta\|\tilde{u}_n\|^2 - \|d\|_{L^1},$$

$\|\tilde{u}_n\|$ is bounded. It follows that $\|\bar{u}_n\| \to \infty$ and that for a subsequence, $\tilde{u}_n(x) \to w(x)$ for almost every x; this implies that $|u_n(x)| \to \infty$ almost everywhere. By Fatou's Lemma,

$$0 \geq \liminf(f(u_n) - f(0)) \geq \liminf \int_\Omega G(x, u_n(x))\,dx$$

$$\geq \int_\Omega \liminf G(x, u_n(x))\,dx \geq \int_\Omega c(x)\,dx > 0,$$

which gives the contradiction.

Remark 6. The uniformity in (17) is used in the proof above only to guarantee the existence of $d \in L^1(\Omega)$ such that $G(x,u) \geq d(x)$ for almost every x and all u.

As an easy consequence of Theorem 7 and Remark 6, we have the following corollary which generalizes in several ways a result of Berger (cf. [4, 8]).

Corollary 3. *Assume that there exists $d \in L^1(\Omega)$ such that*

$$G(x,u) \geq d(x)$$

for almost every x and all u, and a subset Ω_0 of Ω with positive measure such that for almost every $x \in \Omega_0$,

$$G(x,u) \to +\infty \qquad \text{as } |u| \to \infty.$$

Then problem (P) has a solution.

Corollary 3 can also be derived from Theorem 1 with $\beta = 1$ (which shows that the corresponding functional is coercive). This is done by constructing a function $\widehat{G}(x,u)$ with the following properties: $\widehat{G}$ is Caratheodory, $\widehat{G} \leq G$, $\widehat{G}(x,u)$ is Lipschitz continuous in u with a Lipschitz constant independent of x, $\widehat{G}(x,u) \geq d(x)$ and, for almost every $x \in \Omega_0$, $\widehat{G}(x,u) \to +\infty$ as $|u| \to \infty$.

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Subharmonic Solutions for Second Order Differential Equations

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Subharmonic Solutions for Second Order Differential Equations

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Abstract. We provide sufficient conditions for the existence of subharmonic solutions with prescribed minimal period for non autonomous second order differential equations. The proofs are based on Morse theory.

1. Introduction and notations

We consider the system of second order differential equations

$$\ddot{x}(t) + \nabla G(t,x(t)) = 0\ , \tag{1}$$

where $G : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function, periodic with minimal period $T > 0$ in its first variable and is such that its first and second derivatives with respect to its second variable $D_x G(t,x)$ and $D_x^2 G(t,x)$ are continuous ; we shall write $\nabla G(t,x)$ for $D_x G(t,x)$. When $N = 1$ we write equation (1) in the form

$$\ddot{x}(t) + g(t,x(t)) = 0, \tag{2}$$

and, accorlingly, we define

$$G(t,x) = \int_0^x g(t,s)\ ds.$$

Our purpose is to study the problem of the existence of kT-periodic solutions of (1) or (2) ($k \geq 1$ is an integer) which are not T-periodic.

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These solutions will be found by applying Morse theory to the associated funcional

$$\varphi_k(x) = \int_0^{kT} \left[\frac{|\dot{x}(t)|^2}{2} - G(t,x(t)) \right] dt$$

for $x \in H_k \equiv H^1_{kT}(\mathbb{R};\mathbb{R}^N)$, the Sobolev space consisting of the kT-periodic absolutely continuous fuctions $\alpha : \mathbb{R} \to \mathbb{R}^N$ whose first derivative is in $L^2([0,kT];\mathbb{R}^N)$, equipped with the usual inner product

$$\int_0^{kT} [\, (x(t),y(t)) + (\dot{x}(t),\dot{y}(t)) \,]\, dt.$$

Here $(.,.)$ stands for the euclidean inner product in $\mathbb{R}^N$ and $|.| = (.,.)^{1/2}$. Identifying $\mathbb{R}^N$ with the space of constant functions we may write $H_k = \mathbb{R}^N \oplus \tilde{H}_k$ (orthogonal decomposition) and, for each $x \in H_k$,

$$x(t) = \bar{x} + \tilde{x}(t),$$

where $\bar{x} = \frac{1}{kT}\int_0^{kT} x(t)dt$, so that $\int_0^{kT} \tilde{x}(t)dt = 0$. We will also consider the Banach space $C([0,T];\mathbb{R}^N)$ of continuous functions $x : [0,T] \to \mathbb{R}^N$ equipped with the norm $\|x\|_\infty = \sup_{0\le t\le T} |x(t)|$. We will denote by $\|.\|_2$ the usual L^2- norm.

It is well known that under our regularity assumptions the set of kT-periodic solutions of (1) coincide with the set of critical points of φ_k. Moreover, φ_k is a C^2 functional and $D^2\varphi_k(x)$ is a Fredholm operator, for each $x \in H_k$.

It is clear that a kT-periodic solution of (1), even if it is not T-periodic, needs not have minimal period kT. However, if for example k is a prime number and the property

(H_0) *if* $z(t)$ *is a periodic function with minimal period* qT, q *rational, and* $\nabla G(t,z(t))$ *is a periodic function with minimal period* qT, *then* q *is necessarily an integer*

holds, then any kT-periodic solution of (1) which is not T-periodic must have minimal period kT (see [11]) ; these are called subharmonic solutions of (1). For example, if $G(t,x) = a(t)\, G(x)$ or $G(t,x) = G(x) + (h(t),x)$, where $a(t) > 0$ and $h(t)$ have minimal period T, then (H_0) holds. Our main results (theorems 1 and 2) state that under certain conditions upon the function $G(t,x)$ there exist kT-periodic solutions which are not T-periodic, for every k sufficiently large ; under the additional assumption (H_0) this provides subharmonics for (1) with minimal period kT, for every k prime and large.

The typical case we consider is the convex subquadratic case (see theorems 3 and 4) ; this was studied in [11], [17] by the use of a $\mathbb{Z}_p$-index theory (in [17] the superquadratic case is also considered). In [17] it is assumed that (H_0) holds, $G(t,.)$ is convex for every $t \in \mathbb{R}$, $G \ge 0$, $G(t,0) = 0$ and that there exist positive constants $a_1,a_2,a_3,a_4,\alpha,\mu$ with $1 < \alpha \le \mu < 2$, such that

$$a_1 |x|^\alpha - a_2 \le G(t,x) \le a_3 |x|^\mu + a_4$$

for every $(t,x) \in \mathbb{R} \times \mathbb{R}^N$. However the examples given in section 3 show that the main ideas contained in theorems 1 and 2 may apply to other situations where neither convexity nor subquadratic growth hold. We also note that the periodic case (i.e., $G(t,x+\tau) = G(t,x)$ for some $\tau > 0$) was treated in [6] using some ideas developed in this paper.

The paper is organized as follows : in section 2 we recall some definitions and theorems for the estimate of the Morse index of critical points at critical levels of inf-sup type and then prove our main abstract result. In section 3 we apply the ideas of section 2 to equations (1) and (2) in several different situations.

2. A general result

We start by recalling the following definitions : let X be a real Hilbert space with inner product $<.,.>$ and $\varphi : X \to \mathbb{R}$ be a C^2 function. We let $D^2\varphi(x)$ denote the unique bounded self-adjoint operator in X such that $< D^2\varphi(x)y,z > = \varphi''(x)(y)(z)$ for every $x,y,z \in X$ and assume that $D^2\varphi(x)$ is a Fredholm operator for every $x \in X$. Let x_0 be a critical point of φ ; we define the Morse index [augmented Morse index] $m_\varphi(x_0)$ $[m^*_\varphi(x_0)]$ of x_0 as the supremum of the dimensions of the vector subspaces of X over which $D^2\varphi(x_0)$ is negative definite (semi-negative definite). We also define the nullity $v_\varphi(x_0) = m^*_\varphi(x_0) - m_\varphi(x_0)$; x_0 is called non-degenerate if $v_\varphi(x_0) = 0$.

When applied to the functional φ_k, $k \geq 1$, defined in section 1 we simply write $m_k(x_0)$, $m^*_k(x_0)$, $v_k(x_0)$; in this case we can use the following alternative (equivalent) approach : for every $\sigma \in S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and every kT-periodic solution x(t) of (1) define $J(kT,\sigma,x)$ $[J^*(kT,\sigma,x)]$ as the number of negative [non positive] real numbers λ, counted with their multiplicity, for which there exists a nontrivial solution of the problem

$$v''(t) + (D^2_x G(t,x(t)) + \lambda)\, v(t) = 0$$
$$v(t + kT) = \sigma\, v(t).$$

Then $m_k(x) = J(kT,1,x)$ and $m^*_k(x) = J^*(kT,1,x)$; notice that $v_k(x) \leq 2N$. Moreover the function $J(kT,.,.) : S^1 \times H_k \to \mathbb{N}$ is lower semi-continuous. (see [1]).

Lemma 1. *Let* x(t) *be a* T*-periodic (hence* kT *-periodic) solution of* (1) *such that* $m_1(x) \geq 1$. *Then*

$$\lim_{k\to\infty} m_k(x) = +\infty.$$

Proof . Consider the eigenvalue problem

$$v''(t) + (D^2_x G(t,x(t)) + \lambda)\, v(t) = 0 \qquad (3)$$
$$v(t + kT) = v(t) . \qquad (4)$$

By a result of Bott [2] there exists a non trivial solution of (3), (4) if and only if there exists a nontrivial solution v(t) of (3) verifying

$$v(t + T) = \sigma\, v(t) \qquad (5)$$

for some $\sigma \in S^1$, $\sigma^k = 1$ (see the proof of Proposition 2.1 (iv) in [1]). Besides, one has

$$m_k(x) = \sum_{\sigma^k=1} J(T,\sigma,x) \qquad (6)$$

Assume $m_1(x) \geq 1$. Then also $J(T,\sigma,x) \geq 1$ for $\sigma \in S^1$ and $|\sigma - 1| \leq \varepsilon$ (for small $\varepsilon > 0$). Now, given $M \in \mathbb{N}$, choose $k_0 \geq M$ such that $|e^{2\pi ij/k_0} - 1| \leq \varepsilon$ for every $j \in \{1,...,M\}$. Then, if $k \geq k_0$ we have $J(T,e^{2\pi ij/k},x) \geq 1$, $j \in \{1,...,M\}$. From (6) we get $m_k(x) \geq M$ and this proves the lemma. ■

<u>Remark 1</u>. Let C be a compact subset of H_1 consisting of critical points of φ_1 such that $m_1(x) \geq 1$ for every $x \in C$. Then also $J(T,\sigma,x) \geq 1$ for every $x \in C$, $|\sigma - 1| \leq \varepsilon$, $\sigma \in S^1$, if ε is small enough. The preceding argument then shows that $m_k(x) \to +\infty$ as $k \to \infty$, uniformly in $x \in C$.

Lemma 2. *Assume* N = 1 *and let* x(t) *be a* T*-periodic solution of* (2) *such that* $m_1(x) = 0$. *Then*

$$m_k(x) = 0 \qquad \text{for every } k \geq 1$$

and either $v_k(x) = 0$ *for every* $k \geq 1$ *or* $v_k(x) = 1$ *for every* $k \geq 1$.

Proof . Denote by $\lambda_k(x)$ the first eigenvalue of (3), (4). It is clear that $\lambda_k(x) \le \lambda_1(x)$ for every $k \ge 1$; but from the theory of Hill's equation (see [8]) one knows that situation (3), (5) cannot occur when $\lambda < \lambda_1(x)$ (in fact all Floquet multipliers σ of (3), (5) belong to $]0,+\infty[\setminus \{1\}$ if $\lambda < \lambda_1(x)$). Hence we deduce that

$$\lambda_1(x) = \lambda_k(x)$$

for every $k \ge 1$. Now, from the very definition of the Morse index we have $m_1(x) = 0$ if and only if $\lambda_1(x) \ge 0$ and so we have $m_k(x) = 0$ for every $k \ge 1$. If $\lambda_1(x) = 0$ then $v_k(x) = 1$ for every $k \ge 1$ since 0 is the first eigenvalue, which is simple. If otherwise $\lambda_1(x) > 0$, then $v_k(x) = 0$ for every $k \ge 1$. ■

Next we recall two results which provide estimates for the Morse index of some class of critical points. Given a Hilbert space X and a C^2 function $\varphi : X \to \mathbb{R}$ we shall say that φ verifies the Palais-Smale condition (in short (PS) condition) over X if any sequence (x_n) in X such that $(\varphi(x_n))$ is bounded and $\nabla\varphi(x_n) \to 0$ has a convergent subsequence in X. For each $R > 0$ and $x \in X$ we denote by $B_R(x)$ the open ball centered at x with radius R and by $S_R(x)$ its boundary. Also, we assume that $D^2\varphi(x)$ is a Fredholm operator for every critical point x of φ.

Lemma 3. *(Mountain Pass theorem) Let φ be as above, assume that φ satisfies the (PS) condition over X and has only isolated critical points. Suppose that there exist $R > 0$ and $x_0, x_1 \in X$ such that $\|x_0 - x_1\| > R$ and*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf_{S_R(x_0)} \varphi . \tag{7}$$

Then there exists a critical point x of φ such that $x \ne x_0$ and

$$m_\varphi(x) \le 1 \le m^*_\varphi(x) .$$

(For a proof see [9]).

Remark 2. Let us recall that condition (7) holds if x is an isolated local minimum of φ, provided φ satisfies the (PS) condition and $\varphi(u_n) \to -\infty$ for some unbounded sequence (u_n) in X (see [3,Theorem 5.10]) .

Lemma 4 *(Saddle Point Theorem). Let φ be as above, assume that φ satisfies the (PS) condition over X and has only isolated critical points. Let $X = X_1 \oplus X_2$, X_1 and X_2 being closed subspaces of X with* $\dim X_1 = d$, $1 \le d < \infty$ *and assume that for some* $R > 0$ *one has*

$$\sup_{S_R(0) \cap X_1} \varphi < \inf_{X_2} \varphi \tag{8}$$

Then there exists a critical point x of φ such that

$$m_\varphi(x) \le d \le m^*_\varphi(x) .$$

(See [7] or [9]).

From this we can deduce the following

Lemma 5. *Assume $N = 1$ and that for some $k > 1$ the functional φ_k associated to equation (2) satisfies the (PS) condition over H_k and the geometric assumption (8) of lemma 4 [resp : (7) of lemma 3]. Let d be as in lemma 4 [resp : $d \equiv 1$ if (7) holds]. Assume moreover that the (non empty) set Z of critical points of φ is compact in H_1 and that, for every $x \in Z$, either*

$$m_k(x) \ge d + 1 \tag{9}$$

or

$$m_k(x) = 0 = m^*_k(x) . \tag{10}$$

Then equation (2) has a kT-periodic solution which is not T-periodic.

Proof . Let Z_0 and Z_1 denote the subsets of Z whose points verify (10) and (9) respectively. From our hypothesis, Z_0 is finite and Z_1 is compact. Assume by contradiction that (2) has only T-periodic solutions. A compactness and continuity argument shows that we can fix positive constants α,β such that

$$x \in Z_0, |x - z| \le \alpha \Rightarrow (D^2\varphi_k(z)h,h) \ge \beta \|h\|^2$$

for every $h \in H_k$ and

$$x \in Z_1, |x - z| \le \alpha \Rightarrow (D^2\varphi_k(z)h,h) \le - \beta \|h\|^2$$

for every $h \in E_x$ where E_x is some finite dimensional subspace of H_k with $\dim E_x \ge d + 1$. Here $(.,.)$ stands for the inner product in H_k.

Consider the situation (7), take x_0,x_1,R as in lemma 3 and choose

$$0 < \varepsilon < \min \{\beta/2, \frac{1}{3}(\max \{\varphi_k(x_0),\varphi_k(x_1)\} - \inf_{S_R(x_0)} \varphi_k)\}.$$

According to a perturbation theorem of [10], we can choose a C^2 functional ψ such that

- ψ satisfies the (PS) condition over H_k ;
- ψ has only non degenerate critical points (in particular, they are isolated) ;
- $\psi(z) = \varphi(z)$ whenever $\mathrm{dist}\,(z,Z) \ge \alpha/2$;
- $\sup_{z \in H_k} \{|\varphi_k(z) - \psi(z)| + \|\nabla\varphi_k(z) - \nabla\psi(z)\|_{H_k} + \|D^2\varphi_k(z) - D^2\psi(z)\|_{\mathcal{L}(H_k)}\} < \varepsilon.$

It follows from our choice of ε that

$$\max \{\psi(x_0),\psi(x_1)\} < \inf_{S_R(x_0)} \psi,$$

and then from lemma 3 we can take a critical point z of ψ with Morse index one. Now take $x \in Z$ such that $\|z - x\| = \mathrm{dist}\,(z,Z) \le \alpha$. Then either

$$(D^2\psi(z)h,h) \ge (\beta/2) \|h\|^2$$

for every $h \in H_k$ (if $x \in Z_0$) or

$$(D^2\psi(z)h,h) \le - (\beta/2) \|h\|^2$$

for every $h \in E_x$ (if $x \in Z_1$). Since $\dim E_x \ge d + 1 \ge 2$ we get a contradiction in both cases.

Finally, if situation (8) holds, we proceed as before by choosing

$$0 < \varepsilon < \min \{\beta/2, \frac{1}{3}(\inf_{X_2} \varphi_k - \sup_{S_R(0)\cap X_1} \varphi_k)\}$$

where R, X_1, X_2 are as in lemma 4. ■

Now we can state our main general results.

Theorem 1. *Let* $N = 1$, *consider the functional* φ_k $(k \ge 1)$ *associated to equation* (2) *and assume that* φ_k *satisfies the* (PS) *condition over* H_k *for every* $k \ge 1$ *and either*

(a) *for every* $k \ge 1$, φ_k *satisfies the geometric assumption* (7) *of lemma 3 ;*

or

(b) *for every* $k \ge 1$, φ_k *satisfies the geometric assumption* (8) *of lemma 4.*

Assume moreover that the (non empty) set Z *of critical points of* φ_1 *is compact and has the following property :*

(H) *if* $x \in Z$ *and* $m_1(x) = 0$, *then* $\nu_1(x) = 0$.

Then there exists $k_0 \in \mathbb{N}$ *such that for every* $k \ge k_0$ *equation* (2) *has a* kT-*periodic solution which is not* T-*periodic.*

Proof . Let $\tilde{Z}_0 = \{x \in Z : m_1(x) = 0\}$ and $\tilde{Z}_1 = Z \setminus \tilde{Z}_0$. By (H), $\tilde{Z}_0$ is finite and $\tilde{Z}_1$ is compact. Hence, by lemma 2, we may fix $k_0 \in \mathbb{N}$ such that (9) holds for every $x \in \tilde{Z}_1$ and $k \ge k_0$ (see Remark 1) and (10) holds for every $x \in \tilde{Z}_0$. Then lemma 5 can be applied. ■

Remark 3. In case situation (b) holds with $d \geq 2$ and Z, Z_1 are both compact (hence Z_0 is also compact), we can drop assumption (H) in theorem 1 since it follows from lemma 2 that $m_k^*(x) < d$ for any $x \in Z_0$, $k \geq 1$ and then the above arguments apply.

Using the same arguments together with lemma 1, one can prove the following result for system (1), with $N \geq 1$.

Theorem 2. *Assume that the functional* φ_k $(k \geq 1)$, *associated to system* (1), *satisfies the* (PS) *condition over* H_k *for every* $k \geq 1$ *and that either situation* (a) *or* (b) *of theorem 1 hold. Assume moreover that* Z, *the (nonempty) set of critical points of* φ_1, *is compact and*

(H') $\quad m_1(x) \geq 1$ *for every* $x \in Z$.

Then there exists $k_0 \in \mathbb{N}$ *such that for every* $k \geq k_0$ *equation* (1) *has a* kT-*periodic solution which is not* T-*periodic.*

Proof . Simply note that now $Z = \tilde{Z}_1$, use remark 1 and proceed as in lemma 5. ■

Remark 4. It is easily seen that Z is compact in H_1 if and only if Z is bounded in $C([0,T]; \mathbb{R}^N)$.

Next we give a sufficient condition for (H') in theorem 2 to hold.

Lemma 6. *Assume that*

(H") $G(t,.)$ *is convex, for every* $t \in [0,T]$, *and there are no* T-*periodic solutions* $x(t)$ *of* (1) *such that* $D_x G(t,x(t)) \equiv 0$.

Then $m_1(x) \geq 1$ *for any* T-*periodic solution* $x(t)$ *of* (1).

Proof . Let $x \in Z$ and denote by $\lambda_1(x)$ the first eigenvalue of (3), (4) with $k = 1$. It is well known that

$$\lambda_1(x) = \min \{ \int_0^T [|\dot{y}(t)|^2 - (D_x^2 G(t,x(t))y(t),y(t))]dt : y \in H_1, \|y\|_2 = 1 \}.$$

Taking constant functions $y(t) \equiv y \in \mathbb{R}^N$ we get $\lambda_1(x) \leq 0$; and in fact $\lambda_1(x) < 0$ since otherwise $D_x^2 G(t,x(t)) \equiv 0$, which contradicts (H"). But $\lambda_1(x) < 0$ means precisely that $m_1(x) \geq 1$ and we are done. ■

Remark 5. It is clear that (H") holds if $D_x^2 G(t,.)$ is positive definite for every $t \in [0,T]$. Also, if an a priori bound $\|x\|_\infty < R$ for $x \in Z$ is known, we only require the strict convexity for $G(t,.)$ on the ball $B_R(0)$ of $\mathbb{R}^N$.

3. Applications

In this section we apply the above theorems to a few special cases of equations (1) and (2).

Theorem 3. *Consider equation* (1) *with* $N \geq 1$, *assume that* $G(t,x)$ *satisfies* (H") *and*

(i) *there exists a positive constant* K *such that*

$$|\nabla G(t,x)| \leq K$$

for every $(t,x) \in \mathbb{R} \times \mathbb{R}^N$;

(ii) $\lim_{|x| \to \infty} \int_0^T G(t,x)\, dt = +\infty$.

Then the conclusion of Theorem 2 holds true.

Proof . For every $k \geq 1$ write $H_k = \mathbb{R}^N \oplus \tilde{H}_k$ (see section 1). From (i) (resp. (ii)) it follows that φ_k (resp : $-\varphi_k$) is coercive in $\tilde{H}_k$ (resp. : $\mathbb{R}^N$) ; thus we are in situation (b) of theorem 2. In order to verify the (PS) condition, let $(x_n) \subseteq H_k$ be such that

$$\left|\int_0^{kT} \left[\frac{|\dot{x}_n(t)|^2}{2} - G(t,x_n(t)) \right] dt \right| \leq M \qquad (11)$$

$$\left|\int_0^{kT} [(\dot{x}_n(t),\dot{y}(t)) - (\nabla G(t,x_n(t)),y(t))]\, dt \right| \leq \varepsilon_n \|y\|_{H_k} \qquad (12)$$

for every $n \geq 1$, $y \in H_k$, where M, $\varepsilon_n > 0$ and $\varepsilon_n \to 0$. Taking $y = \tilde{x}_n$ in (12) and using (i) we get that $(\|\dot{x}_n\|_2)$ is bounded. Then $(\|\tilde{x}_n\|_\infty)$ and, from (11), $(|\int_0^{kT} G(t,x_n(t))dt|)$ are also bounded. From the convexity assumption we derive

$$G\left(t, \frac{\bar{x}_n}{2}\right) \leq \frac{1}{2} G(t,x_n(t)) + \frac{1}{2} G(t,- \tilde{x}_n(t)) \qquad (13)$$

so that $(\int_0^{kT} G(t, \frac{\bar{x}_n}{2})\, dt)$ is bounded and then from (ii), $(|\bar{x}_n|)$ is bounded. Hence $(\|x_n\|_{H_k})$ is bounded and from classical arguments we can find a convergent subsequence (see e.g. [13]).

To end the proof it remains to show that any sequence (x_n) of T-periodic functions such that

$$\ddot{x}_n(t) + \nabla G(t,x_n(t)) = 0 \qquad (14)$$

is bounded in H_1. Multiplying (14) by $\tilde{x}_n(t)$ and using (i) we get $(\|\dot{x}_n\|_2)$ bounded. Being G convex, one has the inequality

$$G(t,y) \leq G(t,0) + (\nabla G(t,y),y) \quad \text{for } (t,y) \in \mathbb{R} \times \mathbb{R}^N,$$

and from (13) and (14) we get

$$\int_0^T G\left(t, \frac{\bar{x}_n}{2}\right) dt \leq \frac{1}{2} \int_0^T \left[|\dot{x}_n(t)|^2 + G(t,0) + G(t,- \tilde{x}_n(t)) \right] dt . \qquad (15)$$

The result then follows as before. ∎

Theorem 4. *Consider equation* (2), *assume that* G(t,x) *satisfies* (H"), (ii) *and*

(iii) $\lim_{|x| \to \infty} \frac{G(t,x)}{x^2} = 0$ *uniformly in* t ;

(iv) $\lim_{|x| \to \infty} \sup \frac{g(t,x)}{x} \leq M < (\frac{2\pi}{T})^2$ *uniformly in* t.

Then the conclusion of theorem 1 holds true.

Proof . Writing $H_k = \mathbb{R} \oplus \tilde{H}_k$, conditions (ii) and (iii) show that we are again in situation (b) of theorem 1. We only sketch the proof of the Palais-Smale condition which combines the arguments in [5] and [14]. Consider (11) and (12) above (where $\nabla G(t,x) = g(t,x)$). From (12), taking $y \equiv 1$, we get

$$\int_0^{kT} g(t,x_n(t))dt \to 0. \qquad (16)$$

We claim that

$$\min_{0 \leq t \leq kT} |x_n(t)| \leq c \qquad (17)$$

for some constant $c > 0$. If not then, passing to a subsequence if necessary, we would have $\min |x_n(t)| \to +\infty$. Since (H") implies that the function sign(x)g(t,x) is bounded from below, we get from (16) that $(\int_0^{kT} |g(t,x_n(t)|\, dt)$ is bounded (see[5]). Then from (12), taking $y = \tilde{x}_n(t)$, we get that $(\|\dot{x}_n\|_2)$ is bounded and, from (11), $(\int_0^{kT} G(t,x_n(t))dt)$ is also bounded. Being $(\|\tilde{x}_n\|_\infty)$ bounded, using (13) and (ii) we reach a contradiction.

Hence (17) holds. Let us prove now that $(|\bar{x}_n|)$ is bounded. If not, for a subsequence, $|\bar{x}_n| \to +\infty$ and we get from (11) and (iii) that

$$\tilde{x}_n(t) / \bar{x}_n \to 0 \qquad \text{in } H_k$$

(see [14]). But then $|x_n(t)| = |\bar{x}_n|\,|1 + \tilde{x}_n(t) / \bar{x}_n| \to +\infty$ uniformly in t and this contradicts (17). Hence $(|\bar{x}_n|)$ is bounded and from (11) and (iii) $(\|\dot{x}_n\|_2)$ is also bounded.

Finally, let (x_n) be a sequence of T-periodic solutions of (2). A similar argument show that (17) holds (use (15)). Now suppose that $|||x_n||| \equiv \|x_n\|_2 + \|\dot{x}_n\|_2 + \|\ddot{x}\|_2 \to +\infty$ (for some subsequence) and let $z_n(t) = \frac{x_n(t)}{|||x_n|||}$. We proceed as in [16] and use (iv) to write

$$g(t,x) = g_1(t,x).x + g_0(t,x) \tag{18}$$

where g_0, g_1 are functions such that $|g_0(t,x)| \le c_1$ and $-c_2 \le g_1(t,x) \le c_3 < (\frac{2\pi}{T})^2$ for every $(t,x) \in \mathbb{R}^2$ and some positive constants c_i, i = 1,2,3 (notice that, by (H"), g(t,x) / x is bounded below for |x| large). Dividing both members of (14) by $|||x_n|||$, taking limits and using (18) we see that $z_n(t) \to z(t)$ in $C^1([0,T];\mathbb{R})$, $z \not\equiv 0$ and for some function $\alpha(t)$ such that $-c_2 \le \alpha(t) \le c_3 < (\frac{2\pi}{T})^2$ we have

$$\ddot{z}(t) + \alpha(t)\, z(t) = 0\,,$$

$$z(0) - z(T) = 0 = \dot{z}(0) - \dot{z}(T)$$

From Sturm-Liouville theory it follows that $z(t) \neq 0$ for every $t \in [0,T]$. Then $z_n(t)$ has constant sign for n large enough and since $x_n(t) = |||x_n|||\, z_n(t)$ we have a contradiction with (17). This shows that $(|||x_n|||)$ is bounded and ends the proof of the theorem. ■

Example 1. Theorem 3 applies to g(t,x) = arctg x - h(t) where h(t) is a continuous T-periodic function such that $-\frac{\pi}{2} < \frac{1}{T}\int_0^T h(t)dt < \frac{\pi}{2}$.

Example 2. Theorem 4 applies to $g(t,x) = \frac{x}{(1+x^2)^{1/4}} + h(t)$ for any continuous T-periodic function.

Remark 6 : Let $g(t,x) = \frac{x}{(1+x^2)^{1/4}} + \frac{x}{1+x^2} + h(t)$, where h(t) is any continuous T-periodic function. Here all assumptions of theorem 4 are satisfied except for the convexity hypothesis (H"). Nevertheless we still have a priori bounds for the T-periodic solutions of (2) and, for every k ≥ 1, φ_k satisfies the (PS) condition over H_k ; therefore theorem 1 can be applied whenever h(t) is such that condition (H) holds. Note that, for any h(t), equation (2) has a T-periodic solution (this follows readily from the Saddle Point theorem) and thus condition (H) is of generic type with respect to h(t) (see [10]).

Remark 7. It is proved in [18, example 2] by means of phase-plane methods that equation (2) has (at least) a T-periodic solution if $g(t,x) = \frac{x + h(t)}{1 + x^2}$, where h(t) is a continuous T-periodic function. This follows also from the Saddle Point theorem and again we may apply Theorem 1 whenever condition (H) holds (notice also that (H_0) holds for this case). This remark applies also to $g(t,x) = \frac{x}{1+x^2} + h(t)$ provided $\int_0^T h(t)\,dt = 0$.

We now study an Ambrosetti-Prodi type situation.

Theorem 5. *Consider the following equation*

$$x''(t) + g(t,x(t)) = s , \qquad (P)_s$$

where $s \in \mathbb{R}$ *is a parameter,* $g \in C(\mathbb{R}^2;\mathbb{R})$ *is* T*-periodic in its first variable and verifies the regularity assumptions of section* 1. *Suppose that*

(i) $\lim_{|x|\to+\infty} g(t,x) = +\infty$ *uniformly in* t ;

(ii) $g'_x(t,.)$ *is strictly increasing for each* $t \in \mathbb{R}$;

(iii) $\lim_{x\to+\infty} g'_x(t,x) \le A < (\frac{2\pi}{T})^2$ *for every* $t \in \mathbb{R}$;

(iv) $|G(t,x)| \le K_1 e^{-x} + K_2$ *for some positive constants* K_1, K_2 *and every* $t \in \mathbb{R}$, $x \le 0$.

Then there exists a constant $s_0 \in \mathbb{R}$ *such that for every* $s \ge s_0$ *we can find* $k(s) \in \mathbb{N}$ *such that* $(P)_s$ *has a* kT*-periodic solution which is not* T*-periodic, for any* $k \ge k(s)$.

Proof . Without loss of generality we may assume that g(t,x) is positive. For the sake of clarity we divide the proof into several steps.

<u>Step 1</u>. It is known that under the sole assumption (i) there exists $s_1 \ge \max_{t\in\mathbb{R}} g(t,0)$ such that $(P)_s$ has zero, one or two T-periodic solutions according to whether $s < s_1$, $s = s_1$ or $s > s_1$ respectively (see [4]). Moreover, assumptions (ii) and (iii) imply that there are precisely one or two solutions for $s = s_1$ or $s > s_1$ respectively and, in the latter case, they are both non degenerate ; this was proved in [12] for the dissipative Duffing equation with $A < (\frac{\pi}{T})^2$, but it is immediately seen from the proof given there that this still holds true for our problem. In order to calculate

Morse indexes we shall obtain those solutions (for s sufficiently large) by using variational methods applied to the functional

$$\varphi_k(x) = \int_0^{kT} \left[\frac{|\dot{x}(t)|^2}{2} - G(t,x(t)) + sx(t) \right] dt ,$$

$x \in H_k$, as in [15].

<u>Step 2</u> (minimization). For a given $s > K_1$ (see (iv)) consider the closed convex subset of H_1 :

$$C = \{x \in H_1 : \|\dot{x}\|_2 \le 2sT^{3/2}, \bar{x} \le 0\}.$$

It is easily seen that φ_1 is coercive in C and, since it is a weakly lower semi continuous functional, we can find $u \in C$ such that

$$\varphi_1(u) \le \varphi_1(x) \qquad (19)$$

for every $x \in C$. From (iv) we can estimate

$$\begin{aligned} \varphi_1(u) &\le \min \{\varphi(a), a \in]-\infty,0]\} \\ &\le TK_2 + T \min \{K_1 e^{-a} + sa : a \in]-\infty,0]\} \\ &= TK_2 + Ts\,(1 - \log \frac{s}{K_1}) . \end{aligned} \qquad (20)$$

On the other hand, it follows from (iii) that

$$G(t,x) \le A_1 \frac{x^2}{2} + A$$

for every $(t,x) \in \mathbb{R}^2$, where $A_2 > 0$, $0 < A_1 < (\frac{2\pi}{T})^2$, and this implies that φ_1 is bounded below (in fact coercive) on $\tilde{H}$ by a constant which does not depend on s. From (20) we can thus find s_0 large enough so that for $s \ge s_0$ the function $u = u_s$ is such that

$$\bar{u} \ne 0 . \qquad (21)$$

From now on we fix s such that (21) holds. Choose $x = u \pm \varepsilon$, ε small, in (19) and take limits to obtain

$$\int_0^T g(t,u(t))dt = sT.$$

Now choose $x = (1-\varepsilon)\,\tilde{u} + \bar{u}$ in (19), $\varepsilon > 0$ small, and take limits to get

$$\int_0^T |\dot{u}(t)|^2\,dt \le \int_0^T g(u(t))\,\tilde{u}(t)\,dt \le sT\,\|\tilde{u}\|_\infty \le sT^{3/2}\,\|\dot{u}\|_2\,,$$

so that $\|\dot{u}\|_2 \le sT^{3/2} < 2sT^{3/2}$. Hence we conclude that u belongs to the interior of C and thus is an isolated local minimum for φ_1.

Step 3 (Mountain Pass Theorem). Let u be an isolated local minimum of φ_1. It is easily seen that φ_1 satisfies the (PS) condition over H_1; moreover,

$$\varphi_1(a_n) \to -\infty \text{ whenever } (a_n) \in \mathbb{R},\ a_n \to +\infty$$

(this facts remain true for φ_k, $k \ge 1$). Then, according to lemma 3 (see remark 2) we can find our second T-periodic solution v of $(P)_s$ and we have

$$m_1(u) = 0 = \nu_1(u),$$
$$m_1(v) = 1,\ \nu_1(v) = 0.$$

According to lemmas 1 and 2 we can fix k(s) so large that

$$m_k(u) = 0 = \nu_k(u), \tag{22}$$
$$m_k(v) \ge 2 \tag{23}$$

for any $k \ge k(s)$.

Step 4. Take any $k \ge k(s)$ and assume by contradiction that u,v are the only kT-periodic solutions of $(P)_s$ (in particular they are isolated in H_k). For each $n \ge 1$ consider the critical groups $C_n(\varphi_k,u)$ (see [9]). Since $\nu_k(u) = 0$ we have

$$\dim C_n(\varphi_k,u) = \delta_{n,m_k(u)}\,,$$

where δ stands for the Kronecker symbol ([9; corollary 8.3]). From (22) we get $C_0(\varphi_k,u) \ne 0$ and by [9, theorem 8.6] u is a local minimum for φ_k. But then we can apply lemma 3 to φ_k (see step 3) in order to get a second solution - which is precisely v - such that $m_k(v) \le 1$. This contradicts (23) and ends the proof of the theorem. ■

Example 3. Let a(t), h(t) be continuous functions with minimal period T, a(t) > 0, and λ be a positive constant with $\lambda < (\frac{2\pi}{T})^2$. Then if $\frac{1}{T}\int_0^T h(t)\,dt$ is sufficiently large, equation

$$x''(t) + a(t)\,e^{-x(t)} + \lambda x(t) = h(t)$$

admits infinitely many subharmonics with minimal period kT, k prime.

Our last theorem extends partially corollary 8 in [18].

Theorem 6. *Consider equation* (2) *where* g(t,x) *is* T*-periodic in its first variable and satisfies the regularity assumptions of section* 1. *Suppose that* g(t,x) *is bounded below,* $g'_x(t,x) > 0$ *and there exists a positive constant* r_1 *such that*

$$\operatorname{sign}(x) \int_0^T g(t,x)\,dt > 0 \tag{24}$$

for every $t \in \mathbb{R}$, $|x| \ge r_1$. *Then the conclusion of theorem 1 holds true .*

Proof . Again, we divide the proof in several steps and use an argument similar to the one in theorem 5.

Step 1 (a priori bounds). For a given $k \ge 1$ let x(t) be a $\tau \equiv kT$-periodic solution of (2) and m > 0 be such that $-m < \min_{\mathbb{R}^2} g$. Since $\int_0^\tau g(t,x(t))\,dt = 0$, we have $|x(t_0)| < r_1$ for some $t_0 \in [0,\tau]$. Multiplying (2) by $\tilde{x}(t)$ we get

$$\|\dot{x}\|_2 \le \tau^{3/2}\,m \equiv c\,. \tag{25}$$

Hence we have the a priori bound

$$\|x\|_\infty < r_1 + \tau^2 m \equiv r$$

for every τ-periodic solution of (2). Next we make the following remark: if f(t,x) is a function such that $f(t,x) \ge g(t,r_1)$ for every (t,x) in

$\mathbb{R} \times [r_1,+\infty[$ and $f \geq -m$ in $\mathbb{R}^2$, then we still have the bound (25) and $\min x(t) < r_1$ for every τ-periodic solution $x(t)$ of

$$x''(t) + f(t,x(t)) = 0 . \qquad (26)$$

Since the set Z of T-periodic solutions of (2) is a priori bounded and $m_1(x) \geq 1$ for every $x \in Z$, we can choose $k_0 \in \mathbb{N}$ such that

$$m_k(x) \geq 2$$

for every $x \in Z$ and $k \geq k_0$ (see remark 1 and remark 4). In the sequel we fix $k \geq k_0$ and assume by contradiction that all kT-periodic solutions of (2) are T-periodic. We shall denote $\tau = kT$.

Step 2 (truncation). Let $\lambda > 0$ be a small positive number to be chosen below and consider the function

$$f(t,x) = \begin{cases} g_x'(t,r)(x-r) + g(t,r) & x \geq r \\ g(t,x) & -r \leq x \leq r \\ \theta_\lambda(t,x) & -2r \leq x \leq r \\ \theta_\lambda(t,-2r) - \lambda(x+2r) & x \leq -2r \end{cases}$$

where θ_λ is such that $f \in C(\mathbb{R}^2;\mathbb{R})$ is T-periodic in its first variable, its first derivative with respect to its second variable is continuous and $-m \leq \theta_\lambda(t,x) \leq g(t,-r)$.

Now we claim that if

$$0 < \lambda < (c\,\tau^{3/2})^{-1} \left(-\int_0^\tau g(t,-r)\, dt \right) , \qquad (27)$$

$u(t)$ is a τ-periodic solution of (26) and $u(t_1) \equiv \max u(t)$, then either

(a) $u(t_1) < -2r$,

or

(b) $u(t_1) > -r_1$.

Indeed, suppose by contradiction that $-2r \leq u(t_1) \leq -r_1$. By the remark made in step 1, we know that $\|\dot{u}\|_2 < c$; then we have

$$-2r - c\,\tau^{1/2} \leq u(t) \leq -r_1 \qquad (28)$$

for every $t \in [0,\tau]$. But our choice of λ implies that

$$\int_0^\tau (\theta_\lambda(t,x) - \lambda(x+2r))\, dt < 0$$

for any $-2r - c\tau^{1/2} \leq x \leq -2r$, so that we have $\int_0^\tau f(t,x)\, dt < 0$ in $[-2r - c\,\tau^{1/2}, -r_1]$. Since $\int_0^\tau f(t,u(t))dt = 0$, we get a contradiction with (28). This proves the claim.

Now (a) means that $u(t)$ is the unique τ-periodic (in fact, T-periodic) solution of

$$\ddot{u} - \lambda\, u(t) = \theta_\lambda(t,-2r) + \lambda\, 2r. \qquad (29)$$

On the other hand if (b) holds, and according to the remark in step 1, we have $\|u\|_\infty < r$ and $u(t)$ is a τ-periodic solution of (2). Hence we conclude that if $u(t)$ is a τ-periodic solution of (26) verifying a) then any other τ-periodic solution of (26) must be a solution of (2).

Step 3 (the modified problem). Setting $F(t,x) = \int_0^x f(t,x)\, ds$, consider the functional

$$\Psi(x) = \int_0^\tau \left[\frac{\dot{x}^2(t)}{2} - F(t,x(t)) \right] dt ,$$

defined on H_k. Since the function $f(t,x)$ is coercive we may proceed as in step 2 in the proof of theorem 5 and choose

$$u = u_\lambda(t) \in C \equiv \{x(t) \in H_k : \|\dot{x}\|_2 \leq 2c,\ \bar{x} \leq 0\}$$

such that

$$\Psi(u) \le \Psi(x) \quad \text{for every } x \in C.$$

Now, denoting $c(r) = 4r \max \{|g(t,x)|, t \in \mathbb{R}, -r \le x \le 0\}$, and $\bar{g} = \frac{1}{T}\int_0^T g(t,-r)\,dt < 0$, we have

$$-F(t,x) \le c(r) - x\,g(t,-r) + \lambda\,\frac{(x+2r)^2}{2}$$

for every $t \in \mathbb{R}$, $x \le -2r$, so that

$$\begin{aligned}\Psi(u) &\le \min\{\Psi(a) : a \in]-\infty,0]\}\\ &\le \tau\,c(r) + \tau \min\{\tfrac{\lambda}{2}(a+2r)^2 - \bar{g}\,a : a \in]-\infty,0]\}\\ &= \tau\,c(r) + \tau\,(2r\bar{g} - \frac{\bar{g}^2}{2\lambda}),\end{aligned}$$

and this last expression tends to $-\infty$ as $\lambda \to 0$. Since $\|\dot{u}\|_2 \le 2c$ we may choose λ so small that ((27) holds and)

$$\bar{u} = \bar{u}_\lambda < -2r - 2c\,\tau^{1/2}. \tag{30}$$

In particular we have $\bar{u} < 0$ and we can proceed as in step 2 of the proof of theorem 5 in order to prove that u is a local minimum of Ψ and hence a τ-periodic solution of (26). Moreover from (30) we see that situation (a) above holds and from (29) we get that u is non degenerate (in particular, u is isolated in H_k).

Since we have found an isolated local minimum u for Ψ and, as it is easily seen, Ψ satisfies the (PS) condition over H_k, we have the geometric setting of lemma 3. According to our previous remarks we have $m_\Psi(u) = m^*_\Psi(u) = 0$ and $m_\Psi(v) \ge 2$ for every critical point $v \ne u$ of Ψ. It follows from lemma 5 applied to Ψ that equation (26) admits a kT-periodic solution v which is not T-periodic. Necessarily $v \ne u$ and v is a kT-periodic solution of (2), contrary to our assumption. ■

Corollary. *Consider equation*

$$x''(t) + g(x(t)) = h(t), \tag{31}$$

where $g \in C^1(\mathbb{R};\mathbb{R})$ *is bounded below,* $g'(x) > 0$ *and* $h(t)$ *is a continuous function with minimal period* T. *If*

$$\bar{h} \in \text{range}(g), \tag{32}$$

equation (31) *admits subharmonic solutions with minimal period* kT, *for every* k *large and prime.*

Remark 8. Let us notice that condition (32) is also necessary for the existence of a subharmonic solution of (31). Moreover, setting $m = \min_{\mathbb{R}} g$ and being $r_1 > 0$ such that $\operatorname{sign} x\,(g(x) - \bar{h}) > 0$ for $|x| \ge r_1$ we have the a priori bound

$$\|x\|_\infty < r_1 + \sqrt{T}\,(2\int_0^T |h(t)|\,dt - Tm) \equiv R$$

for every T-periodic solution of (31). According to remark 5 (see also the proof given above) it is sufficient to assume a strict monotonicity on the interval $[-R,R] \subseteq \mathbb{R}$.

Example 4. In [18; example 3] it is shown that equation $\ddot{x}(t) + e^{x(t)} = h(t)$ has infinitely many subharmonics provided that $h(t) > 0$. The above corollary asserts that it is sufficient to have $\bar{h} > 0$.

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