INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

SMR.451/18

SECOND COLLEGE ON VARIATIONAL PROBLEMS IN ANALYSIS

(29 January - 16 February 1990)

Infinite cup length and category of free loop spaces with applications

E.R. Fadell
Department of Mathematics
University of Wisconsin
Madison, WI 53706
USA

These are preliminary lecture notes, intended only for distribution to participants

MAIN BUILDING STRADA COSTIERA, 11 TEL. 22401 TELEFAX 224163 TELEX 460392 ADRIATICO GUEST HOUSE VIA GRIGNANO, 9 TEL. 22424 TELEFAX 22458 TELEX 460449
MICROPROCESSOR LAB. VIA BEIRUT, 31 TEL. 22470 TELEFAX 224163 TELEX 460392 GALILEO GUEST HOUSE VIA BEIRUT, 7 TEL. 22401 TELEFAX 22453 TELEX 460392

Infinite Cup Length in Free Loop Spaces with an Application to a Problem of the N-body Type.

E. Fadell* * and S. Husseini *

I. INTRODUCTION

For 1-connected manifolds M with non-trivial, finitely generated cohomology, the free loop space ΛM has infinite (Ljusternik-Schnirelmann) category [FH] and the proof does not depend on the cup length in the cohomology of ΛM. Nevertheless, it is still useful to know that over some fields, depending on M, the cup length is infinite. We will show in this note, that for spheres S^m, ΛS^m has infinite cup length over $\mathbb{Z}_2$ and for complex projective space CP^n, ΛCP^n has infinite cup length over $\mathbb{Z}_r$, where r divides $n+1$. Incidentally, ΛCP^n has only finite cup length over the rational field [VS]. The first result will allow us to compute the relative category [F1] of the pair $(\Lambda M, \Lambda N)$, where M is a wedge of spheres and N a "subwedge". For our application, we will need to compute the category of a certain subspace of ΛS^n described as follows: Let $\mathbb{Z}_2 = \{1, \varsigma\}$ act on ΛS^m by the action $(\varsigma q)(t) = -q(t + \frac{1}{2})$, $0 \leq t \leq 1$, where q is considered 1−periodic. Let $\Lambda_0 S^m$ denote the fixed point set under this action. $\Lambda_0 S^m$ fibers over S^m but for m even, this fibration does not admit a section, which is a requirement for the main tool in [FH]. Nevertheless, we show the cup length of $\Lambda_0 S^m$ over $\mathbb{Z}_2$ is infinite and hence the category of $\Lambda_0 S^m$ is infinite. As an application of these results, it follows that the subspace $\Lambda(N)$ of the Sobolev space $W_T^{1,2}(\mathbb{R}^{kN})$ corresponding to the free loops $\Lambda F_N(\mathbb{R}^k)$ on the N-th configuration space of $\mathbb{R}^k$, has infinite category. Furthermore, if we let $\Lambda_0(N)$ denote the subspace of $\Lambda(N)$ which is the fixed point set of the $\mathbb{Z}_2$−action defined above, then $\Lambda_0(N)$ also has infinite category. This result is the key to proving a critical point theorem for the functional associated with a problem of the N-body type, which improves a result of Coti-Zelati [C] who minimizes the appropriate functional to obtain a critical point. In addition to yielding an unbounded sequence of critical values, the theorem (see Section 3) allows a non-autonomous potential $V(q,t)$ which is C^1 and $T/2$-periodic in t.

---

* *Supported also by the Universität Heidelberg, Mathematisches Institut und Institut für Angewandte Mathematik (SFB)

* Both authors were supported in part by NSF under Grant No. DMS-8722295

2. Cup Length in Some Free Loop Spaces

We employ the following notation. I is the unit interval $[0,1]$; M^I is the space of maps from I to a space M; ΛM is the free loop space on M given by $\Lambda M = \{\alpha \in M^I : \alpha(0) = \alpha(1)\}$; and $\Omega(M) = \Omega(M,*)$, the space of based loops, i.e. loops $\alpha \in \Lambda M$ such that $\alpha(0) = \alpha(1) = * \in M$. If we consider the (Hurewicz) fibration.

(1) $\qquad \Omega(M) \longrightarrow M^I \xrightarrow{q} M \times M$

where $q(\alpha) = (\alpha(0), \alpha(1))$, then the diagonal map $\Delta : M \longrightarrow M \times M$ induces the fibration.

(2) $\qquad \Omega M \longrightarrow \Lambda M \xrightarrow{p} M$

where $p(\alpha) = \alpha(0) = \alpha(1)$. We will also make use of the induced fibrations.

(3)

$$\begin{array}{ccccccc} P^1M & \xrightarrow{\hat{i}_1} & M^I & \quad & P^2M & \xrightarrow{\hat{i}_2} & M^I \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle q} & & {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle q} \\ M & \xrightarrow{i_1} & M \times M & & M & \xrightarrow{i_2} & M \times M \end{array}$$

where $i_1(x) = (x,*), i_2(x) = (*,x)$ and P^1M, P^2M are contractible.

The Leray-Serre spectral sequences of (1), (2), p_1 and p_2 will be denoted by $(E^{p,q}, d)$, $(\overline{E}^{p,q}, \overline{d})$, $('E^{p,q}, d')$, $(''E^{p,q}, d'')$.

We consider first the case $M = S^{m+1}$, $m+1$ even, $m \geq 1$ and prove several lemmas under this assumption. If u is a generator of $'E^{m+1,0}_{m+1} = H_{m+1}(M)$, we define $x \in H_m(\Omega M)$ by $d'_{m+1}\, x = u$.

2.1 <u>Lemma</u> $\quad d''_{m+1}\, x = -u$

<u>Proof</u>. This is a simple calculation using the reverse map.

(4)

$$\begin{array}{ccc} P^2M & \xrightarrow{\nu} & P^1M \\ {\scriptstyle p_2} \searrow & & \swarrow {\scriptstyle p_1} \\ & M & \end{array}$$

where $(\nu\alpha)(t) = \alpha(1-t)$. Then, $\nu_0 = \nu|\Omega M$, has the property that $\nu_0^*(x) = -x$ and hence $d''_{m+1}x = d''\nu_0^*(-x) = \quad d'_{m+1}(-x) = u$.

Comparing, the spectral sequences (SS) of p_1 and p_2 with that of q we have:

2.2 <u>Lemma</u>. $d_{n+1}(x) = u \times 1 - 1 \times u$ in $H^{m+1}(S^{m+1}) = E^{m+1,0}_{m+1}$.

We consider next the differential operator $d_{m+1} : E^{m+1,m}_{m+1} \longrightarrow E^{2m+2,0}_{m+1}$.

2.3 <u>Lemma</u> (a) $d_{m+1}\big((u \times 1)x\big) = -u \times u$

(b) $d_{m+1}\big((1 \times u)x\big) = u \times u$

and the kernel of $d_{m+1} : E_{m+1}^{m+1,m} \longrightarrow E_{m+1}^{2m+2,0}$ is generated by $(u \times 1)x - (1 \times u)x$.

Proof: To prove (a) consider
$d_{m+1}\big((u \times 1)x\big) = (-1)^{m-1}(u \times 1)(u \times 1 - 1 \times u) = u^2 \times 1 - u \times u = -u \times u$
(b) follows from a similar argument. Thus

$$d_{m+1}\big((u \times 1)x + (1 \times u)x\big) = 0$$

and an easy argument shows that the kernel of $d_{m+1}^{m+1,m}$ has $(u \times 1)x + (1 \times u)x$ as generator.

We now consider the digram:

(5)

$$\begin{array}{ccc} \Lambda M & \xrightarrow{\widehat{\Delta}} & M^I \\ p \downarrow & & \downarrow q \\ M & \xrightarrow{\Delta} & M \times M \end{array}$$

The $\mathbb{Z}$-cohomoloy of ΩS^{m+1} has the form $H^*(S^m) \otimes H^*(\Omega S^{2m+1})$ where the first factor has generator x in dimension m and the second factor is a divided polynomial algebra with generators $y_1, y_2, \ldots, y_k, \ldots$ in dimensions $2km$. Let $y = y_1$. Recall that d is the differential in the SS for p, and $x \in \overline{E}_{m+1}^{0,m}, y \in \overline{E}_{m+1}^{0,2m}$.

2.4 Lemma $\overline{d}_{m+1}(x) = 0$, $\overline{d}_{m+1}(y) = 2ux$

Proof. First observe that from Lemma 2.2.

$$\overline{d}_{m+1}(x) = \overline{d}_{m+1}\widehat{\Delta}^*(x) = \Delta^* d_{m+1}(x) = \Delta^*(u \times 1 - 1 \times u) = u - u = 0$$

Since, $M^I \sim M$, in the SS for q, we may assume

$$d_{m+1}(y) = (u \times 1)x + (1 \times u)x \in E_{m+1}^{m,m-1}$$

and hence

$$\overline{d}_{m+1}(y) = \overline{d}_{m+1}\widehat{\Delta}^*(y) = \Delta^* d_{m+1}(y) = 2ux$$

2.5 Lemma Let $i : \Omega M \longrightarrow \Lambda M$, denote the inclusion map, where $M = S^{m+1}$ as above. Then,

$$i^* : H^q(\Lambda M; \mathbb{Z}_2) \longrightarrow H^q(\Omega M; \mathbb{Z}_2)\ ,\ q \geq 0$$

is surjective.

Proof. Consider the terms. $E_{m+1}^{0,*} = H^*(\Omega M)$ in the spectral sequence for p over $\mathbb{Z}$. Let y_k denote one of the generators of the divided polynomial algebra $H^*(\Omega S^{2m+1})$. Then, using induction,

$$\bar{d}_{m+1}(y_1 y_{k-1}) = k\bar{d}_{m+1}(y_k) = (2ux)y_{k-1} + y.(2uxy_{k-2}) = k\ 2uxy_{k-1}$$

therefore, $\bar{d}_{m+1}(y_k) = 2uxy_{k-1}$. Hence, over $\mathbb{Z}_2$, $\bar{d}_{m+1}(y_k) = 0$. This is sufficient to verify the lemma.

2.6 <u>Theorem.</u> If $M = S^{m+1}$, $m+1$ even, then as algebras,

$$H^*(\Lambda S^{m+1};\mathbb{Z}_2) \simeq H^*(S^{m+1},\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(\Omega S^{m+1};\mathbb{Z}_2)$$

<u>Proof.</u> This is immediate from the Leray-Hirsch theorem and the fact that i^* is an isomorphism in dimensions which are multiplies of m.

The case case when $M = S^{m+1}$ with $m+1$ odd is considerably easier. $H^*(\Omega S^{m+1})$ is the divided polynomial algebra on generators $y_1, \ldots, y_k, \ldots$ and in the SS for p, $\bar{d}_{m+1}\ y_k = 0$, so that i^* is surjective over $\mathbb{Z}$ as well as over $\mathbb{Z}_2$.

2.7 <u>Theorem.</u> If $M = S^{m+1}$, $m+1$ odd, then as algebras over $\mathbb{Z}$,

$$H^*(\Lambda S^m) \simeq H^*(S^{m+1}) \otimes H^*(\Omega S^{m+1})$$

2.8 <u>Corollary</u> If $M = S^{m+1}$, $m \geq 1$, the cup length of $H^*(\Lambda S^{m+1};\mathbb{Z}_2)$ is infinite.

<u>Proof.</u> $H^*(\Lambda S^{m+1};\mathbb{Z}_2)$ contains the divided polynomial algebra over $\mathbb{Z}_2$ on generators $y_1, y_2, \ldots, y_k$ and calculating binomial coefficients mod2 we find that the cup product

$$y_2 \cdot y_4 \cdot y_8 \ldots y_2 k = y_r,\ r = 2^{k-1} - 2$$

is non-zero for all $k \geq 1$.

2.9 <u>Remark.</u> Corollary 2.8 implies that the category of $\Lambda S^{m+1}, m \geq 1$, is infinite. However, the direct argument in [FH] is simpler. Nevertheless, we will need Corollary 2.8 later on to compute a relative cup product. Our next example cannot be handled using [FH].

Let f denote any map $f: M \longrightarrow M$ and consider its graph $1 \times f : M \longrightarrow M \times M$. Let $\Lambda_f M$ denote the total space of the induced by the fibration q as in the following diagram:

$$\begin{array}{ccc} \Omega M & & \Omega M \\ \downarrow & & \downarrow \\ \Lambda_f M & \xrightarrow{\hat{f}} & M^I \\ p_f \downarrow & & \downarrow q \\ M & \xrightarrow{1\times f} & M \times M \end{array}$$

An important case for us in the application to be given in Section 3, is $M = S^{m+1}$, $m+1$ even, and f the antipodal map. (If $m+1$ is odd, f is homotopic to the identity and $\Lambda_f M \sim \Lambda M$). Notice that in this case p_f does not admit a section which is why [FH] does not apply.

Let $\widetilde{d}$ denote the differential in the SS over $\mathbb{Z}$ for the fiber map p_f. The analogue of Lemma 2.4 is:

2.10 <u>Lemma.</u> $\widetilde{d}_{m+1}(x) = 2u$, $\widetilde{d}_{m+1}(y) = 0$

<u>Proof.</u> $\widetilde{d}_{m+1}(x) = \widetilde{d}_{m+1}\widehat{f}^*(x) = (1\times f)^* d_{m+1}(x) = (1\times f)^*(u\times 1 - 1\times u) = 2u$. Furthermore, $\widetilde{d}_{m+1}(y) = \widetilde{d}_{m+1}\widehat{f}^*(y) = (1\times f)^* d_{m+1}(y) = (1\times f)^*((u\times 1)x + (1\times u)x) = ux - ux = 0$.

2.11 <u>Lemma.</u> Let $i : \Omega M \longrightarrow \Lambda_f M$, where $M = S^{m+1}$, $m+1$ even, and f the antpodal map. Then,

$$i^* : H^q(\Lambda_f M; \mathbb{Z}_2) \longrightarrow H^q(\Omega M; \mathbb{Z}_2), q \geq 0$$

is surjective. Furthermore, over $\mathbb{Z}$, the image of

$$i^* : H^*(\Lambda_f M) \longrightarrow H^*(\Omega M)$$

contains the divided polynomial algebra in $H^*(\Omega M)$.

<u>Proof.</u> It follows by induction that $\widetilde{d}_{m+1} y_k = 0$ over $\mathbb{Z}$ where, as above, the y_k are generators of the polynomial algebra $H^*(\Omega S^{2m+1})$. This observation suffices to prove the lemma.

2.12 <u>Corollary.</u> $\Lambda_f M$, where $M = S^{m+1}$, $m \geq 1$ has infinite cup length over $\mathbb{Z}$ and $\mathbb{Z}_2$ and hence $\Lambda_f M$ has infinite category.

2.13 <u>Corollary</u> $H^*(\Lambda_f M; \mathbb{Z}_2) \simeq H^*(S^{m+1}, \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(\Omega S^{m+1}; \mathbb{Z}_2)$, as algebras.

Our next example is the computation on the cup length of ΛCP^n. We will make use of the fact that ΩCP^n has the same homotopy type as $S^1 \times \Omega S^{2n+1}$. Working over $\mathbb{Z}$ and employing the diagrams (3), (4) and (5), let x denote a generator of $H^1(\Omega CP^n)$, u a generator of $H^2(CP^n)$, and $y_1, y_2, \dots y_k, \dots$ the generators of the divided polynomial algebra in $H^*(\Omega CP^n)$ corresponding to $H^*(\Omega S^{2n+1})$. Also let $y_1 = y$, to conform to some previous notation. We may assume that $d_2'(x) = u$ in the SS for p_1. Then, Lemmas 2.1 - 2.2 obtain with only notational adjustments and $d_2''(x) = -u$, $d_2(x) = u \times 1 - 1 \times u$, and $\overline{d}_2(x) = 0$, $\overline{d}_2(u^k x) = 0$, $k = 1, \dots, n$ and the differential operators $\overline{d}_j$ are all trivial for $2 \leq j \leq 2n-1$. At the level $\overline{E}_{2n}^{**}$, we have

$$\overline{E}_{2n}^{i,0} =< u^i >, \overline{E}_{2n}^{i,1} =< u^i x >, \overline{E}_{2n}^{0,n} =< y >,$$

$i = 0, \dots, n$, where <> indicates "generated by".

We will need the following in the SS for q in (5).

2.14 Lemma. Let $u_1 = u \times 1$ and $u_2 = 1 \times u$ in $H^*(CP^n \times CP^n)$. Then, in the SS for q the element

$$w = (u_1^n + u_1^{n-1}u_2 + \ldots + u_1u_2^{n-1} + u_2^n)x \in E_2^{2n,1}$$

is a d_2 cocycle, i.e., $d_2(w) = 0$. w is a generator of kernel d_2 chosen so that $d_{2n}(y) = w$. Therefore, in the SS for p in (5) we have $\bar{d}_{2n}(y) = \bar{d}_{2n}(y_1) = (n+1)u^n x$.

Proof. $d_2w = (u_1^n + u_1^{n-1}u_2 + \ldots + u_1u_2^{n-1} + u_2^n)d_2(x) = (u_1^n + u_1^{n-1}u_2 - \ldots u_1u_2^{n-1} + u_2^n)(u_1 - u_2) = u_1^{n+1} - u_2^{n+1} = 0$.

On the other hand, if

$$(a_1u_1^n + a_2u_1^{n-1}u_2 + \ldots + a_nu_1u_2^{n-1} + a_{n+1}(u_2^n)(u_1 - u_2) = 0$$

we have, by equating coefficients, $a_1 = a_2 = \ldots = a_{n+1}$ and w generates ker d_2. Thus, $d_2y = \pm w$ since $H^{2n+1}(M^I) = 0$ and w cannot survive. There is no loss of generality if we stipulate that $d_2(y) = w$. Finally, in the SS for p, we have

$$\bar{d}_{2n}(y) = \bar{d}_{2n}\hat{\Delta}^*(y) = \Delta^*\bar{d}_{2n}y = (n+1)u^n x.$$

2.15 Lemma. On the SS for p we have

$$d_{2n}(y_k) = (n+1)u^n x y_{k-1}$$

Proof. We use induction on k.

$$d_{2n}(y_1 \cdot y_{k-1}) = (n+1)u^n x y_{k-1} + y_1 d_{2n} y_{k-1}$$

$$= (n+1)u^n x y_{k-1} + y_1(n+1)u^n x y_{k-2} = (n+1)u^n x[y_{k-1} + (k-1)y_{k-1}]$$

$$= k(n+1)u^n x y_{k-1} = k\, d_{2n}(y_k)$$

Hence, $d_{2n}(y_k) = (n+1)u^n x y_{k-1}$

2.16 Corollary. If r is a prime which divides $n+1$, then in the SS for p over $\mathbb{Z}_r$ we have $d_{2n}(x) = 0$ and $d_{2n}(y_k) = 0$ for all $k \geq 1$. Hence, the inclusion map $i : \Omega CP^n \longrightarrow \Lambda CP^n$ induces surjections

$$i^* : H^q(\Lambda CP^n; \mathbb{Z}_r) \longrightarrow H^q(\Omega CP^n; \mathbb{Z}_r)$$

and hence, as algebras,

$$H^*(\Lambda CP^n; \mathbb{Z}_r) \simeq H^*(CP^n; \mathbb{Z}_r) \otimes_{\mathbb{Z}_r} H^*(\Omega CP^n; \mathbb{Z}_r)$$

We need to extend some of these results to configuration spaces. First let $M = F_N(\mathbb{R}^k)$, the N-th configuration space of Euclidean k-space $\mathbb{R}^k$. Recall, that

$$F_N(\mathbb{R}^k) = \{x_1, \dots, x_N) \in (\mathbb{R}^k)^N, x_i \neq x_j \text{ for } i \neq j\}$$

Also, the projection $p_N : F_N(\mathbb{R}^k) \longrightarrow F_{N-1}(\mathbb{R}^k)$ given by $p_N(x_1, \dots, x_N) = (x_1, \dots x_{N-1})$ is locally trivial with fiber $\mathbb{R}^k - A_N$, where A_N a set of $(N-1)$ points. In particular. $p_2 : F_2(\mathbb{R}^k) \longrightarrow \mathbb{R}^k$, with fiber $\mathbb{R}^k - 0$.

Hence $F_2(\mathbb{R}^k) \sim \mathbb{R}^k - 0$ and we have for $k \geq 3$

$$H^*(\Lambda F_2(\mathbb{R}^k); \mathbb{Z}_2) \simeq H^*(S^{k-1}; \mathbb{Z}_2) \underset{\mathbb{Z}_2}{\otimes} H^*(\Omega S^{k-1}; \mathbb{Z}_2)$$

so that $\Lambda F_2\mathbb{R}^k$ has infinite cup length over $\mathbb{Z}_2$. It is easy to see that p_N admits a section for $N \geq 3$. In fact we will produce an equivariant section which will be needed by the next example.

2.17 <u>Lemma.</u> For $N \geq 3$, $p_N : F_N(\mathbb{R}^k) \longrightarrow F_{N-1}(\mathbb{R}^k)$ admits a section σ with the property that $\sigma(-x) = -\sigma(x)$.

<u>Proof.</u> Let

$$\alpha = \alpha(x_1, \dots, x_{N-1}) = \min_{i \neq j} |x_i - x_j|.$$

Define

$$x_N = x_N(x_1, \dots, x_{N-1}) = \left(1 - \frac{\alpha}{2\,|x_2 - x_1|}\right) x_1 + \left(\frac{\alpha}{2\,|x_2 - x_1|}\right) x_2$$

and set

$$\sigma(x_1, \dots, x_{N-1}) = (x_1, \dots, x_{N-1}, x_N)$$

2.18 <u>Theorem.</u> $\Lambda F_N(\mathbb{R}^k)$ has infinite cup length over $\mathbb{Z}_2$ for $k \geq 3, N \geq 2$.

<u>Proof.</u> By the previous lemma $\Lambda p_N : \Lambda F_N(\mathbb{R}^k) \longrightarrow \Lambda F_{N-1}(\mathbb{R}^k)$ admits a section for $N \geq 3$ and hence by induction the result follows.

We now consider the configuration space analogue of Corollary 2.12. We define $\Lambda_A F_N(\mathbb{R}^k)$ as a pull-back by the diagram

$$\begin{array}{ccc} \Lambda_A F_N(\mathbb{R}^k) & \longrightarrow & [F_N(\mathbb{R}^k)]^I \\ {\scriptstyle p_A} \downarrow & & \downarrow {\scriptstyle q} \\ F_N(\mathbb{R}^k) & \xrightarrow{I \times A} & F_N(\mathbb{R}^k) \times F_N(\mathbb{R}^k) \end{array}$$

where $A(x_1, \dots, x_N) = (-x_1, \dots, -x_N)$. Thus, $\Lambda_A F_N(\mathbb{R}^k)$ is the space of paths $q = (q_1, \dots, q_N)$ in $F_N(\mathbb{R}^k)$ such that $q_i(1) = -q_i(0)$. Then, the fibration

$p_N : F_N(\mathbb{R}^k) \longrightarrow F_{N-1}(\mathbb{R}^k), N \geq 3$ induces

$$\bar{p}_N : \Lambda_A F_N(\mathbb{R}^k) \longrightarrow \Lambda_A F_{N-1}(\mathbb{R}^{k-1}).$$

$\bar{p}_N$ admits a section using the section σ of Lemma 2.17. It is easy to identify $\Lambda_A F_2(\mathbb{R}^k)$, up to homotopy type, with $\Lambda_f S^{k-1}$ of Corollary 2.12. Combining the remarks we obtain:

2.19 Theorem. For $N \geq 2, k \geq 3$, the cup length of $\Lambda_A F_N(\mathbb{R}^k)$ is infinite.

Our final computation concerns the relative category of a certain pair which can be estimated by considering $H^*(X;A)$ as a module over $H^*(X)$. In [BR], Bahri and Rabinowitz exploited a purely topological result that the free loop spaces $\Lambda F_3(\mathbb{R}^k)$ and $\Lambda F_2(\mathbb{R}^k)$ were not of the same homotopy type to prove a theorem of the 3-body type concerning the existence of an unbounded sequence of critical values without a symmetry condition on the potential (see Section 3). This topological result is derived from a result of Vigué-Poirier and Sullivan [VS] to the effect that the rational Betti numbers of $\Lambda F_3(\mathbb{R}^k)$, $k \geq 3$, were unbounded, while those of $\Lambda F_2(\mathbb{R}^k)$ were bounded. The following result (theorem) provides an alternative tool for the Bahri-Rabinowitz theorem and will, hopefully, play a role in the case $N > 3$.

First we recall one of the definitions of relative category introduced in [F1], [F2].

2.20 Definition Let (X, A) be a topological pair. A categorical cover for (X, A) of length n is an $(n+1)$-tuple of open sets $(V_0, V_1, \ldots, V_n)$ such that $\bigcup V_j \supset X$, V_0 deforms in X to A relative to A, and $V_i, i \geq 1$, deforms in X to a point. $\mathrm{cat}(X, A)$ is the minimum length of such categorical covers if such categorical covers exist. Otherwise, set $\mathrm{cat}(X, A) = \infty$.

The next result is the analogue for cup length in this setting, using the fact that over any commutative ring of coefficients, $H^*(X, A; R)$ is a module over $H^*(X; R)$. Although, the result depends on coefficients we will not display it in the notation.

2.21 Proposition. [F1]. If there exist n elements $u_1, \ldots, u_n$ in $H^*(X)$ of positive dimension such that the product $u_1 u_2 \ldots u_n$ is not in the annihilator of $H^*(X, A)$, then $\mathrm{cat}(X, A) > n$.

Now, let $M = S_1 \vee \ldots \vee S_m$ denote a wedge of spheres of dimension ≥ 2 and M' a "subwedge" which we take to be $S_1 \vee \ldots \vee S_k$, $k < m$. We employ Proposition 2.21 and Corollary 2.8 to prove the following.

2.22 Theorem. $\mathrm{cat}(\Lambda M, \Lambda M') = \infty$.

Proof. We work with $\mathbb{Z}_2$ coefficients.

Consider the diagram

$$\begin{array}{ccccccc}
o \longrightarrow H^q(\Lambda M, \Lambda M') & \xrightarrow{j^*} & H^q(\Lambda M) & \xrightarrow{i^*} & H^q(\Lambda M') & \cdots & o \\
 & & \Big\downarrow r^* & & \nearrow o & & \\
 & & H^q(\Lambda S_m) & & & &
\end{array}$$

where i^* surjects because $\Lambda M'$ is a retract of ΛM and r^* injects, where $r : \Lambda M \longrightarrow \Lambda S_m$ is a retraction which takes $\Lambda M'$ to a point. Since $H^*(\Lambda S_m)$ has infinite cup length the result follows.

2.23 Remark. In the Ljusternick-Schnirelmann method it is useful to know that when the categroy of a space X is inifinite, there are compact subsets of arbitrarily high category in X. When the cup length of X using singular cohomology is infinite over some coefficient field, this is automatic [FH]. We are indebted to Luis Montejano for suggesting the use of "infinite dimensional topology" to verify that when X is an ANR (metric) and has infinite category, then X has compact subsets of arbitrarily high category. For example, if X is a Hilbert manifold modelled on a Hilbert space H, then by a result of D. Henderson [H], $X = P \times H$, where P is a locally finite polyhedron. If X has infinite category, then so does P. Since P is σ-compact, it is now an exercise to show that P has subpolyhedra of arbitrarily high category in P.

In the next section it will be necessary to apply some of the computations of this section to the corresponding Sobolev spaces. Let $W_T^{1,2}(\mathbb{R}^{kN})$ denote the Sobolev space of T periodic functions $q : \mathbb{R} \longrightarrow \mathbb{R}^{kN}$ which are absolutely continuous and have square summable first derivatives. q can be represented by $q = (q_1, \ldots, q_N)$ where each $q_i : \mathbb{R} \longrightarrow \mathbb{R}^k$. The inner product for $W_T^{1,2}(\mathbb{R}^{kN})$ is given by

$$\langle f, g \rangle = \int_0^T < \dot{f}(t), \dot{g}(t) > dt + \int_0^T < f(t), g(t) > dt.$$

Let $C_T^0(\mathbb{R}^{kN})$ denote the Banach space of continuous T-periodic functions with the uniform norm. We may readily identify $C_T^0(\mathbb{R}^{kN})$ with $\Lambda\mathbb{R}^{kN}$. It is a well-known fact [K] that the inclusion $i : W_T^{1,2}(\mathbb{R}^{kN}) \longrightarrow C_T^0(\mathbb{R}^{kN})$ is a continuous injection, whose image is dense in $C_T^0(\mathbb{R}^{kN})$. Define an open set $\Lambda(N) \subset W_T^{1,2}(\mathbb{R}^{kN})$ as follows:

$$\Lambda(N) = \{(q_1, \ldots, q_N) \in W_T^{1,2}(\mathbb{R}^{kN}) : q_i(t) \neq q_j(t) \text{ for } 1 \leq t \leq T\}$$

Then,

$$\bar{i} = i|\Lambda(N) : \Lambda(N) \longrightarrow \Lambda F_N(\mathbb{R}^k) \subset \Lambda\mathbb{R}^{kN}$$

where $\Lambda F_N(\mathbb{R}^k)$ is an open subset of $\Lambda\mathbb{R}^{kN}$. Then, by a theorem of Palais [P], $\bar{i}$ is a homotopy equivalence. Thus $\Lambda(N)$ has both infinite cup length over $\mathbb{Z}_2$ and infinite category. Now, introduce an action of $\mathbb{Z}_2 = \{1, \varsigma\}$ on $W_T^{1,2}(\mathbb{R}^{kN})$ by $(\varsigma q)(t) = -q(t - \frac{T}{2}), 0 \le t \le 1$. Let E_0 denote the fixed point set of this action, namely those q such that $\varsigma q = q$. Then E_0 is a closed Hilbert subspace of $W_T^{1,2}(\mathbb{R}^{kN})$. Let C_0 denote the corresponding subspace of $C_T^0(\mathbb{R}^{kN})$. Then, again E_0 continuously injects into C_0, with image dense in C_0. Let $\Lambda_0(N) = \Lambda(N) \cap E_0$ and $\Lambda_0 F_N(\mathbb{R}^{kN}) = C_0 \cap \wedge F_N(\mathbb{R}^k)$. Then, by the same argument as above, $\Lambda_0(N)$ and $\Lambda_0 F_N(\mathbb{R}^k)$ have the same homotopy type. But $\Lambda_0 F_N(\mathbb{R}^k)$ may be identified with $\Lambda_A F_N(\mathbb{R}^k)$ of the theorem 2.19. Thus, $\Lambda_0(N)$ has infinite cup length over $\mathbb{Z}_2$ and infinite category. Summarizing:

2.25 Theorem: Let

$$\Lambda(N) = \{q_1, \ldots, q_N) \in W_T^{1,2}(\mathbb{R}^{kN}) : q_i(t) \ne q_j(t), 0 \le t \le T\}$$

*

$$\Lambda_0(N) = \{q_1, \ldots, q_N) \in \Lambda(N) : q_i(t) = -q_i(t + \frac{T}{2}), 0 \le t \le T, 1 \le i \le N\}.$$

Then, both $\Lambda(N)$ and $\Lambda_0(N)$ have infinite cup length over $\mathbb{Z}_2$ and hence infinite category (with compact subsets of arbitrarily high category).

3. A Hamiltonian System of the N-body Type

Consider a potential function $V : F_N(\mathbb{R}^k) \times \mathbb{R} \longrightarrow \mathbb{R}$ of the following form:

$$V(q_1, \ldots, q_N) = \frac{1}{2} \sum_{1 \le i \ne j \le N} Vij((q_i - q_j), t)$$

and the following properties for $1 \le i \ne j \le N, q \in \mathbb{R}^k - 0, 0 \le t \le T$.

*

(V_1) $V_{ij} \in C^1((\mathbb{R}^k - \{0\}) \times \mathbb{R}, \mathbb{R}), V_{ij}(q,t) \le 0$.

(V_2) $V_{ij}(q,t) = V_{ji}(q,t)$ and $V_{ij}(q,t) = V_{ij}(q, t + \frac{T}{2})$.

(V_3) $V_{ij}(q,t) \longrightarrow -\infty$ as $q \longrightarrow 0$ uniformly in t.

(V_4) There exists $U_{ij} \in C^1(W - 0; \mathbb{R})$ on a neighborhood W of 0 in $\mathbb{R}^k$ such that:

$$a) U_{ij}(q) \longrightarrow +\infty \text{ as } q \longrightarrow 0$$

$$b) - V_{ij}(q,t) \ge |U'_{ij}(q)|^2, q \in W - 0, t \in [0,T].$$

(V_4), introduced by Gordon [G], is called the Strong Force Condition.

Consider the following Hamiltonian system

$$(HS) \qquad m\ddot{q} + V_q(q,t) = 0, \; q = (q_1, \ldots, q_N)$$

where $m = (m_1, \ldots, m_N)$ is the mass vector with $m_i > 0$. The functional corresponding to (HS) has the form

$$(*) \quad I(q) = \sum_{i=1}^{N} \frac{m_i}{2} \int_0^T |\dot{q}_i(t)|^2 dt - \int_0^T V(q_1(t), \ldots, q_N(t), t)\, dt$$

The arguments employed do not depend on the values m_i so we assume the masses $m_i = 1$ and write

$$(*) \quad I(q) = \frac{1}{2} \int_0^T |\dot{q}|^2 dt - \int_0^T V(q,t)\, dt$$

If we let E denote the Sobolev space $W_T^{1,2}(\mathbb{R}^{kN})$ of T-periodic, absolutely continuous functions with L^2 derivatives, then if (V_1) holds, $I(q)$ is C^1 and bounded from below by 0 on the open subset

$$\Lambda(N) = \{q \in W_T^{1,2}(\mathbb{R}^{kN}) : q_i(t) \neq q_j(t), 1 \leq i \neq j \leq N, 0 \leq t \leq T\}$$

which corresponds to $W_T^{1,2}(F_N(\mathbb{R}^k))$. We also define a closed subspace E_0 of E as follows: Let $\mathbb{Z}_2 = \{1, \varsigma\}$ denote the group of order 2 with non-trivial element ς. Define the action of $\mathbb{Z}_2$ on E by $(\varsigma q)(t) = -q(t + \frac{1}{2})$. Then,

$$E_0 = \{q \in E, \varsigma q = q\}$$

We also set $\Lambda_0(N) = E_0 \cap \Lambda(N)$.

3.1 Theorem. If the potential V satisfies $(V_1) - (V_4)$, then $(*)$ possesses an unbounded sequence of citical values.

3.2 Remark. The Coti-Zelati result [C] proves that when V is autonomous and T-periodic, that the minimum of $(*)$ is a critical value. The Bahri-Rabinowitz result [BR] for $N = 3$, assumes no symmetry such as $V_{ij} = V_{ji}$, but imposes conditions on behaviour of V and V' at infinity.

Before proceeding with the proof of Theorem 3.1 we observe that

$$(1) \qquad V(-q,t) = V(q,t) \quad q \in \Lambda(N), t \in [0, T].$$

and

$$(2) \qquad I(\varsigma q) = I(q), q \in \Lambda(N).$$

3.3 Proposition. Let I_0 denote $I|\Lambda_0(N)$. Then critical points of I_0 are critical points of I.

Proof. This is a general phenomenon. Namely if a functional I is invariant under the action of a finite group G, then critical points of the restriction I_0 to the fixed point set of the action are always critical points of I. In our case, if $u \in E$, then $u - \varsigma u \in E_0$ and

$$I'(q)(u + \varsigma u) = 2I'(q)(u) \qquad q \in \Lambda_0(N)$$

and hence if q is a critical point for I_0, $I'(q)$ vanishes on E.

Theorem 3.1 now follows from the following theorem

3.4 Theorem. If V satisfies $(V_1) - (V_4)$, then $I_0 = I|\Lambda_0(N)$ possesses an unbounded sequence of critical values.

The proof will be broken down into a series of lemmas.

3.5 Lemma. (Gordon's Lemma [G]) If V satisfies $(V_1) - (V_4)$ and if a sequence q_n in $\Lambda_0(N)$ converges weakly to $q \in E_0$, then if $q \in \partial\Lambda_0(N)$, then $I_0(q_n) \longrightarrow +\infty$, where $\partial\Lambda_0(N)$, is the boundary of $\Lambda_0(N)$ in E_0.

3.6 Lemma. If V satisfies $(V_1)-(V_4)$, I_0 satisfies the Palais-Smale condition (PS) on $\Lambda_0(N)$.

Proof. Let q_n denote a sequence in $\Lambda_0(N)$ such that $I_0(q_n) \longrightarrow s \geq 0$ and $I_0'(q_n) \longrightarrow 0$. Then, we may assume $I_0(q_n) \leq s + 1$ and hence

$$\int_0^T |\dot{q}_n(s)|^2 ds \leq s + 1$$

then, since $q_n \in \Lambda_0(N)$

$$\int_t^{t+\frac{1}{2}} \dot{q}_n(s) ds = q_n(r + \frac{1}{2}) - q_n(t) = -2q_n(t)$$

it follows easily that the sequence q_n is bounded in the $W_T^{1,2}$ norm. Again, by standard arguments [R1], there is a subsequence, also denoted by q_n, such that q_n converges weakly to $q \in E_0$, and Gordon's lemma implies that $q_0 \in \Lambda_0(N)$. Furthermore, I' has the form $I'(q) = q - \mathcal{P}_q$, where $\mathcal{P}q_n$ has a (strongly) convergent subsequence in E_0. Since $I'(q_n) \longrightarrow 0$, it follows that a subsequnce of q_n converges strongly to q and I_0 is (PS) on $\Lambda_0(N)$.

The next lemma merely isolates the deformation theorem we employ (see [R2]).

3.7 Lemma. Let Ω denote an open set in a Hilbert space E and $I : \Omega \longrightarrow R$ a C^1 functional which is bounded from below. Suppose I satisfies (PS) on Ω and we have the condition that when $q_n \longrightarrow q \in \partial\Omega$, $q_n \in \Omega$, then $I(q_n) \longrightarrow +\infty$, i.e. I is unbounded at the boundary $\partial\Omega$. Then, for $c \in \mathbb{R}, \bar{\epsilon} > 0$ and U a neighborhood of $K_c = \{q \in \Omega : I(q) = c$ and $I'(q) = 0\}$, there is an $\epsilon > 0, \epsilon < \bar{\epsilon}$ and a deformation $\phi : \Omega \times I \longrightarrow \Omega$ such that

(1) $\phi_0 =$ identity $, \phi_t : \Omega \longrightarrow \Omega$ is a homeomorphism, $t \in I$.

(2) $\phi(q, t) = q$ if $|I(q) - c| \geq \bar{\epsilon}, t \in I$.

(3) $\phi(q,1) \in I^{c-\epsilon}$ if $q \in I^{c+\epsilon} - U$, where

$$I^a = \{q \in \Omega : I(q) < a\}.$$

(4) If $K_c = \emptyset$ we may take $U = \emptyset$.

3.8 Remark. The proof of Lemma 3.7 requires only a minor modification of the proof of theorem A.4 in [R2]. Following the notation in [R2], let

$$A \equiv \{u \in \Omega \mid I(u) \leq c - \hat{\epsilon}\} \cup \{u \in E \mid I(u) \geq c + \hat{\epsilon}\}$$

and

$$B \equiv \{u \in \Omega \mid c - \epsilon \leq I(u) \leq c + \epsilon\}$$

Then, A is closed in Ω but $A \cup (E - \Omega) = A_1$ is closed in E. B is closed in Ω but also closed in E because I is unbounded at $\partial\Omega$. Furthermore, $A_1 \cap B = \emptyset$ because $\epsilon < \hat{\epsilon}$. After requiring the cutoff function to be 0 on A_1 and 1 on B, the proof proceeds verbatim and the Ω is forced to be invariant under the flow.

Next, the corresponding abstract critical point theorem.

3.7 Lemma Let Ω denote an open set in a Hilbert space E and $f : \Omega \longrightarrow \mathbb{R}$ a C^1 functional which is bounded from below. Suppose f satisfies (PS) on Ω and f is unbounded at the boundary $\partial\Omega$. If cat$\Omega = +\infty$ then f possesses an unbounded sequence of critical values.

Proof. The proof is quite standard after making a few remarks. First, since Ω is a Hilbert manifold, Ω possesses compact subsets of arbitrarily high category (see Section 2.) Thus, if we set $\sum_j = \{X \subset \Omega : \text{cat}X \geq j\}$ and

$$c_j = \inf_{\sum_j} \sup_X f(x), \; j \geq 1$$

we see that the c_j are finite and $c_j \leq c_{j+1}$ with $c_1 = \inf_\Omega f$. The usual arguments apply to show that the c_j are all critical values, and $\lim c_j = +\infty$ (see [R2]).

Proof of theorem 3.3 The proof is an immediate application of Lemma 3.2 to
Lemma 3.7.

Bibliography

[BR] Abbas Bahri and Paul H. Rabinowitz, Periodic solutions of Hamiltonian systems of the 3-body type, CMS Technical Report No. 90-8, University of Wisconsin-Madison.

August 1989

[C] Vittorio Coti-Zelati, Periodic solutions for N-body type problems, Preprint 1989.

[F1] E. Fadell, Cohomological methods in non-free G-spaces. with applications to general Borsuk-Ulam theorems and critical point theorems for invariant functionals, Nonlinear Functional Analysis and its Applications, Reidel 1986, 1-47.

[F2] E. Fadell, Lectures in Cohomological Index Theories of G-spaces with Applications to Critical Point theory, Raccolta Di Universitá della Calabria, 1987, Cosenza, Italy.

[FH] E. Fadell and S. Husseini. A note on the category of the free loop space, Proc. Amer. Math. Soc. 107 (1989), 527-536.

[G] W. Gordon, Conservative dynamical systems involving strong forces. Trans. Amer. Math. Soc. 204 (1975), 113-135.

[H] D. Henderson, Infinite dimensional manifolds are open subsets of Hilbert space, Topology 9 (1970), 25-33.

[K] W. Klingenberg, Lectures on Closed Geodesics, Springer-Verlag, GMW No. 230 (1978).

[P] Richard S. Palais, Homotopy theory of infinite dimensional manifolds, Topology 5 (1966), 1-16.

[R1] Paul H. Rabinowitz, Periodic solutions for some forced singular Hamiltorian systems, to appear, Festschrift in honor of Jürgen Moser.

[R2] Paul H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, CBMS Ser. in Math. 65 (1986), AMS, Providence, R.I.

[VS] M. Vigué-Poirier and D. Sullivan, The homology theory of the closed geodesic problem, J. Diff. Geometry II (1976), 633-644.

Edward Fadell
Department of Mathematics
University of Wisconsin-Madison
Madison, Wisconsin 53706
U.S.A.

Sufian Husseini
Department of Mathematics
University of Wisconsin-Madison
Madison, Wisconsin 53706
U.S.A.

A Note on the Category of the Free Loop Space

E. Fadell and S. Husseini

Abstract

A useful result in critical point theory is that the Ljusternik–Schnirelmann category of the space of based loops on a compact simply connected manifold M is infinite (because the cup length of M is infinite). However, the space of free loops on M may have trivial products. This note shows that, nevertheless, the space of the free loops also has infinite category.

1. Introduction.

It is a standard result that if M is a simply connected compact manifold compact and $\Omega(M) = \Omega(M,x_0)$ is the space of based loops on M (based at x_0), then the Ljusternik–Schnirelmann category cat $\Omega(M) = +\infty$. This follows from the now classical result [Serre, 1] that the real (or rational) cohomology of $\Omega(M)$ has non–trivial cup products of arbitrary high length (see also [2]). An inspection of the proof will convince the reader that compactness is not required for the proof of this result. All that is required is that the real (or rational) cohomology $H^*(M)$ be finitely generated and for some $i > 0$ $H^i(M) \neq 0$. However, for the free loop space $\Lambda(M)$, where

$$\Lambda(M) = \{\alpha \in M^I; \alpha(0) = \alpha(1)\} ,$$

*Supported in part by the NSF under Grant DMS–8722295.

it isn't necessarily the case that the cohomology of $\Lambda(M)$ has non–trivial cup products. This is a relatively recent result of M. Vigue–Poirrier and D. Sullivan [3], where, for example, the reduced real cohomology of the free loops on the 2–sphere S^2 has trivial cup products. In view of this fact, it is natural to inquire about that category of the free loop space $\Lambda(M)$. We will show that when M satisfies the preceding conditions:

1. $\mathrm{cat}\,\Lambda(M) = +\infty$
2. $\Lambda(M)$ contains compact subsets C such that $\mathrm{cat}_{\Lambda(M)}C$ is arbitrarily large.

This will allow a direct application of the Ljusternik–Schnirelmann method to, for example, functionals defined on the Sobolev space

$$W_T^{1,2} = \{f: [0,T] \to \mathbb{R}^n - 0,\ f(0) = f(T)\}\ .$$

See Section 4 and [4], [5].

The basic result is the following property of Hurewicz fibrations [6], which we prove in the next section.

<u>Theorem</u>. Let $F \xrightarrow{i} E \xrightarrow{p} B$ denote a Hurewicz fibration which admits a section $\sigma: B \to E$, and F,B and E are 0–connected. If $Q \subset F$ is any subset of F, then

$$\mathrm{cat}_F Q \leq \mathrm{cat}_E Q\ .$$

When applied to the fibration $\Omega(M) \to \Lambda(M) \to M$ we obtain:

<u>Corollary</u>: Let M denote a simply connected manifold (not necessarily compact) such that the real or rational cohomology $H^q(M)$ is finitely generated for each q and $H^i(M) \neq 0$ for some $i > 0$. Then $\mathrm{cat}\,\Lambda(M) = \infty$.

Section 3 considers the category of the free loop space on configuration spaces.

2. Results.

We recall first a basic lemma for Hurewicz fibrations [7]. If $F \xrightarrow{i} E \xrightarrow{p} B$ is a Hurewicz fibration, then there is a lifting function $\lambda:\Omega_p \to E^I$, where $\Omega_p = \{(x,\omega) \in E\times B^I : p(x) = \omega(0)\}$, where $\lambda(x,\omega)(0) = x$; $p(\lambda(x,\omega)(t)) = \omega(t)$, $0 \le t \le 1$. λ induces $\tilde{\lambda}: E^I \to E^I$ by setting $\tilde{\lambda}(\alpha) = \lambda[\alpha(0),p\alpha]$.

2.1 Lemma [7]. $\tilde{\lambda} \sim \mathrm{id}$ preserving projections, i.e. there is a homotopy $\Gamma: E^I \times I \to I$ such that $\Gamma_0 = \mathrm{id}$, $\Gamma_1 = \tilde{\lambda}$ and $p\Gamma(\alpha,s)(t) = p\alpha(t)$ for $\alpha \in E^I$, $s,t \in I$.

2.2 Proposition. Let $F \xrightarrow{i} E \xrightarrow{p} B$ denote a Hurewicz fibration with base points $x_0 \in F$, $b_0 \in B$, $F = p^{-1}(b_0)$. We assume that $\Omega_p: \Omega(E,x_0) \to \Omega(B,b_0)$ admits a section σ. If Y is any space and $f: Y \to F$ is a map homotopic in E to the constant x_0, then f is homotopic in F to x_0.

Proof. Let $A: Y \to E^I$ denote a homotopy such that $A(y)(0) = x_0$, $A(y)(1) = f(y)$. Consider the homotopy $\hat{A}: Y\times I \to F$ given by

$$\hat{A}(y,s) = \Gamma(A(y),s)(1),\ \ 0 \le s \le 1,\ \ y \in F .$$

Note that $\hat{A}(y,0) = f(y)$, $\tilde{A}(y,s) \in F$ for $0 \le s \le 1$. Let $\tilde{A}(y,1) = g(y)$ so that $f \sim g: Y \to F$.

Now, define $C: Y \to E^I$ by $C(y) = \sigma pA(y)$ and observe that $pC(y) = pA(y)$. Now, define a homotopy $\hat{C}: Y\times I \to F$ by

$$\hat{C}(y,s) = \Gamma(C(y),s)(1),\ \ 0 \le s \le 1,\ \ y \in F.$$

Note that $\hat{C}(y,0) = x_0$, $\hat{C}(y,1) = g(y)$, and $\hat{C}(y,s) \in F$, $0 \le s \le 1$. Thus, $f \sim x_0: Y \to F$ and the proposition follows.

2.3 Remark. The reader will note the similarity between Proposition 2.2 and the classical result that $i_*: \pi_*(F) \to \pi_*(E)$ is injective when p admits a section [8]. Proposition 2.2 may also be thought as a situation when the fiber is totally non–homotopic to zero.

Now we recall that a set Q in Y is categorical in Y if the inclusion $i: Q \to Y$ is homotopic in Y to the constant map $y_0 \in Y$.

2.3 Corollary. Let $F \xrightarrow{i} E \xrightarrow{p} B$ denote a Hurewicz fibration as in 2.2 with E 0–connected. If U is categorical in E, then $U \cap F$ is categorical in F. Consequently, for any subset $Q \subset F$

$$\mathrm{cat}_F Q \leq \mathrm{cat}_E Q .$$

Proof. The first part follows from Proposition 2.2, while the second part is immediate from the definition of category, namely, $\mathrm{cat}_Y X$ is the minimum number of categorical open sets in Y which cover X.

2.5 Definition. A manifold M will be called admissible if M is simply connected, the real (or rational) cohomology $H^*(M)$ is finitely generated and for some $i > 0$, $H^i(M) \neq 0$.

2.5 Corollary. Let M denote a simply connected manifold with base point x_0. Then, we have the fibration with section

$$\Omega(M,x_0) \xrightarrow{i} \Lambda(M) \underset{\sigma}{\overset{p}{\rightleftarrows}} M$$

where $p(\omega) = \omega(0)$ and $\sigma(x) = \tilde{x}$ the constant loop at $x \in M$. If $Q \subset \Omega(M,x_0)$ is any subset

$$\mathrm{cat}_{\Omega(M,x_0)}Q \le \mathrm{cat}_{\Lambda(M)}Q \,.$$

In particular, if M is admissible, then $\mathrm{cat}\,\Lambda(M) = \infty$.

2.6 Remark. Thus for example $\Lambda(S^2)$ has infinite category but trivial cup products over $\mathbb{R}([3])$.

2.7 Remark. Suppose F is a closed subset of E and $Q \subset F$. Then, the reverse inequality $\mathrm{cat}_E Q \le \mathrm{cat}_F Q$ holds whenever E is an ANR (normal). This is the case in Corollary 2.4 and hence there the inequality is actually an equality.

Now, we consider the question of compact subsets of $\Lambda(M)$ of arbitrarily high category.

2.8 Lemma. Suppose X is a space such that for some field $\mathbb{F}$ the cup length of X over $\mathbb{F}$ using singular cohomology is $\ge k$, then X has a compact subset of category $> k$.

Proof. We mention first that throughout we employ singular homology and cohomology with coefficients in $\mathbb{F}$ and will make use of the universal coefficient theorem isomorphism

$$\gamma: H^q(X) \longrightarrow \mathrm{Hom}_{\mathbb{F}}(H_q(X);\mathbb{F}) \,.$$

Let $w = \alpha_1\alpha_2\cdots\alpha_k \in H^q(X)$ denote a non-trivial cup product of length k. Then, $\gamma(w) \ne 0$ and hence there is a singular cycle ζ such that $\gamma(w)([\zeta]) \ne 0$. Let A denote the (compact) support of ζ. Then, it is easy to check that if $i: A \to X$ is the inclusion map, $i^*(w) = i^*(\alpha_1)\cdots i^*(\alpha_k)$ in non-zero in $H^q(A)$. Thus the cup-length of A in X is $\ge k$ and $\mathrm{cat}_X A > k$.

2.9 Corollary. Let M denote an admissible manifold with base point x_0. Then, the space of based loops $\Omega(M,x_0)$ (and hence the space of free loops $\Lambda(M)$) contains compact subsets of arbitrarily high category.

3. Configuration Spaces.

If M is any space the k–th configuration space of X, $k \geq 1$, is defined by (see [9])

$$F_k(M) = \{(x_1,\ldots,x_k),\ x_i \in M,\ x_i \neq x_j,\ \text{for } i \neq j\}\ .$$

We will make of the following propositions. Cohomology will be over a field $\mathbb{F}$ of coefficients

3.1 If M is a manifold (without boundary), and $k \geq 2$ then we have locally trivial fibrations

(i)
$$F_{k-1}(M-Q) \longrightarrow F_k(M) \xrightarrow{p} M$$

where $Q \in M$ and $p(x_1,\ldots,x_k) = x_k$; and

(ii)
$$(M-Q_{k-1}) \longrightarrow F_k(M) \xrightarrow{q} F_{k-1}(M)$$

where $Q_{k-1} \subset M$ is a subset of $k-1$ elements and $q(x_1,\ldots,x_k) = (x_1,\ldots,x_{k-1})$.

3.2. If M is a simply connected manifold, $\dim M = m \geq 3$ and $H^i(M)$ if finitely generated over a field $\mathbb{F}$ for each i, then for $k \geq 1$ $F_k(M)$ is simply connected and $H^i(F_k(M))$ and $H^i(\Omega F_k(M))$ are finitely generated over $\mathbb{F}$ for each i.

We prove the next proposition.

3.3 <u>Proposition</u>. If M is a simply connected manifold, $\dim M = m \geq 3$, then for $k \geq 2$, the configuration space $F_k(M)$ is admissible.

<u>Proof</u>: Because of 3.2 and the fact that $F_k(M)$ is finite dimensional we need only show that for some $j > 0$, the real cohomology $H^j(F_k(M)) \neq 0$.

Case 1. $H^i(M) \neq 0$ for some $i \geq 1$. Choose i maximal so that $H^i(M) \neq 0$ and $0 \neq v \in H^i(M)$. We proceed by induction on k and employ the cohomology spectral sequence of the fibration (i) of 3.1. Choose $u \in F_{k-1}(M-Q)$ of maximal dimension so that $u \neq 0$. Then, in the E_2-term of the spectral sequence $u \otimes v \neq 0$ and has dimension > 0. It is easy to see that $u \otimes v$ "survives" to E_∞ and contributes a non-zero element to $H^j(F_k(M)), j \geq i$.

Case 2. $H^i(M) = 0$ for all $i > 0$. For $k = 2$ we employ the spectral sequence of the fibration (i) of 3.1 to see that $\mathbb{R} = H^{m-1}(M-Q) \approx H^{m-1}(F_2(M))$. For $k \geq 3$, we employ induction on k and the spectral sequence of the fibration (ii) of 3.1, together with the argument in Case 1 to obtain the desired result.

3.4 <u>Proposition</u>. If M is a simply connected manifold, $\dim M \geq 3$, then for $k \geq 2$, $\operatorname{cat} \Lambda F_k(M) = \infty$ and $\Lambda F_k(M)$ contains compact subsets of arbitrarily high category.

4. An Application.

In [4], Rabinowitz used the main result of section 2 (Corollary 2.9) in the special case where $M = \mathbb{R}^M - \{0\}$ to prove the existence of infinitely many periodic solutions of a certain Hamiltonian system. In this section we give an alternative argument for a key proposition in his treatment based upon a general abstract critical point theorem which is the analogue of a previous "linking" result in [10] which was done in the context of a relative cohomological equivariant index theory which will be replaced here by relative (Ljusternik–Schnirelmann) category theory introduced in [11] and [12].

We review first one version of relative category. If (E,A) is a topological pair with $A \neq \phi$ and closed in E, then for $A \subset X \subset E$ we define the relative category $\operatorname{cat}_E(X,A)$ as follows. A <u>categorical cover</u> of (X,A) consists an open (in E) set $W \supset A$ and open sets $\{V_j\}$ such that

1. $W \cup (\cup V_j) \supset X$

2. There is a homotopy of pairs $H:(W,A) \times I \to (E,A)$ such that $H_0(x) = x$ and $H_1(x) \in A$, $x \in A$.

3. Each V_j is contractible to a point in E.

4.1 <u>Definition</u>. $\mathrm{Cat}_E(X,A) = n$ if (X,A) admits a categorical cover $\{W,V_j\}$ with n sets V_j and n is minimal with this property. If no such finite categorical cover exists we set $\mathrm{cat}_E(X,A) = \infty$.

4.2 <u>Remark</u>. If $A = \phi$, $\mathrm{cat}_E(X,\phi) = \mathrm{cat}_E X$ has its usual meaning.

The following properties are immediate

4.3 <u>Proposition</u>

a) $A \subset X_1 \subset X_2$ implies $\mathrm{cat}_E(X_1,A) \le \mathrm{cat}_E(X_2,A)$

b) $A \subset X_1, X_2 \subset E$ implies

$$\mathrm{cat}_E(X_1 \cup X_2,A) \le \mathrm{cat}_E(X_1,A) + \mathrm{cat}_E X_2$$

Relative category may be used to define a "linking" concept as follows.

4.4 <u>Definition</u>. Let A and B denote disjoint closed sets in a space E. If

$$\mathrm{cat}_E(E-B,A) < \mathrm{cat}_E(E,A)$$

we say that A and B <u>link</u> (in the category sense). If, in addition, $\mathrm{cat}_E(E,A) = +\infty$, we say that A and B <u>strongly link</u>.

We review next a local form of the Palais–Smale condition $(PS)_s$. Let Λ denote an open set in a Banach space a $f:\Lambda \to \mathbb{R}$ a C^1–functional. f is said to satisfy $(PS)_s$ if any sequence $q_j \in \Lambda$ satisfying $f(q_j) \to s$ and $f'(q_j) \to 0$ is precompact. $(PS)_s$ is used crucially in the following deformation theorem ([4],[13]). We will use that notation $K_c = \{q \in \Lambda, f'(c) = 0 \text{ and } f(q) = c\}$. Also $f^a = \{q \in \Lambda: f(q) < a\}$.

4.5 Proposition. Let Λ denote an open set in a Banach space E and $f:\Lambda \to \mathbb{R}$ a C^1–functional. Suppose f satisfies $(PS)_s$ for all $s > \alpha$. Then, for any critical point $c > \alpha$, U a neighborhood of K_c, and $\bar{\varepsilon} > 0$, there is an $\varepsilon > 0$ and a deformation $\varphi:\Lambda \times I \to \Lambda$ such that

(1) φ_0 = identity, $\varphi_t:\Lambda \to \Lambda$ is a homeomorphism, $t \in [0,1]$.

(2) $\varphi(q,t) = q$ if $|f(q) - c| \geq \bar{\varepsilon}$.

(c) $\varphi(q,1) \in f^{c-\varepsilon}$ if $q \in (f^{c+\varepsilon} - U)$.

If $K_c = \emptyset$, we may take $U = \emptyset$.

We may now state our abstract critical point theorem.

4.6 Theorem Let Λ denote an open set in a Hilbert (or Banach) space E such that Λ contains compact subsets of arbitrarily large category and let $f:\Lambda \to \mathbb{R}$ denote a C^1–functional. Suppose further that there are disjoint closed sets A and B in Λ such that

1. $\mathrm{cat}_E(\Lambda - B, A) < \mathrm{cat}_E(\Lambda, A) = \infty$, i.e. A and B strongly link.
2. $\mathrm{cat}_E A < \infty$
3. $\sup_A f < \inf_B f$
4. f is $(PS)_s$ for all $s > \sup_A f$

then, f possesses an unbounded sequence of critical values.

Proof. For each integer $j \geq 0$ let

$$\Sigma_j = \{X \mid A \subset X \subset \Lambda,\ \mathrm{cat}_\Lambda(X,A) \geq j\}\ .$$

Observe that for each j, there is a compact set Y such that $\mathrm{cat}_E Y \geq j + \mathrm{cat}_E A$ and hence $\mathrm{cat}_E(A \cup Y, A) \geq j$. Thus, Σ_j is non–empty and f is bounded on $A \cup Y$. Hence, we may define

$$c_j = \inf_{X \in \Sigma_j} \sup_X f(x), \qquad j \geq 0$$

where

$$c_0 \leq c_1 \leq c_2 \leq \ldots c_j \leq c_{j+1} \leq \ldots\ .$$

Let $m = \mathrm{cat}_\Lambda(E-B,A)$. If $\mathrm{cat}_\Lambda(X,A) \geq m+1$, then $X \cap B \neq \phi$ for, otherwise $(X,A) \subset (\Lambda-B,B)$ and $\mathrm{cat}_\Lambda(X,A) \leq m$. Therefore,

$$\sup_A f = c_0 < c_{m+1}$$

i.e. a "jump" occurs at index $m+1$. We now show that each $c_j, j > m$, is a critical value. Let $c = c_j, j > m$ and consider K_c. Choose $\bar{\varepsilon} < \frac{1}{2}(c_{m+1} - c_0)$ and an open set $U \supset K_c$ such that $U \subset f^{-1}(c_{m+1} - \bar{\varepsilon}, \infty)$ and $\mathrm{cat}_\Lambda U = \mathrm{cat}_\Lambda K_c$. Now let $\varepsilon > 0$ and φ the deformation given by Proposition 4.5. Observe that φ remains fixed on A throughout the deformation. Take $X \in \Sigma_j$ so that $\mathrm{cat}_\Lambda(X,A) \geq j$ and $\sup_X f < c + \varepsilon$. Now

$$\mathrm{cat}_\Lambda(X,A) \leq \mathrm{cat}_\Lambda(X-U,A) + \mathrm{cat}_\Lambda(U)\ .$$

If $K_c = \phi$, then $\text{cat}_\Lambda(X,A) = \text{cat}_\Lambda(X-U,A) \geq j$. On the other hand, $f(\varphi_1(X-U),A) < c-\varepsilon$. since, $\text{cat}_\Lambda(\varphi_1(X-U),A) = \text{cat}_\Lambda(X-U,A) \geq j$, this would force $c_j < c - \varepsilon = c_j - \varepsilon$ which is a contradiction. Thus $K_c \neq \phi$.

To show that the c_j are unbounded we proceed as follows. Let $\bar{c} = \sup c_j$. $\bar{c}$ is again a critical value. Let K denote the set of all critical points q such that $c_{m+1} \leq f(q) \leq \bar{c}$. The $(PS)_s$ condition for all $s > c_0$ forces K to be compact and $K \cap A = \phi$. Suppose $\text{cat}_\Lambda K = k > 0$. Again choose $\bar{\varepsilon} < \frac{1}{2}(c_{m+1}-c_0)$ and an open set $U \supset K$ such that $U \subset f^{-1}(c_{m+1}-\bar{\varepsilon},\infty)$ and $\text{cat}_\Lambda U = k$. Furthermore, ε and φ will be as in Proposition 4.5, with $c = \bar{c}$. Choose an index j such that $c_j > \bar{c} - \varepsilon$ and $X \in \Sigma_{j+k}$ such that

$$\sup_X f < c_{j+k} + \varepsilon < \bar{c} + \varepsilon .$$

Then

$$\text{cat}_\Lambda(X,A) \leq \text{cat}_\Lambda(X-U,A) + \text{cat}_\Lambda(U)$$

so that $\text{cat}_\Lambda(X-U,A) \geq j$. But, then $\text{cat}_\Lambda(\varphi_1(X-U),A) \geq j$ and $\varphi_1(X-U,A) \subset f^{\bar{c}-\varepsilon}$. This force $c_j < \bar{c} - \varepsilon$ which is a contradiction.

As an application of Theorem 4.6, we give an alternative proof of a result of P. Rabinowitz which he used to prove a slightly more general result [4]. The setting is the following Hamiltonian system.

$$\text{(HS)} \qquad \ddot{q} + V_q(t,q) = 0$$

where the potential function $V(t,q)$ satisfies the following conditions:

(V1) $V(t,q)$ is a C^1–function from $\mathbb{R} \times \Omega \to \mathbb{R}$, $\Omega = \mathbb{R}^n - \{0\}$, $n \geq 3$, which is T–periodic in t.

(V2) $V(t,q) < 0$ and $V(t,q) \to 0$, $V_q(t,q) \to 0$ as $|q| \to 0$, uniformly in $t \in [0,T]$.

(V3) $V(t,q) \to -\infty$ as $q \to 0$, uniformly in $t \in [0,T]$.

(V4) There is a neighborhood N of 0 in $\mathbb{R}^n$ and a C^1–function $U: N - \{0\} \to \mathbb{R}$ such that $U(q) \to \infty$ as $q \to 0$ and $-V(t,q) \geq |U_q(q)|^2$ for $q \in N - \{0\}$ and all $t \in [0,T]$.

The period $T>0$ will be fixed throughout the remainder of this section and $E_T = W_T^{1,2}(\mathbb{R},\mathbb{R}^n)$ will denote the Sobolev space of T–periodic functions with square summable first derivatives, under the norm

$$\|q\| = \left(\int_0^T |\dot{q}|^2 dt + [q]^2\right)^{1/2}$$

where $\dot{q} = \frac{dq}{dt}$ and

$$[q] = \frac{1}{T}\int_0^T q(t)dt .$$

Set

$$\Lambda = \Lambda_T = \{q \in E_T \,|\, q(t) \neq 0 \text{ for all } t \in [0,T]\} .$$

Λ is an open subset of E_T and Λ has the same homotopy type as the space $\Lambda(\mathbb{R}^n-0)$ of free loops on $\mathbb{R}^n-0$. $\mathbb{R}^n-0$ is identified with the constant loops in $\mathbb{R}^n-0$.

Corresponding to (HS) is the functional $I:\Lambda \to R$ given by

$$I(q) = \int_0^T (\tfrac{1}{2}|\dot{q}|^2 - V(t,q))dt\,, \qquad q \in \Lambda\,.$$

Critical points of I give classical T–periodic solutions of (HS) (see [4]). We set

$$I^{\varepsilon} = \{q \in \Lambda: I(q) < \varepsilon\}\,.$$

4.7 Proposition Assuming (V1) – (V3), there is an $\varepsilon > 0$ and an $R > 0$ such that if $B(0,R)$ is the open ball of radius R, $A = \mathbb{R}^n - B(0,R)$, $B = \Lambda - I^{\varepsilon}$, then

1. $\sup_A I < \varepsilon$
2. I^{ε} is deformable into A
3. $\mathrm{cat}_{\Lambda_T}(E-B,A) = 0$, $\mathrm{cat}_{\Lambda_T}(\Lambda,A) = +\infty$.

Proof. First we choose a decreasing sequence $\varepsilon_m > 0$ such that $\varepsilon_m \to 0$ and a corresponding increasing sequence $R_m > 0$ such that $R_m \to +\infty$ with the property that

$$|V(t,q)| < \varepsilon_m \text{ implies } |q| > R_m\,.$$

Choose an index k such that

(1) $\qquad R_k - [2\varepsilon_k]^{1/2}T > [2\varepsilon_k]^{1/2}T$

and

(2) $\qquad -\int_0^T V(t,q)dt < \frac{1}{2}\varepsilon_k T$, for $|q| \geq R_k$

and set $\varepsilon = Te_k$, $R = R_k$, $A = \mathbb{R}^n - B(0,R)$, $B = \Lambda - I^{\varepsilon}$. If $q \in I^{\varepsilon}$,

(3) $$\|\dot{q}\|_{L_2}^2 = \int_0^T |\dot{q}|^2 \le 2I(q) \le 2T\varepsilon_k$$

and hence

(4) $$\|\dot{q}\|_{L_2} \le (2\varepsilon)^{1/2} .$$

Now, write $q = [q] + Q$, where $Q(t) = q(t) = [q]$ and

(5) $$[q] = \frac{1}{T} \int_0^T q(t)dt .$$

Recall that

(6) $$\|Q\|_{L_\infty} = \max_t |Q(t)|$$

and the general inequality [4],

(7) $$\|Q\|_\infty \le T^{1/2} \|\dot{q}\|_{L_2} .$$

Hence,

(8) $$\|Q\|_\infty \le (2T\varepsilon)^{1/2} .$$

Now, consider a constant loop q, with $|q| \ge R$.

$$(9) \qquad I(q) = -\int_0^T V(t,q)dt < \frac{\varepsilon}{2} < \varepsilon$$

and hence $A = \mathbb{R}^n - B(0,R) \subset I^\varepsilon$ and

$$(10) \qquad \sup_A I < \varepsilon .$$

Consider now the homotopy, $H: I^\varepsilon \times [0,1] \to \Lambda$, where

$$(11) \qquad H(q,s) = [q] + (1-s)Q, \qquad 0 \le s \le 1$$

which is fixed on constant loops q. For $q \in I^\varepsilon$, it is easy to verify that

$$(12) \qquad |[q]| \ge R - (2T\varepsilon)^{1/2} > (2\varepsilon T)^{1/2}$$

and using (8)

$$(13) \qquad |Q(t)| \le (2T\varepsilon)^{1/2} \text{ for all } t \in [0,1] .$$

This forces

$$(14) \qquad [q] + (1-s)Q(t) \ne 0, \ \ 0 \le s, t \le 1 .$$

Thus the homotopy has range in Λ and deforms I^ε into the subspace $\mathbb{R}^n - B(0,\rho)$, $\rho = (2\varepsilon T)^{1/2}$. If one follows H by a radial homotopy, we obtain a deformation of I^ε to $\mathbb{R}^n - B(0,R)$, with $\mathbb{R}^n - B(0,R)$ fixed throughout the composite homotopy. This also shows that $cat_\Lambda(\Lambda - B, A) = 0$. Finally since $cat_\Lambda A = 2$ and $cat\ \Lambda = +\infty$, it is clear that $cat_\Lambda(\Lambda, A) = +\infty$.

4.8 Theorem (Rabinowitz [4]) If we assume that V satisfies (V1)–(V4), the function I possesses an unbounded sequence of critical values.

Proof: Rabinowitz [4] verifies that I satisfies $(PS)_s$ for all $s > 0$ and we will not repeat the argument. In view of Proposition 4.7, the proof is now a direct application of Theorem 4.6.

References

1. Serre, J. P., Homologie singulier des espaces fibrés, Ann. of Math., 54 (1951), 425–505.

2. Schwartz, J. T., Generalizing the Lusternik–Schnirelmann theory of critical points, Comm. Pure and Applied Math. 17 (1964), 307–315.

3. Vigué–Poirrier, M. and Sullivan, D., The homology theory of the closed geodesic problem, J. Diff. Geom., 11 (1976), 633–644.

4. Rabinowitz, P., Periodic solutions for some forced singular Hamiltonian systems, to appear in the Festschrift fur Jurgen Moser.

5. Ambrosetti, A. and Coti Zelati, V., Critical points with lack of compactness and singular dynamical systems, to appear in Annali Matematica Pura et Applicata.

6. Hurewicz, W., On the concept of fiber space, Proc. Nat. Acad. Sci., 41 (1955), 956–961.

7. Fadell, E., On fiber spaces, Trans. Amer. Math. Soc., 90 (1959), 1–14.

8. Spanier, E., Algebraic Topology, McGraw–Hill, 1966.

9. Fadell, E. and Neuwirth, L., Configuration Spaces, Math. Scand. 10 (1962), 111–118.

10. Fadell, E. and Husseini, S., Relative cohomological index theories, Advances in Math. 64 (1987), 1–31.

11. Fadell, E., Cohomological methods in non–free G–spaces with applications to general Borsuk–Ulam theorems and critical point theorems for invariant functionals, Nonlinear Functional Analysis and its Applications, Reidel 1986, 1–47.

12. Fadell, E., Lectures in Cohomological Index Theories of G–spaces with Applications to Critical Point Theory, Raccolta Di Universetà della Calabria, 1987, Cosenza, Italy.

13. Rabinowitz, P., Minimax Methods in Critical Point Theory with Applications to Differential Equations, CBMS Report No. 65 (1984).

The University of Wisconsin–Madison.