INTERNATIONAL ATOMIC ENERGY AGENCY
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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

SMR.451/3

SECOND COLLEGE ON VARIATIONAL PROBLEMS IN ANALYSIS

(29 January - 16 February 1990)

The dual variational principle and elliptic problems with discontinuous nonlinearities

A. Ambrosetti
Scuola Normale Superiore
Pisa, Italy

and

M. Badiale
Dipartimento di Matematica
Università di Padova
Padova, Italy

Reprinted from JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS Vol. 140, No. 2, June 1989
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The Dual Variational Principle and Elliptic Problems with Discontinuous Nonlinearities*

A. AMBROSETTI

Scuola Normale Superiore, Pisa, Italy

AND

M. BADIALE

Dipartimento di Matematica, Università di Padova, Padova, Italy

Submitted by Frank H. Clarke

Received July 10, 1987

INTRODUCTION

The main purpose of this paper is to study elliptic boundary value problems of the type

$$\begin{cases} -\Delta u = f(u) + p(x), & x \in \Omega \\ u = 0, & x \in \partial\Omega, \end{cases} \qquad (*)$$

where Ω is a bounded domain in $\mathbf{R}^N$ and f has, possibly, "upward" discontinuities.

The idea is to find solutions of $(*)$ by using Clarke's Dual Action Principle [5]. This approach has a remarkable smoothing effect, in the sense that it allows one to look for solutions of $(*)$ as critical points of a functional which, in spite of the discontinuity of f, is C^1.

The Variational Principle is discussed in Section 1 and is applied in Section 2 to $(*)$, leading one to proofs of existence and multiplicity results for various kinds of nonlinearities. Among other things, we can find the results of both [4, 5] and [11] in a quite direct way.

Notations. Ω is a bounded domain in $\mathbf{R}^N$, $N \geqslant 2$, with smooth boundary $\partial\Omega$;

$$|\cdot|_q \text{ denotes the norm in } L^q(\Omega),\ q \geqslant 1;$$

$$(\cdot,\cdot) \text{ denotes the scalar product in } L^2(\Omega);$$

* Supported by the Ministero P.I. (40%), Gruppo Naz. "Calcolo delle Variazioni."

0022-247X/89 $3.00

λ_j and ϕ_j, $j \in \mathbf{N}$, satisfy $0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots$, $\phi_j \in W_0^{1,2}(\Omega)$, and

$$-\Delta\phi_j = \lambda_j \phi_j,$$

where Δ stands for the Laplace operator.

We will also take $\phi_1 > 0$ and $|\phi_j|_2 = 1$.

If E is a Hilbert space and $J \in C^1(E; \mathbf{R})$, $J'(u)$ will denote the gradient of J at u.

$c_1, c_2, \ldots$ stand for possibly different, positive, constants.

$\to$ denotes strong and $\rightharpoonup$ weak convergence.

1. The Variational Principle

We suppose $f: \mathbf{R} \to \mathbf{R}$ is a measurable function satisfying

($f1$) there is a set $A \subset \mathbf{R}$ with no finite accumulation points, such that $f \in C(\mathbf{R} - A)$;

($f2$) there is an $m \geqslant 0$ such that $h(s) := ms + f(s)$ is strictly increasing.

We notice that for a problem such as (*) one can always suppose that f is bounded from below [above] as $s \to +\infty$ $[-\infty]$: otherwise a truncation argument and the maximum principle can be used. Therefore the main restriction imposed in ($f2$) is concerned with the discontinuity points. In fact ($f2$) implies

$$f(a-) \leqslant f(a) \leqslant f(a+) \qquad \forall a \in A,$$

where $f(a\pm) = \lim_{s \to a\pm} f(s)$.

We set

$$T_a = [f(a-), f(a+)] \qquad (a \in A)$$

and

$$\hat{f}(s) = \begin{cases} f(s) & \forall s \notin A \\ T_a & \forall a \in A. \end{cases}$$

Let $p \in L^2(\Omega)$ be given and consider the Dirichlet boundary value problem (*).

We say that v is a solution of (*) if

$$v \in W_0^1(\Omega) \cap W^{2,2}(\Omega)$$

and

$$-\Delta v(x) - p(x) \in \hat{f}(v(x)) \qquad \text{a.e. in } \Omega. \tag{1}$$

Without loss of generality, we can suppose that h in ($f2$) satisfies

$$h(s) \to +\infty \; [-\infty] \qquad \text{as } s \to +\infty \; [-\infty]. \tag{2}$$

Moreover, from now on, we will take $A = \{a\}$. This will simplify the notations. The arguments in the general case are quite similar.

By ($f2$) and (2) it is possible to define a single-valued function $g: \mathbf{R} \to \mathbf{R}$ by setting

$$g(t) = \begin{cases} a, & & \text{if } t - ma \in T_a \\ s, & \text{with } h(s) = t & \text{if } t - ma \notin T_a. \end{cases}$$

In particular, one has

$$g(t) = \xi \qquad \text{iff} \quad t - m\xi \in \hat{f}(\xi). \tag{3}$$

It is easy to verify that $g \in C(\mathbf{R})$. Set $G(t) = \int_0^t g(\tau)\, d\tau$.

Let $E = L^2(\Omega)$. For all $m \geqslant 0$ we can define a linear self-adjoint operator $K: E \to E$ by

$$v = K\psi \qquad \text{iff } -\Delta v + mv = \psi, \; v \in W_0^1(\Omega)$$

and a functional $J: E \to \mathbf{R}$ by

$$J(u) = \int_\Omega \{G(u) - \tfrac{1}{2}uKu - uKp\}\, dx.$$

Our main result is:

Theorem 1. *Let ($f1$)–($f2$) be satisfied. Then:*

(i) *$J \in C^1(E, \mathbf{R})$ and if $J'(u) = 0$ then $v = K(u+p)$ is a solution of (*).*

(ii) *If either*

(α) *$-p(x) \notin T_a$ for a.e. $x \in \Omega$, or*

(β) *u is a local minimizer of J, then the level set $\Omega_a = \{x \in \Omega: v(x) = a\}$, $v = K(u+p)$, has Lebesgue measure $|\Omega_a| = 0$ and therefore v satisfies*

$$-\Delta v(x) = f(v(x)) + p(x) \qquad \text{a.e. in } \Omega.$$

Proof. From (2) it follows that

$$|g(t)| \leqslant c_1 + c_2 |t|. \tag{4}$$

Then $|G(t)| \leqslant c_3 + c_4 |t|^2$ and $G(u) \in L^1(\Omega)$ $\forall u \in E$. Moreover, from the regularity theory of elliptic equations one has that $K\psi \in W_0^1(\Omega) \cap W^{2,2}(\Omega)$

$\forall \psi \in E$. Hence J is well defined in E. From (4) it follows [8, Thm. 3.7] that $J \in C^1(E, \mathbf{R})$.

Let $u \in E$ be such that $J'(u) = 0$ and set $v = K(u+p)$. Then $v \in W_0^{1}(\Omega) \cap W^{2,2}(\Omega)$ and

$$g(u) = K(u+p) = v. \tag{5}$$

The definition of K implies

$$-\Delta v + mv = u + p. \tag{6}$$

From (3) and (5) we infer that $u(x) - mv(x) \in \hat{f}(v(x))$ a.e. in Ω. This and (6) show that v is a solution of (*).

Next, we set

$$\Omega_a = \{x \in \Omega : v(x) = a\}.$$

Since $v \in W^{2,2}(\Omega)$, a theorem of Stampacchia [10] applies and $-\Delta v(x) = 0$ a.e. in Ω_a. From (1) it follows that

$$-p(x) \in \hat{f}(v(x)) = T_a \qquad \text{a.e. in } \Omega_a$$

and this proves (ii) in case (α) holds.

Last, suppose (β) and let $-p(x) \in T_a$ a.e. in Ω_a.

Set $T_a = [b_1, b_2]$, $T^+ = [b_1, \frac{1}{2}(b_1+b_2)]$, $T^- = T_a - T^+$, and $\Omega^\pm = \{x \in \Omega : -p(x) \in T^\pm\}$.

Define $\chi \in L^2(\Omega)$ by

$$\chi(x) = \begin{cases} 1, & x \in \Omega^+ \\ -1, & x \in \Omega^- \\ 0, & x \in \Omega - \Omega_a. \end{cases}$$

For $\varepsilon > 0$ small enough one has

$$-p(x) + \varepsilon\chi(x) \in T_a \qquad \text{a.e. in } \Omega_a \tag{7}$$

and

$$\frac{d}{d\varepsilon} J(u + \varepsilon\chi) = (g(u+\varepsilon\chi), \chi) - \varepsilon(\chi, K\chi) - (\chi, K(u+p)).$$

But from $u(x) + p(x) = -\Delta v(x) + mv(x)$ it follows that $u(x) + p(x) = ma$ a.e. in Ω_a and thus

$$(g(u+\varepsilon\chi), \chi) = \int_{\Omega_a} g(u+\varepsilon\chi)\chi = \int_{\Omega_a} g(ma - p + \varepsilon\chi)\chi. \tag{8}$$

From (7), (8) and $g(t) = a$ if $t - ma \in T_a$, we infer

$$(g(u+\varepsilon\chi), \chi) = a \int_{\Omega_a} \chi.$$

Moreover

$$(\chi, K(u+p)) = (\chi, v) = \int_{\Omega_a} \chi v = a \int_{\Omega_a} \chi.$$

Then we find

$$\frac{d}{d\varepsilon} J(u+\varepsilon\chi) = -\varepsilon(\chi, K\chi). \tag{9}$$

Since u is a minimizer of J, (9) implies $(\chi, K\chi) = 0$. Setting $K\chi = \psi$, one has $(\chi, K\chi) = |\text{grad}\, \psi|_2^2$, hence $(\chi, K\chi) = 0$ iff $\chi \equiv 0$, namely iff meas $\Omega_a = 0$. This completes the proof. ∎

Remark 2. In all the above arguments the Laplace operator $-\Delta$ can be substituted by any elliptic variational operator, as well as one can deal with more general nonlinearities like $f(x, s)$.

Remark 3. In Theorem 1 one can take $E = L^\alpha(\Omega)$, $\alpha > 1$, according to the fact that $G(t) \cong t^\alpha$ as $|t| \to \infty$. One would have $K\psi \in W_0^{1}(\Omega) \cap W^{2,\alpha}(\Omega)$ $\forall \psi \in L^\alpha(\Omega)$; the rest remains unaffected.

Remark 4. Theorem 1 is based on Clarke's Dual Variational Principle [6]. Such a principle has been used to overcome the indefiniteness of the Action integral in Hamiltonian systems (see, e.g., [7]); the new feature here is that it allows one to deal with a smooth functional although f is discontinuous.

A possible interest of our approach is that we can apply to J the standard critical point theory.

In Section 2 we will indicate how to proceed in the concrete situations. To limit the paper to a reasonable length, we will discuss, rather than all the possible results, some examples only.

2. EXAMPLES

For simplicity, in the sequel we will always take $m = 0$.

Our first application deals with a case in which J is coercive and is related to [11].

EXAMPLE 5. Let f satisfy $(f1)$–$(f2)$ and let k, $0<k<\lambda_1$, be such that

$$(f3) \qquad |f(s)| \leqslant c_1 + k\,|s|.$$

We will show that

$\forall p \in L^2(\Omega)$ (*) *has a solution* v *satisfying* (3).

Indeed, $(f3)$ implies

$$G(t) \geqslant \frac{1}{2k}\,t^2 - c_2\,|t|.$$

Moreover, one has $(u, Ku) \leqslant (1/\lambda_1)\,|u|_2^2$. Hence

$$J(u) \geqslant \frac{1}{2k}\,|u|_2^2 - \frac{1}{2\lambda_1}\,|u|_2^2 - c_3\,|u|_2.$$

Since $k<\lambda_1$, then J is bounded from below and coercive on E. Since K is compact in E, then $\exists u \in E$: $J(u) = \min_E J$. Applying Theorem 1, the claim follows.

Our next example is a problem at resonance.

EXAMPLE 6. Let f satisfy $(f1)$–$(f2)$ and

$$(f4) \qquad f(s) = \lambda_1 s + b(s), \qquad \text{with } b_\pm = \lim_{s\to\pm\infty} b(s) \in \mathbf{R}.$$

Then (*) *has a solution provided*

$$b_- \int_\Omega \phi_1 < -(p, \phi_1) < b_+ \int_\Omega \phi_1. \tag{10}$$

In this case we shall apply a "linking" theorem. Let $W = \{w \in E: (w, \phi_1) = 0\}$ and $E\,(=L^2(\Omega)) = \mathbf{R}\phi_1 \oplus W$. Using $(f4)$ and (10) it is easy to check that

$$\begin{cases} J(t\phi_1) \to -\infty & \text{as } |t| \to \infty \\ \inf_W J > -\infty. \end{cases} \tag{11}$$

Moreover, one shows:

LEMMA 7. *J satisfies*

$(\mathrm{PS})_c$ *if* $u_n \in E$ *is such that* $J(u_n) \to c$ *and* $J'(u_n) \to 0$ *then* $\exists u^* \in E$: $J(u^*) = c$ *and* $J'(u^*) = 0$.

The proof of Lemma 7 requires some technicality and is postponed to the Appendix. Assuming the validity of Lemma 7, we can apply Theorem 1.2 of [9]. Actually in [9] such a theorem is proved under the stronger assumption that J satisfies (PS) (namely: *if* $J(u_n) \to c$ *and* $J'(u_n) \to 0$ *then* u_n *has a converging subsequence*). However, it is readily verified that $(\mathrm{PS})_c$ suffices, as already shown in [3] in the case of the Mountain-Pass theorem. Thus J has a critical point which gives rise, through Theorem 1, to a solution of (*). This proves the claim.

EXAMPLE 8. Let $a \neq 0$, $p = 0$, and f satisfy $(f1)$–$(f2)$ and

$$(f5) \qquad \begin{cases} f(s) = o(s) & \text{at } s = 0, \\ f(s) \cong |s|^{\sigma-1}s & \text{as } |s| \to \infty,\ 1 < \sigma < (N+2)/(N-2). \end{cases}$$

Then (*) *has a solution* $v \neq 0$.

The details of the proof are omitted, because it is based on the arguments of [2] and those of Lemma 7. In fact, letting $E = L^\alpha(\Omega)$, α the conjugate exponent of $\sigma+1$, one shows that: (i) J satisfies $(\mathrm{PS})_c$; and (ii) the Mountain-Pass theorem [2] applies. As for (i), one first proves as in [2] that $|u_n|_\alpha \leqslant$ const and then uses the same arguments of Lemma 7.

The following is an example with a continuous but not smooth nonlinearity. The smoothing effect of the Variational Principle will allow one to handle this case in a rather direct way, too.

EXAMPLE 9. Let $k > 1$ and consider the Dirichlet problem

$$-\Delta u = \mathrm{sign}(u) \cdot |u|^{1/k} \qquad \text{in } \Omega,\ u = 0 \text{ on } \partial\Omega. \tag{12}$$

We will show that (12) *has infinitely many solutions.*

Here $G(t) = (1/(k+1))\,|t|^{k+1}$. Take $X = L^{k+1}(\Omega)$ and $J(u) = \int_\Omega \{G(u) - \frac{1}{2}uKu\}\,dx$.

Since $\int G(t) = (1/(k+1))\,|u|_{k+1}^{k+1}$, it follows readily that J is coercive and bounded below on X. To show that (PS) holds, let $u_n \in X$ be such that $J(u_n) \to c$ and $J'(u_n) = G'(u_n) - Ku_n \to 0$. By the former we deduce that $|u_n|_{k+1} \leqslant c_1$; since $K\colon X \to W^{2,k+1}(\Omega)$, by the Sobolev embedding theorem it follows that $Ku_n \to z$ in X^*, up to a subsequence. Then $G'(u_n) \to z$ and since G' is strictly increasing, we infer that $u_n \to \hat{z}$, too. This proves (PS).

Next, since f is odd then J is even and the Lusternik–Schnirelman theory applies. We assume the reader is familiar with such a theory and use standard notations (see, for example, [1]).

The critical levels

$$l_m = \inf_{\gamma(A) \geqslant m}\ \sup_A J \qquad (m \in \mathbf{N})$$

($\gamma(A)$ is the "genus" of A) carry critical points $u \neq 0$ provided $l_m < 0$.

Now, let us notice that $G \in C^2$ implies, via a well-known result on Nemitski operators [8, Thms. 3.7 and 3.4], that J is C^2 on X. Moreover $G''(0)=0$ yields: $J''(0)[\psi, \psi] = -(\psi, K\psi)\ \forall \psi \in X$.

For any fixed $m \in \mathbf{N}$, let $X_m = \text{span}\{\phi_1, ..., \phi_m\}$. For $\psi \in X_m$, one has

$$(\psi, K\psi) \geqslant \frac{1}{\lambda_m} |\psi|_2^2,$$

hence (X_m being finite-dimensional)

$$J''(0)[\psi, \psi] \leqslant -c_1 |\psi|_{k+1}^2 \qquad \forall \psi \in X_m.$$

Therefore, $\forall \varepsilon > 0$ small enough, letting $A_{m,\varepsilon} = \{\psi \in X_m : |\psi|_{k+1} = \varepsilon\}$, it follows that

$$\sup_{A_{m,\varepsilon}} J < 0.$$

Since $\gamma(A_{m,\varepsilon}) = m$, then $l_m < 0$. Applying the Lusternik–Schnirelman critical point theory, we conclude that J has infinitely many critical points on X, corresponding to solutions of (12) through Theorem 1 and Remark 3.

APPENDIX

Lemma 7 will be proved in several steps. For simplicity, we will take $p = 0$.

First, some remarks are in order. Letting $u_n = t_n \phi_1 + w_n$, $w_n \in W$, and substituting in (11), it follows that $|u_n|_2 \leqslant c_1$. Hence, up to a subsequence, $u_n \rightharpoonup u^*$ in $L^2(\Omega)$.

Let $v^* = Ku^*$,

$$\Gamma = \{x \in \Omega : v^*(x) = a\}, \qquad \Omega^* = \Omega - \Gamma$$

and

$$\psi(x) = \begin{cases} 1 & \text{if } x \in \Gamma \\ 0 & \text{if } x \in \Omega^*. \end{cases}$$

From $J'(u_n) \to 0$ and the compactness of K it follows that

$$g(u_n) \to Ku^* = v^* \qquad \text{in } L^2(\Omega) \text{ and a.e. in } \Omega. \tag{13}$$

If we prove that

$$J'(u^*)\ (= g(u^*) - v^*) = 0 \tag{A1}$$

$$J(u_n) \to J(u^*), \tag{A2}$$

Lemma 7 will follow.

It is convenient to discuss separately what happens in Ω^* and in Γ. First of all, we claim

$$u_n \to u^* \qquad \text{in } L^2(\Omega^*). \tag{A3}$$

In fact $v^*(x) \neq a$ for $x \in \Omega^*$; then $f \in C(\mathbf{R} - \{a\})$ and (13) yield

$$u_n \to f(v^*) \qquad \text{a.e. in } \Omega^*. \tag{14}$$

Since $f(s) = \lambda_1 s + b(s)$ with b bounded, we deduce that $|u_n| \leqslant c_1 |g(u_n)| + c_2$. Using also (13), it follows that $|u_n| \leqslant h$ for some $h \in L^2(\Omega)$. Then (14) yields: $u_n \to f(v^*)$ in $L^2(\Omega^*)$. Since $u_n \rightharpoonup u^*$ in $L^2(\Omega)$, (A3) follows.

Since g is asymptotically linear, from (A3) we infer

$$g(u_n) \to g(u^*) \qquad \text{in } L^2(\Omega^*) \tag{15}$$

and

$$\int_{\Omega^*} G(u_n) \to \int_{\Omega^*} G(u^*). \tag{16}$$

Next, to study the behaviour on Γ, we distinguish whether $0 \in T_a$ or not. We first show

$$\text{if } 0 \notin T_a \text{ then } |\Gamma| = 0. \tag{17}$$

In fact, let $T_a = [b_1, b_2]$ with $b_1 > 0$ (if $b_2 < 0$ the proof is similar). As seen in the proof of Theorem 1(ii), one has

$$u^* = -\Delta v^* = 0 \qquad \text{a.e. in } \Gamma. \tag{18}$$

Since $u_n \rightharpoonup u^*$ and using (18), we find

$$\int_\Gamma u_n = (u_n, \psi) \to (u^*, \psi) = \int_\Gamma u^* = 0. \tag{19}$$

On the other hand, (13) yields, in particular, that $g(u_n) \to a$ a.e. on Γ. This, the continuity, and the strict monotonicity of g readily imply that $\liminf u_n(x) \geqslant b_1$ for a.e. $x \in \Gamma$. As seen before, $|u_n| \leqslant h \in L^2(\Omega)$. Then Fatou's lemma yields

$$\liminf \int_\Gamma u_n \geqslant b_1 |\Gamma|.$$

This and (19) prove (17).

Proof of (A1). If $0 \notin T_a$, (15) and (17) imply

$$g(u_n) \to g(u^*) \qquad \text{in } L^2(\Omega)$$

and (A1) follows from (13).

If $0 \in T_a$, then (18) implies: $g(u^*(x)) = g(0) = a = v^*(x)$ for a.e. $x \in \Gamma$, and again (A1) holds. ∎

Proof of (A2). If $0 \notin T_a$, (17) holds and (16) becomes

$$\int_\Omega G(u_n) \to \int_\Omega G(u^*).$$

Since K is compact, (A2) follows.

If $0 \in T_a$, then $G(s) = as$ for $s \in T_a$; by arguments similar to those employed before, one shows that

$$|G(u_n) - au_n| \to 0 \qquad \text{a.e. in } \Gamma$$

and, as a consequence,

$$\int_\Gamma |G(u_n) - au_n| \to 0. \tag{20}$$

From (16), (19), and (20), we infer

$$\int_\Omega G(u_n) = \int_{\Omega^*} G(u_n) + \int_\Gamma G(u_n) \to \int_{\Omega^*} G(u^*).$$

Since $u^* = 0$ a.e. in Γ and $G(0) = 0$, then $\int_{\Omega^*} G(u^*) = \int_\Omega G(u^*)$ and (A2) follows in this case, too.

The proof of Lemma 7 is now complete. ∎

References

1. A. Ambrosetti and P. H. Rabinowitz, *J. Funct. Anal.* **14** (1973), 349–381.
2. A. Ambrosetti and P. N. Srikanth, *J. Math. Phys. Sci.* **18** (1984), 441–451.
3. H. Brezis, J. M. Coron, and L. Nirenberg, *Comm. Pure Appl. Math.* 33 (1980), 667–689.
4. G. Cerami, *Rend. Circ. Mat Palermo* **32** (1983), 336–357.
5. K. C. Chang, *J. Math. Anal. Appl.* **80** (1981), 102–129.
6. F. Clarke, *C. R. Acad. Sci* **287** (1978), 951–952; *J. Differential Equations* **40** (1981), 1–6.
7. F. Clarke and I. Ekeland, *Comm. Pure Appl. Math.* 33 (1980), 103–116.
8. G. Prodi and A. Ambrosetti, "*Analisi Non Lineare,*" Quaderni della Scuola Normale Superiore, Pisa, 1973.

9. P. H. Rabinowitz, Some minimax theorems and applications to nonlinear partial differential equations, *in* "Nonlinear Analysis: A Collection of Papers in Honor of E. H. Rothe" (L. Cesari, R. Kannan, and H. F. Weinberger, Eds.), Academic Press, San Diego, CA.
10. G. Stampacchia, *Ann. Inst. Fourier* (*Grenoble*) **15** (1965), 189–258.
11. C. A. Stuart and J. F. Toland, *J. London Math. Soc.* **21** (1980), 329–335.

Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium