INTERNATIONAL ATOMIC ENERGY AGENCY
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SMR.451/4

SECOND COLLEGE ON VARIATIONAL PROBLEMS IN ANALYSIS

(29 January - 16 February 1990)

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Offprint from "Archive for Rational Mechanics and Analysis", Volume 108, Number 2, 1989, pp. 97–109

Existence of Steady Vortex Rings in an Ideal Fluid

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Communicated by H. BREZIS

§ 0. Introduction

Consider an ideal fluid occupying all of R^3 with axisymmetric velocity field q. A vortex ring $\mathscr{A}$ is a toroidal region in R^3 such that $\operatorname{curl} q = 0$ in $R^3 \setminus \mathscr{A}$ while $\operatorname{curl} q \neq 0$ in $\mathscr{A}$.

In cylindrical coordinates, in terms of the Stokes stream function Ψ the problem can be reduced to a free boundary problem on the half plane $\Pi = \{(r, z) : r > 0\}$ of the form (*cf.* § 1):

$$-L\Psi = 0 \quad \text{on } \Pi \setminus A, \tag{0.1}$$

$$-L\Psi = \lambda r^2 f(\Psi) \quad \text{on } A, \tag{0.2}$$

$$\Psi(0, z) = -k \leqq 0, \tag{0.3}$$

$$\Psi_{|\partial A} = 0, \tag{0.4}$$

$$\Psi_r/r \to -W, \quad \Psi_z/r \to 0 \quad \text{as } r^2 + z^2 \to \infty. \tag{0.5}$$

Above, L stands for a second order elliptic differential operator. A is the (*a priori* unknown) cross section of the vortex ring. f is called the "vorticity function" with coupling strength parameter $\lambda > 0$. k is the flux constant measuring the flow rate between the z-axis and ∂A. The constant $W > 0$ is the "propagation speed", namely the limit of the velocity field q at infinity. Subscripts denote partial derivatives.

When $k = 0$ and f is a positive constant, an explicit solution of (0.1–0.5) was found by HILL [12]. It corresponds to a spherical vortex, *Hill's vortex*.

Papers [6, 14] deal with the existence of vortex rings bifurcating from Hill's vortex and [4, 5] study uniqueness questions.

Global existence of vortex rings was first established in [10] to which we also refer for a description of the physical significance of the problem. However, in [10] a nonlinear *eigenvalue* problem is solved and the coupling constant λ arises as a Lagrange parameter which is left undetermined. For physical applications, however, existence results for *fixed* λ, say $\lambda = 1$, are desirable.

Motivated by [10], problem (0.1–0.5) for fixed $\lambda = 1$ has been studied in [13], and, independently, in [1] assuming that the vorticity f is *superlinear*. In both [1] and [13] it is assumed that $f(0) = 0$, even if from the physical point of view a strictly positive vorticity f is more appropriate. Lastly, the case of a *superlinear* f with $f(0) > 0$ *small* is investigated in [8]. Let us point out that when the free boundary problem (0.1–0.5) is translated, as usual, into a semilinear elliptic problem on R^2 by extending $f(s) = 0$ for $s < 0$, then if $f(0)$ is strictly positive the corresponding nonlinearity will be *discontinuous* at 0.

Besides [6] and [14], where $f \equiv$ constant, we do not know any existence results for vortex rings for given strength parameter λ and bounded, positive vorticity function f.

The purpose of this paper is to study such a case. More precisely, in our Theorem 4.1 we establish the existence of a solution Ψ of (0.1–0.5), corresponding to a bounded, symmetric vortex core A, under the assumptions that k, λ, W are prescribed and the vorticity function f is *bounded* and positive, and so gives rise to a *discontinuous* nonlinearity, as in [10].

Our approach would apply to superlinear f as well; also for this case in the present generality the existence of vortex rings would be new, extending the results of [1], [8], [13]. However, to limit the paper to a reasonable length, we discuss in detail only the case of bounded vorticity, which seems to be the most interesting one.

Problem (0.1–0.5) is first approximated by a semilinear Dirichlet boundary value problem on a ball B_R centered in 0, passing then to the limit as $R \to \infty$. The approximate problem is accessible by variational methods and possesses, for R large, two *nontrivial*, cylindrically symmetric solutions: v_R, the absolute minimum of the associated energy; and u_R, corresponding to a "Mountain Pass" critical point [2].

It is worth noting that, strikingly, in the limit the energetically unstable solutions u_R survive, while the stable ones, v_R, diverge. To perform the limit procedure we use the variational characterization of the "Mountain Pass" solution u_R and derive, by arguments somewhat related to those of [16], a uniform bound for $|\nabla u_R|$ in L^2 for a sequence $R_m \to \infty$. When f is superlinear, this bound could be obtained by a more direct argument from the equation itself (*cf.* [1]) but the latter approach does not seem to work in the case of a *bounded* f. In contrast, the approach we use here could be employed to solve more general semilinear elliptic variational problems in R^n under suitable symmetries.

The rest of the paper is divided into 4 sections. In § 1 the problem is described in more detail; in § 2 the existence of solutions of the approximating problems is derived; § 3 contains the *a priori* estimates which enable us to pass to the limit; in § 4 we state the main results.

A preliminary announcement of this paper has appeared in Applied Mathematics Letters.

§ 1. Setting of the problem

As stated in § 0, by axisymmetry the vortex problem can be formulated in the half space $\Pi = \{(r, z) : r > 0\}$. As is shown for example in [10], if q is the velocity field, there is a stream function Ψ such that

$$q = \left(\frac{1}{r}\,\frac{-\partial\Psi}{\partial z},\ 0,\ \frac{1}{r}\,\frac{\partial\Psi}{\partial r}\right).$$

Let L denote the operator

$$L = r\frac{\partial}{\partial r}\left(\frac{1}{r}\,\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2};$$

then the vorticity of the flow, curl q, has cylindrical components $(0, \omega = -r^{-1} L\Psi, 0)$. Finally, the laws of hydrodynamics demand that ω/r is constant on stream-surfaces $\Psi =$ constant. Thus the problem of finding a vortex ring with cross section $A \subset \Pi$, flux constant $k \geqq 0$ and propagation speed $W > 0$, amounts to determining a function $\Psi \in C^1(\Pi) \cap C^2(\Pi \setminus \partial A)$ satisfying (0.1–0.5), for some function f and constants λ, k and W.

Without loss of generality we may take $\lambda = 1$, $W = 2$. We also set $\psi(r, z) = \Psi(r, z) + r^2 + k$, the reduced flow potential, and introduce the functions $h, g : R \to R$

$$h(s) = 0 \quad \text{if } s \leqq 0, \qquad h(s) = 1 \quad \text{if } s > 0;$$

$$g(s) = h(s) f(s).$$

In this notation (0.1–0.5) become:

$$\begin{aligned} -L\Psi &= r^2 g(\psi - r^2 - k) \quad \text{in } \Pi, \\ \psi(0, z) &= 0, \\ |\nabla\psi|/r &\to 0 \quad \text{as } r^2 + z^2 \to \infty. \end{aligned} \tag{1.1}$$

A solution of (1.1) is a $\psi \in C^2(\Pi \setminus \partial A) \cap C^1(\Pi)$ which solves the first equation in (1.1) almost everywhere. By the maximum principle any solution ψ of (1.1) is positive; the set $A = \{\Psi > 0\} = \{(r, z) : \psi(r, z) > r^2 + k\}$ corresponds to the vortex core.

Following Ni [13], we introduce as new unknown the function u, related to ψ by

$$\psi(r, z) = r^2 u(r, z).$$

Then, formally, we have $L\psi = r^2 \Delta u$, where Δ denotes the Laplacian in cylindrical coordinates (r, z) in R^5, with

$$r = \sqrt{x_1^2 + \ldots + x_4^2}, \quad z = x_5.$$

Hence if $u(r, z)$ solves

(P) $$-\Delta u = g(r^2 u - r^2 - k) \quad \text{in } R^5, \quad u \to 0 \quad \text{as } |x| \to \infty,$$

then $\psi(r, z) = r^2 u(r, z)$ is the desired solution of (1.1).

Observe that if the vortex core $\{(r, z) : u(r, z) > 1 + k/r^2\}$ is bounded, then the decay condition "$u \to 0$ as $|x| \to \infty$" implies (0.5).

Let $B(R) = \{x \in R^5 : |x| < R\}$ denote the ball in R^5 centered at $x = 0$ with radius R. It is natural to approximate of (P) by the following boundary value problem:

$$(\mathrm{P})_R \qquad -\Delta u = g(r^2u - r^2 - k) \quad \text{in } B(R), \quad u = 0 \quad \text{on } \partial B(R),$$

This problem has a physical interest in itself. It will be studied in the following section.

§ 2. The approximate problem

Problem $(\mathrm{P})_R$ will be solved by variational methods. We will use standard notations for Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces $H^{m,p}(\Omega)$, for any domain $\Omega \subset R^5$. The norm in $L^2(B(R))$ will be denoted by $|u|_{2,R}$. $H(R)$ will denote the space of cylindrically symmetric u in $H_0^{1,2}(B(R))$ and will be equipped with scalar product and norm, respectively

$$((u, v))_R = \int_{B(R)} \nabla u \cdot \nabla v,$$

$$\|u\|_R^2 = ((u, u))_R.$$

In the sequel we will suppose

(*f*) *f is bounded, continuous, positive and nondecreasing on* $]0, \infty[$.

Let

$$G(r, u) = \int_0^u g(r^2v - r^2 - k)\, dv$$

and define J_R, $E_R : H(R) \to R$ by setting

$$J_R(u) = \int_{B(R)} G(r, u),$$

$$E_R(u) = \tfrac{1}{2} \|u\|_R^2 - J_R(u).$$

Note that E_R is well defined on $H(R)$ and is the difference of a quadratic and a Lipschitz continuous and convex term. Therefore, although E_R is not Fréchet differentiable in $H(R)$, it possesses a set-valued super-gradient $dE_R(u) = u - dJ_R(u) \subset H(R)$ at any point $u \in H(R)$, where dJ_R is the sub-gradient of J_R, represented by g, the maximal monotone extension of the map $u \to g(r^2u - r^2 - k)$ obtained by filling up the jump of g at 0. One has

$$v \in dE_R(u) \Leftrightarrow E_R(u + w) - E_R(u) - ((v, w))_R \leqq o(\|w\|_R), \quad \forall\, w \in H(R).$$

Moreover, the map $u \to dE_R(u)$ is weakly upper semi-continuous, see [9, Prop. 6, p. 105], and compact.

A critical point of E_R is a $u \in H(R)$ such that $0 \in dE_R(u)$.

Lemma 2.1. *$u \in H(R)$ is a critical point of E_R if and only if u is a positive solution of $(\mathrm{P})_R$ almost everywhere.*

Proof. If $0 \in dE_R(u)$ then the results of Section 2.2 of [4] imply readily

$$-\Delta u \in g(r^2u - r^2 - k), \quad u \in H(R) \cap H^{2,2}(B(R)).$$

Let $\Gamma = \{(r, z) : r^2u = r^2 + k\}$. By a theorem of STAMPACCHIA $-\Delta u = -\Delta(k/r^2) = 0$ a.e. on Γ. Since we defined $g(0) = 0$, $-\Delta u = g(r^2u - r^2 - k)$ a.e. in $B(R)$, and u is a solution a.e. of $(\mathrm{P})_R$. By the maximum principle $u > 0$. The converse is obvious. □

Remark 2.2. Actually, for the critical points obtained below one has meas $(\Gamma) = 0$, and therefore the value $g(0)$ could be defined in an arbitrary way. □

Note that $(\mathrm{P})_R$ always has the *trivial* solution $u \equiv 0$. In order to prove the existence of solutions $u \not\equiv 0$ we next derive some lemmas which will enable us to employ variational principles. Some of the arguments are rather standard and will be sketched only.

Lemma 2.3. *Suppose (f) holds. Then*

(i) *for any $R > 0$, E_R is bounded from below, weakly lower semicontinuous and coercive on $H(R)$;*

(ii) *for any $R > 0$ the function $u \equiv 0$ is a (strict) relative minimizer of E_R and for any $\varrho_0 > 0$ there exists $0 < \varrho < \varrho_0$, $\alpha > 0$, such that $E_R(u) \geqq \alpha$, $\forall u : \|u\|_R = \varrho$;*

(iii) *$\exists\, R_0 > 0$ and $u_1 \in H(R_0)$ such that $E_{R_0}(u_1) < 0$. Moreover, setting $u_1 = 0$ outside $B(R_0)$, then $u_1 \in H(R)$ and $E_R(u_1) < 0$, $\forall\, R \geqq R_0$.*

Proof. (i) is trivial because g is bounded.

(ii) From the fact that $g(r, u)$ is monotone in u and vanishes for $r^2u < r^2 + k$, by the Sobolev inequality we have

$$\int_{B(R)} G(r, u) \leqq \int_{B(R)} g(r^2u - r^2 - k)\, u \leqq C \int_{\{x : u(x) \geqq 1\}} u \leqq C \int_{B(R)} |u|^{\frac{10}{3}} \leqq c\, \|u\|_R^{\frac{10}{3}}.$$

Hence (ii) follows.

(iii) Let $\phi \in H(1)$ satisfy $J_1(\phi) > 0$. Scaling $\phi_R(x) = \phi(x/R) \in H(R)$, we have

$$\|\phi_R\|_R^2 = R^3\, \|\phi\|_1^2. \tag{2.0}$$

Moreover, by the monotonicity of g

$$\begin{aligned} J_R(\phi_R) &= \int_{B(R)} \left\{ \int_0^{\phi_R(x)} g(r^2(s-1) - k)\, ds \right\} dx \\ &\geqq R^5 \int_{B(1)} \left\{ \int_0^{\phi\left(\frac{x}{R}\right)} g\left(\left(\frac{r}{R}\right)^2 (s-1) - k\right) ds \right\} d\left(\frac{x}{R}\right) \qquad (2.1) \\ &= R^5\, J_1(\phi), \end{aligned}$$

for all $R \geqq 1$. Hence

$$E_R(\phi_R) \to -\infty \quad (R \to \infty), \tag{2.2}$$

and (iii) follows. □

The next lemma is concerned with the Palais-Smale condition which for non-smooth functionals like E_R can be replaced by the following:

Lemma 2.4. *Let f satisfy (f) and suppose that $u_m \in H(R)$ is a sequence such that*

$$|E_R(u_m)| \leq c, \quad \inf\{\|v\|_R : v \in dE_R(u_m)\} \to 0. \tag{2.3}$$

Then, up to a sub-sequence, $u_m \to u$ in $H(R)$ and $0 \in dE_R(u)$.

Proof. Use the fact that E_R is coercive on $H(R)$ and dE_R is weakly upper semi-continuous and compact. □

In the sequel a sequence u_m in $H(R)$ satisfying (2.3) will be referred to as a PS-sequence. For $u \in H(R)$, $u \geq 0$, we denote by u^* the Steiner symmetrization of u with respect the $z = x_3$ axis, namely $u^* \in H(R)$, $u^*(r, z) = u^*(r, -z)$, u^* is non-increasing in $|z|$ and equi-measurable with u for fixed r:

$$\text{meas}\{z : u^*(r, z) > c\} = \text{meas}\{z : u(r, z) > c\}, \quad \forall\, c \geq 0, \quad r \geq 0.$$

Note that $\|u^*\|_R \leq \|u\|_R$, $J_R(u^*) = J_R(u)$ for $u \in H(R)$. Hence, in particular,

$$E_R(u^*) \leq E_R(u), \quad \forall\, u \in H(R). \tag{2.4}$$

We are now in position to state the main result of this section:

Theorem 2.5. *Suppose (f) holds. Then for $R \geq R_0$ defined in Lemma 2.3(iii) problem $(\mathrm{P})_R$ has at least two positive symmetric solutions $u_R = u_R^*$ and $v_R = v_R^*$ satisfying:*

$$J_R(v_R) = \min\{J_R(u) : u \in H_R\} < 0;$$

$$J_R(u_R) = \inf_{p \in \Lambda(R)} \max\{J_R(p(t)) : 0 \leq t \leq 1\}, \tag{2.5}$$

where

$$\Lambda(R) = \{p \in C([0, 1]; H(R)) : p(0) = 0, p(1) = u_1\}.$$

Moreover, for u_R the free boundary Γ has zero measure.

Proof. By Lemma 2.3(i) J_R attains the minimum on some $v_R \in H(R)$. By Lemma 2.3(iii) $J_R(v_R) < 0$ for R large and hence $v_R \neq 0$. By (2.4) we may assume that $v_R = v_R^*$.

Lemmas 2.4, 2.3(ii) and (iii) enable us to apply the "Mountain Pass" theorem [2] in the form stated in [9] (suitable for Lipschitz functionals) yielding the existence of a critical point $u_R \neq 0$ satisfying (2.5). Similarly, (2.4) and the arguments of [7, Theorem 3.4. p. 403–405] allow us to find a critical point $u_R = u_R^*$ satisfying (2.5) and such that $\partial u_R/\partial z < 0$ for $z > 0$. In particular it follows that meas $(\Gamma) = 0$.

Both u_R and v_R give rise to positive solutions of $(\mathrm{P})_R$ according to Lemma 2.1 (see also Remark 2.2). □

Remarks 2.6. (i) The preceding theorem is related to the results of [3], where an approach based on a dual variational principle is employed. Actually, the approach of [3] furnishes an alternative proof of Theorem 2.5.
(ii) Let us point out that the symmetry of the solutions does not follow (at least in a direct way) from the result by GIDAS, NI & NIRENBERG [11] because g is discontinuous. In [3] a rather simple proof of the symmetry results needed here can be found. □

§ 3. A priori estimates for u_R

In order to obtain *a priori* bounds on suitable critical points u_R characterized by the min-max principle (2.5) and suitable for passing to the limit $R \to \infty$ we need to take a closer look at the mechanism for constructing u_R.

We set

$$\gamma(R) = \inf_{p \in \Lambda(R)} \sup_{u \in p} E_R(u) > 0$$

where $\Lambda(R)$ has been defined in the preceding section.

Recall that, for $R' < R$, we may regard $H(R') \subset H(R)$ (simply extend $u \in H(R')$ by setting $u = 0$ outside $B(R')$) and, still denoting the extended function by u, we conclude that $E_{R'}(u) = E_R(u)$. It follows that $\Lambda(R') \subset \Lambda(R)$, whence $\gamma(R') \geq \gamma(R)$. In other words $\gamma(R)$ is non-increasing, hence a.e. differentiable and

$$\int_{R_0}^{\infty} \left|\frac{d}{dR}\gamma(R)\right| dR \leq \gamma(R_0) - \liminf_{R \to \infty} \gamma(R) \leq \gamma(R_0) < \infty.$$

As a consequence, there is a sequence $R_m \to \infty$ such that

$$\lim_{m \to \infty} R_m \frac{d}{dR}\gamma(R_m) = 0. \tag{3.1}$$

Before stating the *a priori* estimates, we need some preliminary results.

Lemma 3.1. *For $R_0 < R' = sR < R$ and $u \in H(R)$ we let $u_s(x) = u(x/s) \in H(R')$. Then, if $s < 1$ and sufficiently close to 1,*

$$\gamma(sR) = \inf_{p \in \Lambda(R)} \sup_{u \in p} E_{sR}(u_s).$$

Proof. Let us consider the maps

$$u \to \tilde{u} = u(\cdot/s),$$

$$v \to \hat{v} = v(s\cdot).$$

which yield an isomorphism between $H(R)$ and $H(sR)$ and induce mappings $\Lambda(R) \to \Lambda(sR)$ and $\Lambda(sR) \to \Lambda(R)$ as follows: for $p \in \Lambda(R)$ with $p(1) = u_1$

let $\tilde{p} \in \Lambda(sR)$ be the path

$$\tilde{p}(t) = \left(p\left(\frac{t}{s}\right)\right)^{\sim} \quad \text{for } 0 \leqq t \leqq s; \quad \tilde{p}(t) = u_1(\cdot/t) \quad \text{for } s \leqq t \leqq 1.$$

Conversely, for $p \in \Lambda(sR)$ let $\hat{p} \in \Lambda(R)$ be the path

$$\hat{p}(t) = \left(p\left(\frac{t}{s}\right)\right)^{\wedge} \quad \text{for } 0 \leqq t \leqq s; \quad \hat{p}(t) = u_1(t\cdot) \quad \text{for } s \leqq t \leqq 1.$$

It is easy to verify that for all s sufficiently close to 1 and all $s \leqq t \leqq 1$ there results $E_{sR}(u_1(\cdot/t)), E_R(u_1(t\cdot)) < 0$. Moreover, given a path $p \in \Lambda(sR)$, let $q = \tilde{\hat{p}} \in \Lambda(sR)$ be the path obtained composing the above maps $\hat{p}$ and $\tilde{p}$. Note that

$$\sup_{u \in q} E_{sR}(u) = \sup_{u \in p} E_{sR}(u) \geqq \gamma(sR) > 0.$$

Hence if we let $\tilde{\Lambda} = \{\tilde{p} : p \in \Lambda(R)\}$ and define

$$\tilde{\gamma} = \inf_{\tilde{p} \in \tilde{\Lambda}} \sup_{u \in \tilde{p}} E_{sR}(u) = \inf_{p \in \Lambda(R)} \sup_{u \in p} E_{sR}(\tilde{u}),$$

it follows that $\tilde{\gamma} = \gamma(sR)$. □

Proposition 3.2. *Suppose $R \to \gamma(R)$ is differentiable at R, $R > R_0$. Then there is a (positive) solution u_R of $(\mathrm{P})_R$ satisfying*

$$\|u_R\|_R^2 \leqq C \cdot (\gamma(R) + 2R\,|\gamma'(R)| + 5),$$

with a constant C independent of R.

Proof. *Step* 1. We set $u_s(x) = u(x/s)$ for $0 < s < 1$ close to 1. By the preceding lemma, for any $\varepsilon \in]0, 1]$ there exists $p \in \Lambda = \Lambda(R)$ such that

$$\sup_{u \in p} E_{sR}(u_s) \leqq \gamma(sR) + \varepsilon(1 - s^5). \tag{3.2}$$

Moreover, let $u \in p$ satisfy

$$E_R(u) \geqq \gamma(R) - \varepsilon(1 - s^5). \tag{3.2'}$$

From (3.2-2') it follows:

$$E_{sR}(u_s) - E_R(u) \leqq \gamma(sR) - \gamma(R) + 2\varepsilon(1 - s^5). \tag{3.3}$$

First we estimate the left-hand side of (3.3). By (2.1) $J_{sR}(u_s) \leqq s^5 J_R(u)$ and

$$\frac{s^5}{1 - s^5}(J_R(u) - J_{sR}(u_s)) \geqq J_{sR}(u_s).$$

On the other hand, by (2.0) one has

$$\|u_s\|_{sR}^2 = s^3 \|u_s\|_R^2$$

whence

$$\frac{s^5}{1 - s^5}(\|u\|_R^2 - \|u_s\|_{sR}^2) = \frac{s^2 - s^5}{1 - s^5}\|u_s\|_{sR}^2.$$

As a consequence, for $0 < 1 - s$ small,

$$s^5 \frac{E_{sR}(u_s) - E_R(u)}{1 - s^5} \geqq J_{sR}(u_s) - \tfrac{13}{10}\|u_s\|_{sR}^2.$$

This inequality and (3.3) imply that for s close to 1

$$-\tfrac{3}{10}\|u_s\|_{sR}^2 + J_{sR}(u_s) \leqq R\,|\gamma'(R)| + 3\varepsilon. \tag{3.4}$$

From (3.4) we deduce

$$\begin{aligned} E_{sR}(u_s) &= \tfrac{1}{2}\|u_s\|_{sR}^2 - J_{sR}(u_s) \\ &\geqq \tfrac{1}{5}\|u_s\|_{sR}^2 - R\,|\gamma'(R)| - 3\varepsilon, \end{aligned} \tag{3.5}$$

whence:

$$\begin{aligned} s^3\|u\|_R^2 = \|u_s\|_{sR}^2 &\leqq 5(E_{sR}(u_s) + R\,|\gamma'(R)| + 3\varepsilon) \\ &< 5(\gamma(sR) + R\,|\gamma'(R)| + 4\varepsilon) \leqq 5(\gamma(R) + 2R\,|\gamma'(R)| + 5\varepsilon). \end{aligned} \tag{3.6}$$

Step 2. We claim *there is a PS-sequence* $u_m \in H(R)$ *such that*

(i) $E_R(u_m) \to \gamma(R)$;

and

(ii) $\limsup_{m \to \infty} \|u_m\|_R^2 \leqq c^* =: 5[\gamma(R) + 2R\,|\gamma'(R)| + 5] + 1.$

To see this, for $\delta > 0$ set

$$U_\delta = \{u \in H(R) : \|u\|_R^2 \leqq c^* + \delta,\ |E_R(u) - \gamma(R)| \leqq \delta\}$$

and suppose, by contradiction, that for some $\varepsilon^* > 0$ and any $u \in U_{\varepsilon^*}$

$$\inf\{\|v\|_R : v \in dE_R(u)\} > \varepsilon^*.$$

By [9, Lemma 3.4 and Theorem 3.1], corresponding to $c = \gamma(R)$, $\varepsilon_0 = \min\{\varepsilon^*, \gamma(R)\}$, $N = H(R) \setminus U_{\varepsilon^*}$, we can find $\varepsilon \in]0, \varepsilon_0[$ and a homeomorphism $\Phi : H(R) \to H(R)$ such that

$$\Phi(u) = u \quad \text{if } |E_R(u) - \gamma(R)| \geqq \gamma(R); \tag{3.7}$$

$$\begin{aligned} &E_R(\Phi(u)) \leqq E_R(u) \quad \text{for all } u; \\ &E_R(\Phi(u)) \leqq \gamma(R) - \varepsilon \quad \text{if } u \in U_{\varepsilon^*},\ E_R(u) < \gamma(R) + \varepsilon. \end{aligned} \tag{3.8}$$

For $s < 1$ close to 1 choose $p \in \Lambda(R)$ such that

$$\sup\{E_{sR}(u_s) : u \in p\} \leqq \gamma(sR) + (1 - s^5).$$

Then by Step 1 any $u \in p$ where $E_R(u) \geqq \gamma(R) - (1 - s^5)$ satisfies $\|u\|_R^2 \leqq c^*$. In particular, if s is sufficiently close to 1, by using (2.0) and (2.1) we can arrange that for all such u

$$E_R(u) \leqq E_{sR}(u_s) + \varepsilon/2 \leqq \gamma(sR) + (1 - s^5) + \varepsilon/2 \leqq \gamma(R) + \varepsilon,$$

and $u \in U_{\varepsilon^*}$.

Applying Φ to p, by (3.7) we obtain a comparison path $p' = \Phi(p) \in \Lambda(R)$ which satisfies

$$\sup \{E_R(u) : u \in p'\} < \gamma(R). \tag{3.9}$$

In fact, if $E_R(u) \geqq \gamma(R) - (1 - s^5)$ (otherwise there is nothing to prove), by the preceding remarks and (3.8) it follows that $E_R(\Phi(u)) \leqq \gamma(R) - \varepsilon$ for any $u \in p$. Clearly (3.9) contradicts the definition of $\gamma(R)$ and the proof of Step 2 is complete.

The conclusion of Proposition 3.2 now follows immediately from Lemma 2.4. □

Combining Proposition 3.2 and (3.1) with the arguments of [7] we obtain

Corollary 3.3. *There exist a constant c, a sequence $R_m \to \infty$, and a sequence of symmetric solutions $u_m = u_{R_m}$ of $(P)_{R_m}$ with*

$$\|u_m\|_{R_m} < c. \tag{3.10}$$

§ 4. Existence of vortex rings

In this final section we prove the existence of a solution of problem (P), or, equivalently, of problem (0.1–0.5), by a limiting procedure.

Let $u_m \in H(R_m)$ be the sequence found in Corollary 3.3 and set

$$A_m = \{(r, z) \in B(R_m) : r^2 u_m(r, z) > r^2 + k\}.$$

Lemma 4.1. *There exists $R^* > 0$ such that $A_m \subset B(R^*)$ for all integer m.*

Proof. The lemma would follow from Corollary 3.3 and the estimates of [10, § 5.2] or [13, § 5.3]. Below, taking advantage of the boundedness of f, we report a slightly different, short proof, to make the paper as self-contained as possible.

Extend u_m to all R^5 setting $u_m \equiv 0$ outside $B(R_m)$. Fix r_0; then the following estimate holds (hereafter we use the symbol c to denote possibly different constants, independent both of m and r_0):

$$\begin{aligned}
&\text{meas}\,\{z : u_m(r_0, z) \geqq \tfrac{1}{2}\} \\
&\leqq c \int_{-\infty}^{\infty} [u_m(r_0, z)]^{\frac{8}{3}}\, dz \\
&\leqq c \int_{r_0}^{\infty} \left| \frac{\partial}{\partial r} \int_{-\infty}^{\infty} [u_m(r, z)]^{\frac{8}{3}}\, dz \right| dr \leqq c \int_{r_0}^{\infty} \int_{-\infty}^{\infty} |\nabla u_m| \cdot u_m^{\frac{5}{3}}\, dr\, dz \\
&\leqq c \frac{1}{r_0^3} \int_{r}^{\infty} \int_{-\infty}^{\infty} |\nabla u_m| \cdot u_m^{\frac{5}{3}}\, r^3\, dr\, dz \leqq c \frac{1}{r_0^3} \|u_m\|_{R_m} |u_m|_{\frac{10}{3}, R_m}^{\frac{5}{3}} \leqq c \frac{1}{r_0^3} \|u_m\|_{R_m}^{\frac{8}{3}}.
\end{aligned}$$

By use of (3.10)

$$\text{meas}\,\{z : u_m(r_0, z) \geqq \tfrac{1}{2}\} \leqq c r_0^{-3}. \tag{4.1}$$

Moreover,

$$\|\Delta u_m\|_{\infty, R_m} \leqq \sup f < c \tag{4.2}$$

uniformly, and by the L^p-regularity theory the families $\{u_m(. + x_m)\}$ are equibounded in C^1, locally in R^5, for any choice of $\{x_m\}$.

First, let $r_m = \max \{r : (r, z) \in A_m$ for some $z\}$; by symmetry, r_m is achieved for $z = 0$. Set $x_m = (r_m, 0)$; then $u_m(x_m) \geqq 1$, and by equicontinuity there exists $z_0 > 0$ (independent on m) such that $u_m(r_m, z_0) \geqq 1/2$. By (4.1) this implies the uniform bound

$$r_m \leqq c z^{-\frac{1}{3}}.$$

Likewise, choose $x_m = (r_m, z_m) \in A_m$, where $z_m = \max \{z : (r, z) \in A_m$ for some $r\}$. As before we conclude the existence of some $r_0 > 0$ such that $u_m(x) \geqq 1/2$ for $x = (r, z_m)$ and $|r - r_m| \leqq r_0$. But then (4.1) implies that

$$z_m \leqq c r_0^{-3}$$

uniformly, and the conclusion follows. □

Next by (4.2) we also conclude that

(a) u_m converges in $C^{1+\alpha}_{\text{loc}}(R^5)$, $0 < \alpha < 1$, to some u, solving (P).

Let us note that $u \not\equiv 0$; otherwise for m large, $u_m < 1$ on $B(R^*)$, whence $r^2 u_m < r^2 + k$. Since u_m is a solution of $(P)_{Rm}$ and $g(r, z) = 0$ for all $r^2 z < r^2 + k$, it would follow that $u_m \equiv 0$, a contradiction. Moreover remark that $u \not\equiv 0$ implies that the vortex core

$$A = \{(r, z) \in R^5 : r^2 u(r, z) > r^2 + k\}$$

is not empty. Finally, by Lemma 4.1, $A \subset B(R^*)$ is bounded. In addition, (a) and Theorem 2.5 imply that

(b) u is symmetric because the u_m were so; moreover, $\partial u/\partial z > 0$ for $z > 0$. Hence ∂A has zero measure.

Finally, also in view of point (a) above, one has:

(c) $\psi = r^2 u$ is a solution of (1.1) in the sense specified in Section 1.

We can conclude by stating:

Theorem 4.2. *Suppose (f) holds and let u_R, v_R be the solutions of $(P)_R$, R large, found in Theorem 2.5 and Proposition 3.2, respectively. Then*

(i) *there is a sequence $R_m \to \infty$ and $u \in H^{1,2}(R^5)$ such that $u_{R_m} \to u$ in $H^{1,2}$; $u = u(r, z)$ and $\psi = r^2 u$ is a positive, symmetric solution of (1.1) corresponding to a non-empty bounded vortex core;*

(ii) $E_R(v_R) \to -\infty$ *and* $|v_R|_{2,R} \to \infty$.

Proof. (i) This follows from Lemma 4.1 and conclusions (a), (b), (c).

(ii) by (2.2) it follows that $E_R(v_R) \to -\infty$ $(R \to \infty)$.

Finally, let $c > 0$ be a constant such that $G(r, u) < cu^2$. Such a constant exists because g is bounded and $g(r^2u - r^2 - k) = 0$ for all $r^2u < r^2 + k$. Then

$$0 < \tfrac{1}{2} \|v_R\|_R^2 = E_R(v_R) + \int_{B(R)} G(r, v_R) \leqq E_R(v_R) + c\,|v_R|_{2,R}^2 .$$

Hence

$$c\,|v_R|_{2,R}^2 \geqq -E_R(v_R) \to \infty .$$

This completes the proof of the Theorem. □

Remarks 4.3. (i) The arguments concerning the existence of u_R and its convergence to a solution u of (P) work if f is superlinear, as well. However, in such a case, the *a priori* estimates on $|\nabla u_R|_{2,R}$ can be obtained in a more direct way, as, for example, in [1].

(ii) It is clear that the procedure employed above can be used to prove the existence of nontrivial solutions of semilinear elliptic boundary value problems in R^n with bounded nonlinearity, in presence of a suitable symmetry. We leave it to the reader to carry out the details.

(iii) Theorem 4.1 holds if $k \geqq 0$. If $k = 0$, the vortex is spherical. If, in addition, f is identically constant, we would find Hill's spherical vortex, according to the uniqueness result of [4].

Acknowledgments

AMBROSETTI thanks BOB TURNER for several useful discussions.

STRUWE is indebted to the kind hospitality and support of the Scuola Normale Superiore, Pisa, during the writing of this paper.

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(Received March 20, 1989)