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The forced pendulum: A paradigm for nonlinear analysis and dynamical systems

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The forced pendulum: A paradigm for nonlinear analysis and dynamical systems

Jean Mawhin

Abstract. This paper is a revised version of a talk given at the celebration of the sixtieth birthday anniversary of Professor Wolfgang Walter at Karlsruhe in May 1987. We show that, besides its fundamental importance in the development of classical mechanics, elliptic function theory and the modern theory of dynamical systems, the pendulum equation, and in particular the forced pendulum problem has played and still play a very basic role in the development and testing of modern mathematical techniques in nonlinear functional analysis and critical point theory.

Introduction

The study of the mathematical free pendulum constitutes since centuries a classical chapter of every textbook of analytical mechanics. Mgr Lemaître, the famous cosmologist, went further in his lectures on mechanics at the University of Louvain. His lecture notes are entitled: "Leçons de mécanique. Le pendule", and we can read, in their introduction: "An intermediate attitude that we shall follow consists in retaining from the history of science the preeminence given to a particular problem, the motion of the pendulum, and hence to present the fundamental concepts in the setting of this particular problem ... This problem of the pendulum is one of those where the science of mechanics has provided one of its greatest contributions to the edification of modern mathematics, because it can be widely identified with the study of the elliptic functions around which was constructed the theory of functions of a complex variable."

Much more recently, in an extended and highly interesting article of the French "Encyclopediae Universalis" entitled "Systèmes dynamiques différentiables", A. Chenciner uses again the pendulum as a central theme to initiate the reader to the modern theory of dynamical systems: "The first chapters describe in a detailed way examples connected to the pendulum and introduce more and more complex asymptotic behaviors whose analysis will require the more abstract concepts of the last part." Among the heads of the nine chapters of this article, we find: the pendulum without friction: a Hamiltonian system, the pendulum with linear friction: a structurally stable system, periodic perturbations of a frictionless pendulum and area-preserving diffeomorphisms of the plane.

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The aim of this paper is to add to those eloquent examples of the role of the pendulum as a paradigm in classical mechanics and modern theory of dynamical systems (which are briefly recalled in Part One and in the last section of Part Two), some further arguments on the importance of the forced pendulum equation to the development of the fundamental methods of nonlinear functional analysis and of global analysis.

We hope to convince the reader that classical problems always remain a source of inspiration and motivation in the genesis and testing of modern sophisticated mathematical methods. The number and interest of the problems which are left open will confirm Poincaré's famous quotation that there are no solved problems, but only problems which are more or less solved.

I. The free pendulum

1. Galileo's isochronism law (linearization)

Although he undoubtly had precursors, like Ibn Yunus, Oresme, Leonardo, Cardano, Galileo is the first scientist usually associated to the experimental and theoretical study of the pendulum. Everybody knows the story – true or false – of his discovery in 1583 or 1584 of the *isochronism* of the pendulum oscillations, by observing a suspended lamp in the cathedral of Pisa. Galileo's relations with church happened to be less happy later in his life!

The first written document of Galileo on the isochronism of the pendulum is a letter of 1602 to Guidobaldo del Monte: "You will pardon my insistence in wishing to convince you of the truth of the proposition that the motions in the same quadrant of a circle are made in equal times." In his famous *"Dialogo"* of 1632, he writes: "The same pendulum makes its oscillations with the same frequency, or very little different, almost imperceptibly, whether these are made through large ones or very small ones along a given circumference." This is of course a wrong statement or, in a positive way, we can consider it as a very early manifestation of what is maybe the very first basic tool in nonlinear science, the *linearization*. We know indeed that the isochronism is not a property of the solutions of the differential equation of the pendulum with length l

$$u'' + (g/l)\sin u = 0, \tag{1}$$

but of its linearized form

$$u'' + (g/l)\,u = 0. \tag{2}$$

The precise expression $T = 2\pi(l/g)^{1/2}$ for the period of the solutions of (2) will be first given by Newton in his *"Principia"*, in 1687.

Notice that some *applications* of the pendulum were already present in Galileo's work, in particular to medicine with his "pulsilogium" for checking the pulse of a patient, to navigation with the determination of the longitude, and to horology with the regulation of mechanical clocks.

2. Relation between period end amplitude (series and elliptic functions)

If Guidobaldo del Monte already expressed in 1602 a good deal of scepticism to Galileo's isochronism claim, it is a Belgian astronomer, Wendelin, who first showed experimentally that the period of oscillations increases with the amplitude of the oscillation and gave fairly precise correspondance tables in his *"Luminarcani Eclipses Lunares"* of 1644. This fact was then mathematically deduced by Huygens in his famous *"Horologium Oscillatorum"* of 1673, and the same Huygens also observed the nonlinear phenomenon of *synchronization* between two pendulums attached on the same thin string. Two hundred and fifty years later, van der Pol and Appleton will discover an analogous phenomenon in electrical circuits and the theory will be initiated by van der Pol.

The mathematical relation between the period T and the amplitude A is expressed by Euler in 1736 in his *"Mechanica"* by the series

$$T = 2\pi(l/g)^{1/2}\left[1 + \sum_{k=1}^{\infty}\left(\frac{1\cdot 3\ldots(2k-1)}{2\cdot 4\ldots(2k)}\right)^2 \sin^{2k}(A/2)\right] \tag{3}$$

and Poisson, in his *"Traité de Mécanique"* of 1811, will analyze the pendulum equation by using a method of development in *series of power of a small parameter.* Relation (3) will be formulated by Legendre in 1825 *("Traité des fonctions elliptiques")* and by Jacobi in 1829 *("Fundamenta nova theoriae functionem ellipticarum")* through elliptic integrals and functions, namely

$$T = 4(l/g)^{1/2}K(k) = 4(1/g)^{1/2}sn^{-1}(1,k)$$

where

$$K(k) = \int_0^{\pi/2} \frac{dx}{(1-k^2\sin^2 x)^{1/2}}$$

is the complete elliptic integral of first kind, sn^{-1} the inverse sinus-amplitude function and $k^2 = \sin^2(A/2)$. The importance of the elliptic functions in the present renewal of the *integrable mechanical systems* has not to be emphasized here.

3. Phase plane analysis (qualitative theory)

The quantitative theory of the free pendulum was completed by the expression of the solutions of equation (1) in terms of elliptic functions. We owe to Poincaré in 1881 the *qualitative* study of the solutions of nonlinear differential equations, namely, in the case of equation (1), the topological description of the orbits $(u(t), u'(t))$ of the solutions of (1) in the phase-plane (u, u'). The corresponding picture, with the stable equilibria $(2k\pi, 0)$ *(centers)*, the unstable equilibria

$((2k+1)\pi, 0)$ *(saddle points)*, the nonconstant periodic solutions *(closed orbits)*, the rotatory solutions and the separatrices connecting unstable equilibria *(heteroclinic orbits* or *homoclinic orbits* if we identify, modulo 2π, the unstable equilibria, i.e. if we work on the natural cylindrical phase manifold). As often, the history has taken the opposite way with respect to Poincaré's methodology in attacking nonlinear differential equations: the quantitative study of the free pendulum has preceded the qualitative one.

The qualitative approach is particularly useful in the discussion of the damped free pendulum equation

$$u'' + cu' + (g/l)\sin u = 0$$

whose solution cannot be expressed in terms of known functions, which is also the case for the equation

$$u'' + cu' + a\sin u = b$$

which occurs in the study of synchronous motors (Edgerton, Fourmarier, Tricomi).

II. The forced pendulum

1. Duffing's heuristic approach (approximations and jumps)

The study of the forced pendulum equation

$$u'' + a\sin u = b\sin(\omega t) \tag{4}$$

where $a \neq 0$ and $\omega > 0$ has a much more recent history than that of the free pendulum equation. It can be traced to Duffing's monograph *"Erzwungene Schwingungen bei veränderlicher Eigenfrequenz"* published in 1918 and motivated by questions of applied mechanics. The main question is to analyze the phenomenon of resonance when the period of the oscillations of the unforced equation depends upon the amplitude. The forced pendulum is considered as the simplest model which exhibits this behavior.

Duffing starts by replacing equation (4) by its first nonlinear approximation

$$u'' + au - au^3/6 = b\sin(\omega t) \tag{5}$$

and uses then a *first-order Galerkin* or *harmonic balance method* which consists in obtaining an approximate solution of the form

$$u_0(t) = A\sin(\omega t)$$

by inserting it in (5), developing the left-hand member in Fourier series and equating the Fourier coefficient of $\sin(\omega t)$ of the left-hand and the right-hand member. This gives an algebraic third order equation in A, with coefficients depending upon a, b and ω, which provides therefore an approximate relation between the frequency of the forcing term and the approximate amplitude of the periodic response, which

is of course supposed to exist (the *frequency-amplitude curve*). The study of this curve allowed Duffing to discover the typically nonlinear phenomenon of the *amplitude jump* of the periodic response when the external frequency varies, a phenomenon which has been "rediscovered" and reinterpreted recently in the frame of *catastrophe theory*.

Let us notice that Duffing also considers the influence of a damping term (proportional to u') in (5), that he gives a very detailed treatment of the free equation (1) using the elliptic functions of Weierstrass and that he also finds, in an appendix, his frequency-amplitude curve by using the *Ritz approximation method*, i.e. a variational approximation method which consists in replacing the minimization of the corresponding Hamiltonian action

$$J(u) = \int_0^{2\pi/\omega} \left[(1/2)(u'(t))^2 - au^2(t) + a\frac{u^4(t)}{24} + b\sin(\omega t)\,u(t)\right] dt$$

by the minimization of the real function of A obtained by replacing $u(t)$ by $A\sin(\omega t)$ in J.

2. Hamel's mathematical treatment (direct method of the calculus of variations, fixed point method, Liapunov-Schmidt reduction)

From the mathematical viewpoint, a fundamental contribution was given in 1922 by G. Hamel (a student of Hilbert) in the Mathematische Annalen. Hamel starts by giving the first existence proof of a T-periodic solution of equation (4) (with $T = 2\pi/\omega$) by the *direct method of the calculus of variations* elaborated in the beginning of the century by Hilbert to legitimate the Dirichlet principle in potential theory. To this effect, Hamel tries to minimize the Hamiltonian action of (4) given by

$$J(u) = \int_0^T [(1/2)(u'(t))^2 + a\cos u(t) + bu(t)\sin(\omega t)]\,dt$$

over a suitable space of T-periodic functions. He makes the basic observation that J can be written in the equivalent form

$$J(u) = \int_0^T [(1/2)(u'(t))^2 + a\cos u(t) + \omega^{-1}bu'(t)\cos(\omega t)]\,dt,$$

from which it is evident that J is bounded from below and such that

$$J(u + 2\pi) = J(u). \tag{6}$$

Because of this condition, one can, without loss of generality, restrict itself to the solutions such that $|u(0)| \leq 2\pi$, so that, by using the equation, one see that all the possible corresponding critical points are such that $|u(t)|$, $|u'(t)|$ and $|u''(t)|$ are uniformly bounded by quantities depending only upon a, b and T. The result then follows from the Hilbert method. We shall come back later to recent reformations and developments of this approach in the setting of modern critical point theory.

Hamel uses then the Ritz method, like Duffing, to obtain an equation for the approximate value of the first Fourier coefficient of the periodic solutions (assumed to be odd functions). This equation is easily found to be

$$A - 2aJ_1(A) + b = 0$$

where J_1 is a Bessel function, and has a unique solution when $|a| \leq 1$ or when $|a| > 1$ and $|b/a|$ is large, three solutions when $|a| > 1$ and $|b/a|$ is small enough, and so on, according to the oscillatory properties of J_1.

The next step in Hamel's paper consists in using the *symmetry* of (4) (with respect to u and t) to notice that the *odd T-periodic solutions* of (4) can be obtained by extending, using oddness and periodicity, the solutions of (4) which satisfy the *Dirichlet boundary conditions*

$$u(0) = u(T/2) = 0. \tag{7}$$

Problem (4)–(7) is then written in the equivalent form of a *nonlinear integral equation* or *fixed point problem*

$$u(t) = a \int_0^{T/2} K(t,s) \sin u(s)\, ds - b \sin(\omega t) := (Fu)(t), \tag{8}$$

where K is the Green function of the differential operator $-d^2/dt^2$ with the Dirichlet boundary conditions on $[0, T/2]$. Equation (8) is then solved uniquely, for $|a| < 1$, by the *method of successive approximations* (notice that the famous memoir of Banach on the *contraction mapping theorem* also appeared in 1922). Indeed, Hamel already uses here a refinement of the contraction mapping theorem, namely that the method of successive approximations converges to a unique fixed point when the n^{th} iterate F^n of F is a contraction for some n. Notice that (8) is a very early example of what will be called a *Hammerstein equation*.

To study equation (8) when $|a| \geq 1$, Hamel uses the method introduced by E. Schmidt in 1908 to study nonlinear integral equations and called now the *Liapunov-Schmidt method*. He first considers the case of small solutions when b is small, i.e. the solvability of an implicit function problem of the form

$$G(u, b) = 0$$

near the solution $(0, 0)$. When $a \neq n^2$ ($n \in \mathbb{N}^*$), $G'_u(0, 0)$ is invertible as a linear mapping in the space of continuous functions on $[0, T/2]$ verifying the Dirichlet conditions, and the existence and uniqueness of small solutions u for small $|b|$ follows. This is essentially the method already used by Horn for the same problem in

1920, as Hamel notices it. When a is the square of an integer, Hamel, following E. Schmidt, first writes (8) in the equivalent form

$$u(t) - m^2 \int_0^{T/2} K(t,s)\, u(s)\, ds = m^2 \int_0^{T/2} K(t,s)[\sin u(s) - u(s)]\, ds - b \sin(\omega t), \tag{9}$$

where $m^2 = a$. Setting $P_m(t, s) = (4/T)\, m^{-2} \sin(m\omega t) \sin(m\omega s)$, $K_m = K + P_m$,

$$z = (4/T) \int_0^{T/2} u(s) \sin(m\omega s)\, ds, \tag{10}$$

he can then write (9) as

$$u(t) - m^2 \int_0^{T/2} K_m(t,s)\, u(s)\, ds$$

$$= z \sin(m\omega t) + m^2 \int_0^{T/2} K(t,s)[\sin u(s) - u(s)]\, ds - b \sin(\omega t), \tag{11}$$

which is now an implicit function problem of the form

$$G_m(u, b, z) = 0$$

such that

$$(G_m)'_u(0,0,0):\ v \mapsto v - m^2 \int_0^{T/2} K_m(\cdot, s)\, v(s)\, ds$$

is again invertible, in contrast to $G'_u(0, 0)$. The implicit function theorem provides a unique small solution $u = U(z, b)$ of (11) for small b and v, and this u will be a solution of our original problem (9) is z satisfies the *bifurcation equation*

$$z = (4/T) \int_0^{T/2} U(z, b)(s) \sin(m\omega s)\, ds$$

deduced from (10).

The same approach is used by Hamel to study equation (8) with $|a| \geq 1$, $|b|$ small and u is close to a nontrivial solution of the equation

$$u'' + a \sin u = 0$$

expressed in terms of elliptic functions.

We think that this description is sufficient to convince the reader that Hamel's paper is one of the most striking pioneering contributions to nonlinear analysis.

3. The Dirichlet problem (Hammerstein und Lichtenstein variational treatments, Birkhoff-Kellog-Schauder fixed point theory, continuation methods)

The followers of Hamel will concentrate on the Dirichlet problem (4)–(7) and to its generalization

$$\begin{aligned} &u'' + a \sin u = e(t) \\ &u(0) = u(T/2) = 0. \end{aligned} \tag{12}$$

In particular, between 1927 and 1930, Hammerstein gave a very complicated proof of the existence of at least one solution for (12) for all values of a and all continuous functions e by first approximating equation (8) by a finite system of equations in finitely many unknowns via a *Galerkin method*, proving the existence of a solution to the Galerkin equations by a *variational method* (i.e. by minimizing a real function whose gradient zeros correspond to the Galerkin equations) and then going to the limit with the approximate solutions. It is interesting to notice that a much more simple variational treatment was available (but not used) by dwelling on a remarkable paper written by Lichtenstein in 1915 in Crelle's journal and quoted by Hammerstein in his 1930 paper in Acta Mathematica. Denoting by $E(t)$ the unique solution of the linear problem

$$u'' = e(t), \quad u(0 = u(T/2) = 0,$$

and making the change of unknown quantity given by $u = v + E$, we get the equivalent problem

$$\begin{aligned} &v'' + a\sin(v + E(t)) = 0 \\ &v(0) = v(T/2) = 0. \end{aligned} \tag{13}$$

For the more general problem

$$v'' = f(t, v), \quad v(0) = v(T/2) = 0 \tag{14}$$

Lichtenstein proves the existence of a solution when there exists $A \in \mathbb{R}$ such that

$$F(t, v) := \int_0^v f(t, s)\, ds \geq A$$

for all $t \in [0, T/2]$ and $v \in \mathbb{R}$, by showing that the associated action functional given by

$$J(v) = \int_0^{T/2} [(1/2)(v'(t))^2 + F(t, v(t))]\, dt,$$

which is clearly bounded below, achieves its minimum on a suitable space of functions verifying the Dirichlet conditions on $[0, T/2]$. Lichtenstein's theorem can of course be directly applied to (13) and provides a much more simple proof of Hammerstein's result. Notice that, besides the use of a Galerkin argument, the difficulties of Hammerstein treatment comme from the fact that if we start from a variational equation in a Hilbert space H,

$$Lu = g'(u),$$

with L self-adjoint and g a differentiable real function, which corresponds to the gradient of the real function

$$\varphi:\ u \mapsto (1/2)(Lu, u) - g(u),$$

the left-hand member of the equation

$$u - L^{-1}g'(u) = 0$$

(when we assume L to be invertible) is not, in general, the gradient of a real function on H.

Hammerstein's result could also have been proved by using a general existence theorem for nonlinear boundary value problems stated and proved by Birkhoff and Kellogg in their pioneering paper of 1922 (decidedly a great year for nonlinear analysis) in the Transactions of the Americal Mathematical Society where they extend the Brouwer fixed point theorem to the infinite dimensional spaces L^2 and C^k, and therefore pave the way for the more general *Schauder's fixed point theorem*. Indeed, the operator F defined in (8) (with $b\sin(\omega t)$ replaced by a general continuous function $e(t)$) clearly maps the space $C[0, T/2]$) into a compact set of this space (by the Ascoli-Arzela theorem) and hence has at least one fixed point. The application of this method to the pendulum equation will be explicitely mentioned in the thirties by the Italian school (Caccioppoli, Scorza-Dragoni, ...).

In a series of papers published in the thirties, Iglish has used the Liapunov-Schmidt method and the *continuation method* (i.e. global implicit functions theory) to study extensively the *bifurcation* of the solutions of (4) – (7) when the parameters a and b vary, a problem which is still today far from being solved completely and would deserve a further study based upon the modern techiques of bifurcation theory which have been developed recently.

We can finally notice that the Dirichlet problem for the *damped and forced* pendulum equation

$$\begin{aligned} &u'' + cu' + a\sin u = e(t) \\ &u(0) = u(T/2) = 0 \end{aligned} \tag{15}$$

can be treated exactly like the undamped case through the use of the Schauder fixed point theorem (providing existence for each c, a, T and e) but has no variational structure when $c \neq 0$. However, the change of unknown quantity given by

$$u = e^{-(c/2)t} v$$

transforms the problem (15) into the equivalent one

$$\begin{aligned} &v'' - (c^2/4)v + e^{(c/2)t} a \sin(e^{-(c/2)t} v) = e(t), \\ &v(0) = v(T/2) = 0 \end{aligned}$$

which has a variable structure and still verifies the assumptions of the Lichtenstein theorem.

4. The periodic problem: existence of a solution (upper and lower solution method, variational treatments)

In contrast to the Dirichlet problems (12) and (15), the corresponding periodic problems

$$u'' + a\sin u = e(t), \qquad u(0) - u(T) = u'(0) - u'(T) = 0, \tag{16}$$

$$u'' + cu' + a\sin u = e(t), \qquad u(0) - u(T) = u'(0) - u'(T) = 0, \tag{17}$$

are not solvable for all forcing functions s. Indeed, for both problems, the following *necessary condition for sovability*

$$|\bar{e}| := \left| T^{-1} \int_0^T e(t)\, dt \right| \leq |a| \tag{18}$$

is easily obtained by integrating the differential equation over $[0, T]$ and using the boundary conditions. This condition is trivially sufficient when the forcing term is a constant. In contrast to the Dirichlet case, the associated linear operator $-d^2/dt^2$ with the periodic boundary conditions is not invertible, and hence problems (16) or (17) are not reducible to a Hammerstein equation of type (8) with F bounded on the function space. This may be the reason why, except maybe for cases where $e(t) = \varepsilon h(t)$ with $|\varepsilon|$ small, the periodic problem was not considered between Hamel's paper and that of Knobloch, in the Mathematische Zeitschrift of 1963, dealing with the *method of upper and lower solution* for periodic boundary value problems. Knobloch's general result for the problem

$$u'' = f(t, u, u'), \quad u(0) - u(T) = u'(0) - u'(T) \tag{19}$$

with f continuous and such that an estimate of the type

$$|f(t, u, v)| \leq c_1 v^2 + c_2$$

holds for bounded u, is that the existence of T-periodic functions α and β of class C^2 such that the inequalities

$$\alpha''(t) \geq f(t, \alpha(t), \alpha'(t)), \quad \beta''(t) \leq f(t, \beta(t), \beta'(t))$$

and

$$\alpha(t) \leq \beta(t)$$

hold for $t \in [0, T]$, imply the existence of one solution to (19) such that

$$\alpha(t) \leq u(t) \leq \beta(t)$$

for $t \in [0, T]$. The choice of $\alpha(t) = \pi/2$ and $\beta(t) = 3\pi/2$ implies that problems (16) or (17) have at least one solution taking values in $[\pi/2, 3\pi/2]$ for all forcing terms e such that

$$\max_{t \in [0, T]} |e(t)| \leq |a|. \tag{20}$$

If we apply Knobloch's theorem to the equivalent formulation for (17)

$$v'' + cv' + a \sin(v + \tilde{E}(t)) = \bar{e}, \quad v(0) - v(T) = v'(0) - v'(0) = 0 \tag{21}$$

where $\tilde{E}(t)$ is the unique solution of the linear problem

$$u'' = e(t) - \bar{e}, \quad u(0) - u(T) = u'(0) - u'(T) = 0, \quad \int_0^T u(t)\, dt = 0,$$

we obtain existence of a solution if e is such that

$$\operatorname{osc} \tilde{E} := \max_{t \in [0, T]} \tilde{E}(t) - \min_{t \in [0, T]} \tilde{E}(t) \leq \pi \tag{22}$$

and $\bar{e}$ satisfies the condition

$$|\bar{e}| \leq |a| \cos[(1/2) \operatorname{osc} \tilde{E}]. \tag{23}$$

Condition (22) will be in particular satisfied if the L^2-norm of $e - \bar{e}$ is not too large, so that hypotheses (18) and (23) are distinct.

On the other hand, motivated by a result of Castro who used in 1980 a *variational version of the Liapunov-Schmidt method* to prove the existence of a solution for (16) when

$$\bar{e} = 0 \quad \text{and} \quad |a| \leq 1,$$

Willem in 1981 in the Publications du Séminaire d'analyse non linéaire de l'université de Besançon and Dancer in 1982 in the Annali di Mathematica rediscovered independently the *Hamel original variational treatment* (simplified by the explicit use of Sobolev spaces and lower semicontinuity) of the periodic problem (16) and proved the existence of a solution of (16) when

$$\bar{e} = 0 \tag{24}$$

by minimizing the corresponding action functional. Notice that condition (24) is distinct from conditions (18) and (23) and that we do not know, up to now, any condition which would unify them. Notice also that the only known proofs under condition (24) are the variational ones. As, in the periodic case, no transformation is known which, like in the Dirichlet case, would reduce problem (17) with $c \neq 0$ to a periodic problem having variational structure, the question was open to know if (24) still implied the existence of a solution of (17) for all a and all $c \neq 0$. Very recently, Ortega has answered this question negatively by showing that if $T \geq 2\pi\gamma > 0$, there exists $h \in C^\infty(\mathbb{R})$, T-periodic and verifying the condition $\bar{h} = 0$ and $\lambda_0 > 0$ such that, for all $\lambda > \lambda_0$, the problem

$$u'' + \lambda\gamma u' + \lambda \sin u = \lambda h(t), \quad u(0) - u(T) = u'(0) - u'(T) = 0$$

has no solution.

Those results show that, although some contributions of Dancer, Fournier, Kannan, Ortega, Willem and the author provide a good mathematical description of the set of forcing terms e for which problems (16) or (17) have a solution, much remains to be done in obtaining explicit conditions generalizing (20), (23) or (24).

5. The periodic problem: multiplicity of the solution set (topological degree and critical point theory)

If we apply the existence results of the previous section to the special case where $e = 0$, we can easily check that they provide the solution $u = \pi$ (or the geometrically equivalent solution $u=(2k+1)\pi$, $k\in\mathbb{Z}$), i.e. the unstable equilibrium of the pendulum (the physically unobservable one). When $e=0$, we know that there always exist another geometrically distinct T-periodic solution, namely $u=0$, the stable equilibrium. A natural question is then that of the existence of two geometrically distinct solutions of (16) and (17) when one of the conditions (20), (23) or (24) holds. This was done by the author in 1982, when condition (20) holds with strict inequality, using *topological degree*, a way of counting algebraically the number of solutions of an equation. The simple underlying idea can be explained as follows. If we choose to use the Leray-Schauder degree to prove Knobloch theorem then, when strict inequalities hold in its assumptions, we obtain from the proof the supplementary information that the absolute value of the Leray-Schrauder degree $d(E_{\alpha,\beta})$ associated to the corresponding fixed point operator and to the open set of $C([0,T])$

$$E_{\alpha,\beta}=\{u:\ \alpha(t)<u(t)<\beta(t)\ \text{ for all }\ t\in[0,T]\}$$

is equal to one. Now, because of the periodicity of the nonlinear term in the pendulum equation, we can take $(\alpha,\beta)=(\pi/2,3\pi/2)$ or $(\alpha,\beta)=(5\pi/2,7\pi/2)$ or $(\alpha,\beta)=(\pi/2,7\pi/2)$. Now the additivity property of degree tels us that if $D=E_{\pi/2,7\pi/2}\setminus(E_{\pi/2,3\pi/2}\cup E_{5\pi/2,7\pi/2})$, then

$$d(E_{\pi/2,7\pi/2})=d(E_{\pi/2,3\pi/2})+d(E_{5\pi/2,7\pi/2})+d(D),$$

and hence necessarily $d(D)\neq 0$. But then, the Leray-Schauder fixed point theorem implies the existence of a solution in D, i.e. of a solution geometrically distinct from the one in $E_{\pi/2,3\pi/2}$ or its translates by $2k\pi$. A similar argument can be used when conditions (22) and (23) hold with strict inequalities.

When we consider equation (16) under condition (24), the existence of a second solution can be obtained, as shown by the author and Willem in 1984 in the Journal of Differential Equations, by a *minimax method* close to the Ambrosetti-Rabinowitz *mountain pass lemma* and its ancestors in minimal surface theory. Here again, the underlying geometrical idea is rather simple. Because of condition (24), the action functional associated to (16) and given by

$$J(u)=\int_0^T[(1/2)(u'(t))^2+a\cos u(t)+u(t)\,e(t)]\,dt$$

has the periodicity property

$$J(u+2\pi)=J(u) \tag{25}$$

for all u in the suitable Sobolev space of T-periodic functions. Consequently, J has infinitely many minimums, and hence two, and if they are isolated (there is nothing to prove in the other case), there must exist another critical point of J (the mountain pass connecting the two valleys). This critical point provides the second geometrically distinct solution.

This multiplicity result can also be obtained using another minimax method, the *Lusternik-Schnirelmann theory*, as shown recently by Fournier, Willem and the author. The periodicity property (25) of the action suggests to consider J as defined, instead of the Sobolev space considered above, on the infinite-dimensional cylinder obtained by taking the product of the circle S^1 with the corresponding vector subspace of T-periodic functions having mean value zero. Now this Banach manifold has Lusternik-Schnirelmann category equal to two (in contrast to that of a Banach space, which is equal to one), where the *Lusternik-Schnirelmann category* of a topological space S is the least integer k such that S can be covered by k closed contractible subsets. Now, a theorem of Palais ensures that, in our situation, the number of critical points of J is greater or equal to the Lusternik-Schnirelmann category of the underlying manifolds, so that (16) has at least solutions. Such a result can also obtained, as shown independently by Rabinowitz, using, in the usual Sobolev space, a minimax argument on a family of subsets which takes advantage of the periodicity property of the functional. Those Lusternik-Schnirelmann approaches have the advantage, over the mountain pass lemma argument, of giving better multiplicity results for variational systems of second order equations with periodic nonlinearity: the number of distinct periodic solutions in this case will be greater of equal to the dimension of the system plus one, instead of the value two still given by the mountain pass argument.

More generally, the results can be carried to Lagrangian systems with N degrees of freedom

$$\begin{aligned}&(d/dt)(D_y\,L(t,u,u')=D_x\,L(t,u,u')\\&u(0)-u(T)=u'(0)-u'(T)=0\end{aligned} \tag{26}$$

where the Langrangian L, sufficiently smooth, has the form

$$L(t,x,y)=(1/2)(M(t,x)\,y,y)+(1/2)(Ax,x)+V(t,x)+(e(t),x)$$

with $M(t,x)$ symmetric and uniformly positive definite and A symmetric and semipositive definite. Those assumptions easily imply that problem (26) with $V=0$ and $e=0$ only admits constant solutions, which will be necessarily then elements of ker A. The natural periodicity assumption which has now to be made on M and V, and which generalizes the 2π-periodicity of $\sin u$ in the forced pendulum equation, is to assume the existence of a basis $\{a_1,\ldots,a_m\}$ of ker A and of positive numbers $T_1,\ldots,T_m$ (where $1\le m\le N$) such that the relations

$$M(t,x+T_i\,a_i)=M(t,x),\qquad V(t,x)+T_i\,a_i)=V(t,x)\qquad(1\le i\le m)$$

hold for all $t\in[0,T]$ and $x\in\mathbb{R}^N$. Under thoses assumptions, it can be shown by the Lusternik-Schnirelmann arguments sketched above that the following natural generalization of condition (24)

$$(\bar{e},a_i)=0,\qquad(1\le i\le m)$$

implies the existence of at least $m+1$ geometrically distinct solutions of (26) (Mawhin, Fournier, Willem, Rabinowitz). Systems (26) and the assumptions are general enough to contain as special cases the Beletski equations for the libration of satellites and the equations of the forced multiple pendulum.

A sharper multiplicity result can be obtained, under some nondegeneracy assumptions, using *Morse theory*. In this approach, the critical points of a functional φ over a manifold M are detected by analyzing the changes in the topology (namely the homology) of the sets $\varphi^c = \{u \in M: \varphi(u) \leq c\}$ when c crosses a critical value, i.e. the image by φ of a critical point. If u is a critical point of φ with critical value c, its critical groups are defined by $C_n(\varphi, u) = H_n(\varphi^c \cap U, \varphi^c \cap U \backslash \{u\})$, where the H_n denote the relative homology groups and U is a closed neighbourhood of u. The most precise information provided by Morse theory concerns the case where the critical points u are non-degenerate, i.e. when $\varphi''(u)$ is invertible. If it is the case, and if φ is bounded from below and satisfies some compactness condition, Morse theory implies that the number of critical points of φ in M is greater or equal to the sum of the Betti numbers M, i.e. the sum of the dimensions of its homology groups. The functional J associated to (26), i.e.

$$J(u) = \int_0^T L(t, u(t), u'(t))\,dt$$

is bounded from below under the assumptions above and satisfies the periodicity conditions

$$J(u + T_i a_i) = J(u), \qquad (1 \leq i \leq m)$$

for all u in a suitable Sobolev space. It is therefore natural to consider J defined over the product M of an m-dimensional torus T^m and of an infinite dimensional subspace of functions u satisfying the conditions

$$(u, a_i) = 0, \qquad 1 \leq i \leq m.$$

It is easy to show that the sum of the Betti numbers of M is equal to that of the Betti numbers of T^m, i.e. to

$$\sum_{k=0}^{m} \binom{m}{k} = 2^m.$$

Consequently, when all the critical points of J are non-degenerate, their number is not less than 2^m.

Weakening the assumption that A is semi-positive definite in the results described above is important for applications like linearly coupled forced pendulums, some models of multipoint Josephson junctions in solid state physics and systems coming from space discretizations of some boundary value problems for the sine-Gordon equation. The difficulty is that, without this assumption, J is no more bounded from below, but more sophisticated minimax methods than the Lusternik-Schnirelmann one and more elaborate techniques of Morse theory allow to extend the above conclusions, modulo some mild restrictions upon A, as shown very recently by Chang, Fonda and the author. This last extension is closely related to the Conley-Zehnder proof of a *conjecture of Arnold* in symplectic geometry, that they reduce to an estimation of the number of periodic solutions of a Hamiltonian system

$$J z' + D_z H(t, z) = 0$$

where H is periodic in each of its variables.

6. The periodic problem: perturbation of the non-constant solutions (subharmonics, invariant tori and chaotic solutions)

All the global existence results described up to now give periodic solutions which correspond, in the unforced case, to the stable and unstable equilibria of the free pendulum equation: they can be considered as deformations, under the action of the forcing term, of those very special periodic solutions. Much less is known about the effect of a periodic forcing term $e(t)$ on the other trajectories of the free pendulum, and all what is known refers to the special case of the equation (with, say, $a > 0$)

$$u'' + a \sin u = \varepsilon h(t) \tag{27}$$

where h is T-periodic and $\varepsilon > 0$ is a small parameter (perturbation problems).

Using *Morse theory for nondegenerate critical manifolds*, Willem has shown recently that if $T > 2\pi/a$, and ε is sufficiently small, the closed orbits of (27) with $\varepsilon = 0$ having period jT for some $j \in \mathbf{N}^*$ generate at least two solutions of period jT for (9). A similar result had been obtained by Sari and Schmitt when the forcing term h satisfies some symmetry conditions. The *Kolmogorov-Arnold-Moser theory* is a powerful tool for the study of the perturbation of the qualitative phase-plane portrait of the pendulum equation.

Recently, much attention has been paid to the *perturbation of the homoclinic orbits* of the free pendulum equation under the action of a small periodic forcing term. In particular, Kirchgraber, in an unpublished paper of 1982 and Palmer, in an article published in 1984 in the Journal of Differential Equations, have considered the following special case of (27)

$$u'' + \sin u = \varepsilon \sin t. \tag{28}$$

If $e_0 = -\pi$, $e_1 = \pi$, $s(t)$ and $-s(t)$ respectively denote two consecutive unstable equilibriums of (28) and the connecting homoclinic orbits, and if $\eta_0(t)$ and $\eta_1(t)$ are the 2π-periodic solutions of (28) generated by e_0 and e_1 respectively, Palmer proves, for ε small, the existence of solutions $\sigma_+(t)$ and $\sigma_-(t)$ near $s(t)$ and $-s(t)$ respectively, such that

$$\sigma_+(t) - \eta_0(t) \to 0 \quad \text{as} \quad t \to -\infty, \qquad \sigma_+(t) - \eta_1(t) \to 0 \quad \text{as} \quad t \to +\infty.$$

Moreover, if $\alpha = (\ldots, a_{-1}, a_0, a_1, \ldots)$ is a doubly infinite sequence with $a_j \in \{0, 1\}$ for all integers j, Palmer prove that for $\varepsilon > 0$ sufficiently small and m sufficiently large, there is a unique solution u_a of (28) along which the pendulum rotates counterclockwise during the time segment $[(2k-2)m\pi, 2km\pi]$ if $a_k = 1$ and clockwise if $a_k = 0$. This shows how the forced pendulum leads us to the exploding domain of *chaotic motions* where a random behavior can come from a deterministic system with very few degrees of freedom. The importance of pendulum-like equations in the study of those fascinating phenomena is examplified by the variants of the forced pendulum equation, like

$$u'' + cu' + a \sin u = b \sin(s - t),$$

$$u'' + c(u^2 - 1)u' + a \sin u = b \cos t,$$

which regularly occur in popular articles or books about class.

We are far away from the isochronous oscillations of the suspended lamp in the cathedral of Pisa and from the commonsense expression "regular as a clockwork", but we see that, although chaos came from order, there is still some order in chaos.

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