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Generic results for the existence of nondegenerate periodic solutions of some differential systems with periodic nonlinearities

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Generic Results for the Existence of Nondegenerate Periodic Solutions of Some Differential Systems with Periodic Nonlinearities

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Introduction

The infinite dimensional version of Sard's theorem, due to Smale [9], is very useful in the study of boundary value problems of the form

$$A(u) = f$$

where A is a C^1-Fredholm map from a Banach space E to a Banach space F. In some applications, the existence of a solution is proved only when f belongs to a closed subspace G of F. In this case, it can be interesting to prove that the set of regular values of A in G is residual in G. This is not possible, if $G \neq F$, by a direct application of Smale's result. A first concrete example is given by the periodic problem for the forced pendulum equation

$$u'' + A \sin u = f(t), \; u(0) - u(T) = u'(0) - u'(T) = 0,$$

which is solvable when f has mean value zero (see [10] and [7] for a more general version). Under this assumption, the periodic problem has at least two geometrically distinct solutions. An other proof of this result, using a generalization of the Poincaré-Birkhoff theorem, was recently given by J. Franks [5]. A second example is a periodic problem motivated by the Conley-Zehnder solution of the Arnold conjecture [2]. Consider a Hamiltonian $H : \mathbb{R}^N \to \mathbb{R}$ which is periodic in each variable. The associated periodic problem

$$Ju' + \nabla H(u) = f(t), \; u(0) = u(T),$$

is solvable when f has mean value zero (see [1], [3], [6] for more general results).

Let C_T^0 be the space of continuous T-periodic functions with the supremum norm. The main result of this paper (Theorem 2) provides conditions under which all the solutions of the N-dimensional differential system

$$Lx + \nabla V(x) = y,$$

are nondegenerate when y belongs to a dense subset $\mathcal{G}$ of the set of y $\in C_T^0$ with mean value zero. In this equation, L is a linear differential operator acting on T-periodic functions and V a smooth function sublinear at infinity and verifying some periodicity conditions. It is essentially assumed that the kernel and cokernel of L coincide with some subspace Z of the constant mappings in C_T^0, and the condition on V is that, for each ξ on the unit sphere of Z, the set

$$\mathcal{N}(\xi) = \{x \in \mathbb{R}^N : (\nabla V(x), \xi) = 0 \text{ and } D^2V(x)\xi = 0\}$$

is totally disconnected. The idea of the proof consists in considering $L + \nabla V$ as a mapping between suitably defined Banach manifolds and then applying the Sard-Smale theorem.

The assumptions of Theorem 2 are always satisfied for the forced pendulum problem and its generalization

$$x'' + g(x) = f(t)$$

where g is smooth, periodic, of mean value zero, and has a totally disconnected set of zeros. This is the case treated in Theorem 3 and then applied to obtaining a new proof of the result in [7] about the

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existence of T-periodic solutions of this equation when f is close enough to a given $y \in \mathcal{G}$. Another application, related to [4], is also given to the existence of subharmonic solutions when $f \in \mathcal{G}$.

The last section of the paper provides applications of Theorem 2 to the obtention of conditions for the genericity of Morse type multiplicity results for the T-periodic solutions of T-periodic systems of the form

$$(P(t)x')' + Q(t)x + \nabla V(x) = y(t),$$

or of the form

$$Jx' + Q(t)x + \nabla V(x) = y(t),$$

with P a positive-definite matrix function, J the symplectic matrix and V satisfying suitable periodicity conditions.

1 The abstract results

Given $T > 0$, $N \geq 1$ and $k \geq 0$, define

$$C^k_T = \{u \in C^k(\mathbb{R},\mathbb{R}^N) : u \text{ is T-periodic}\}$$

together with the norm

$$|u|_k = \sum_{j=0}^{k} \|u^{(j)}\|_\infty ,$$

so that C^k_T is a Banach space.

Consider now a bounded linear operator $L : C^k_T \to C^0_T$ satisfying

(L_1) There exists a bounded projector $P : C^0_T \to C^0_T$ such that the following sequence is exact :

$$C^k_T \xrightarrow{P_k} C^k_T \xrightarrow{L} C^0_T \xrightarrow{P} C^0_T$$

where P_k denotes the restriction of P to C^k_T. The range of P is included in $\mathbb{R}^N$ which is identified to constant functions.

(L_2) P is L^2-symmetric, i.e.

$$\int_0^T (Px(t),y(t))dt = \int_0^T (x(t),Py(t))dt$$

for every $x,y \in C^0_T$, where $(.,.)$ denotes the usual inner product in $\mathbb{R}^N$.

We shall denote by Y the range of L and by Z the kernel of L, so that $Y = \text{Ker } P$, $Z = R(P)$.

Consider now a function $V \in C^2(\mathbb{R}^N,\mathbb{R})$ satisfying

(V_1) For every $\xi \in Z$ such that $|\xi| = 1$, the set

$$\mathcal{N}(\xi) = \{x \in \mathbb{R}^N : (\nabla V(x),\xi) = 0 \text{ and } D^2V(x)\xi = 0\}$$

is totally disconnected.

Under the previous assumptions, we analyze the problem in C^k_T

(1) $$Lx + \nabla V(x) = y$$

assuming that $y \in Y$.

Definition 1 *We say that $y \in Y$ is <u>regular</u> for problem (1) if every solution x of (1) is non degenerate in the sense that the corresponding linearized problem*

$$Lu + D^2V(x)u = 0$$

only admits the trivial solution in C^k_T.

We shall prove the following result :

Theorem 1 *Under the preceeding assumption, if $k \geq 1$, the set*

$$\mathcal{G} = \{y \in Y : y \textit{ is regular for } (1)\}$$

is residual in Y.

We shall use the following notations

$$M = \{x \in C^k_T : P(\nabla V(x)) = 0\}$$

$$X = \{x \in M : x \text{ is not constant}\} = M \backslash \mathbb{R}^N.$$

Lemma 1 Under the assumptions of theorem 1, X is a C^1-manifold.

Proof : Define the map

$$\Phi : C^k_T \backslash \mathbb{R}^N \to Z$$

by

$$\Phi(x) = P(\nabla V(x))$$

so that $X = \Phi^{-1}(0)$. We have only to prove that $\Phi'(x)$ is onto for each $x \in X$. If this is not the case, there exists $x \in X$ and $\xi \in Z$ such that $|\xi| = 1$ and

(2) $$(\Phi'(x)u,\xi) = 0, \ \forall u \in C^k_T.$$

Since

$$\Phi'(x)u = P(D^2V(x)u)$$

we obtain from (2) and (L_2) that

$$\int_0^T (u(t), D^2V(x(t))\xi)dt = 0, \ \forall u \in C^k_T.$$

The above relation implies that

$$D^2V(x(t))\xi = 0, \ \forall t \in \mathbb{R}.$$

On the other hand

$$\frac{d}{dt}(\nabla V(x(t)),\xi) = (\dot{x}(t), D^2V(x(t))\xi) = 0$$

and, since $x \in X$, $(\nabla V(x(t)),\xi) = 0$. Therefore, for every t, $x(t) \in \mathcal{N}(\xi)$ and (V_1) implies that x is constant, contradicting the fact that $x \in X$. □

Lemma 2 *Under the assumptions of theorem* 1, *the map* $\psi : X \to Y$ *defined by*

$$\psi(x) = Lx + \nabla V(x)$$

is a C^1-Fredholm map of index zero.

Proof : At each $x \in X$, the tangent space is given by

$$T_xX = \{u \in C^k_T : P(D^2V(x)u) = 0\}$$

and the derivative

$$\psi'(x) : T_xX \to Y$$

is defined by

$$\psi'(x)u = Lu + D^2V(x)u.$$

It follows from (L_1) that

$$\mathrm{Ker}(L + D^2V(x)) = \mathrm{Ker}(\psi'(x))$$

$$R(L + D^2V(x)) \cap Y = R(\psi'(x))$$

and, from these identities, we obtain

$$\mathrm{codim}_{C^0_T} R(L + D^2V(x)) = \mathrm{codim}_Y R(\psi'(x)).$$

Using again (L_1), one sees that L is a Fredholm operator of index zero and since $D^2V(x)$ is compact, it follows that $L + D^2V(x)$ is also Fredholm of index zero. Hence

$$\dim \mathrm{Ker}(\psi'(x)) = \mathrm{codim}\, R(\psi'(x)). \qquad \square$$

Proof of theorem 1 : Lemma 1 and lemma 2 allow us to apply Sard-Smale theorem [9] to conclude that

$$\tilde{G} = \{y \in Y : y \text{ is a regular value of } \psi\}$$

is residual in Y. It suffices now to prove that

$$S = \{Lx + \nabla V(x) : x \text{ is constant, } P\nabla V(x) = 0\}$$

has a residual complement in Y since

$$\tilde{G} \cap \complement_Y S \subseteq G.$$

For every $k \geq 1$,

$$S_j = \{Lx + \nabla V(x) : x \text{ is constant, } |x|_k \leq j, P\nabla V(x) = 0\}$$

is a closed subset of Y with empty interior. Hence we obtain that

$$\complement_Y S = \bigcap_{j=1}^{\infty} \complement_Y S_j$$

is residual in Y. □

In addition we assume that

(V_2) $\lim_{|x| \to \infty} \nabla V(x)/|x| = 0$

(V_3) There exists a basis $(z_1, \ldots, z_m)$ of Z such that

$$V(x + z_i) = V(x), \forall x \in \mathbb{R}^N, 1 \leq i \leq m.$$

Theorem 2 *Under the assumptions* $(L_{1\text{-}2})$ *and* $(V_{1\text{-}2\text{-}3})$, *if* $k \geq 1$, *the set* G *is open and dense in* Y.

Proof : By Theorem 1, it is enough to show that $\complement_Y G$ is closed. Assume that $(y_n) \subset \complement_Y G$ converges to $y \in Y$. According to the definition of G there exists sequences (x_n) and (u_n) in C_T^k such that

(3) $$Lx_n + \nabla V(x_n) = y_n$$

(4) $$Lu_n + D^2V(x_n)u_n = 0, \quad |u_n|_k = 1.$$

Writing $x_n = \bar{x}_n + \tilde{x}_n$ where $\bar{x}_n = P_k x_n$, we can assume by (V_3) that

$$\bar{x}_n = \sum_{i=1}^{m} c_i^n z_i, \quad 0 \leq c_i^n < 1, \quad 1 \leq i \leq m.$$

Let us denote by $K : Y \to \text{Ker } P_k$ the generalized inverse of L. We obtain from (3),

(5) $$\tilde{x}_n = K(y_n - \nabla V(x_n)).$$

Assumptions (V_2) implies that $(\tilde{x}_n)$ is bounded in C_T^k. Thus (x_n) is bounded in C_T^k and, by the Ascoli-Arzela theorem, going if necessary to a subsequence, we can assume that $x_n \to x$ in C_T^0, but then, by (5) $x_n \to x$ in C_T^k. Similarly, going if necessary to a subsequence, we can assume that $u_n \to u$ in C_T^k. We obtain from (3) and (4)

$$Lx + \nabla V(x) = y$$

$$Lu + D^2V(x)u = 0, \qquad |u|_k = 1,$$

so that $y \in \complement_Y G$. □

Remark : We shall see in the next section that condition (V_1) is not very restrictive when dim Z = 1. For higher dimensions, let us consider the simple situation when $Z = \mathbb{R}^m$ and

$$V(x) = \sum_{i=1}^{m} V_i(x_i)$$

with V_i real analytic and not identically zero.

If m = 2 and if we assume that

$$[V_i'(x)]^2 + [V_i''(x)]^2 > 0$$

for each $x \in \mathbb{R}$ and $i = 1,2$, then (V_1) holds. Indeed, for $\xi = (\xi_1,\xi_2)$, $\mathcal{N}(\xi)$ is the set of solutions of

$$\xi_1 V_1'(x_1) + \xi_2 V_2'(x_2) = 0,\ \xi_1 V_1''(x_1) = \xi_2 V_2''(x_2) = 0.$$

If $\xi_1\xi_2 \neq 0$, $x \in \mathcal{N}(\xi)$ implies that $V_1''(x_1) = V_2''(x_2) = 0$ and therefore $\mathcal{N}(\xi)$ is discrete. If $\xi = (1,0)$, $V_1'(x_1) = V_1''(x_1) = 0$ and $\mathcal{N}(\xi) = \phi$, and the remaining cases are similar.

If $m = 3$, it follows from the periodicity of V_i (condition V_3) that $V_2''(\alpha) = V_3''(\beta) = 0$ for some $\alpha,\beta \in \mathbb{R}$. Let $(\xi_2,\xi_3) \in \mathbb{R}^2$ be such that $\xi_2^2 + \xi_3^2 = 1$ and

$$\xi_2 V_2'(\alpha) + \xi_3 V_3'(\beta) = 0.$$

Taking $\xi = (0,\xi_2,\xi_3)$, one has

$$\{(x_1,\alpha,\beta) : x_1 \in \mathbb{R}\} \subset \mathcal{N}(\xi)$$

and $\mathcal{N}(\xi)$ is not totally disconnected.

2 Forced pendulum type equations

In this section we consider the existence of T-periodic solutions of the equation

$$\ddot{x} + g(x) = y(t). \tag{6}$$

We assume that $y : \mathbb{R} \to \mathbb{R}$ is a continuous T-periodic function such that

$$\int_0^T y(t)dt = 0$$

and that $g : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function such that, setting

$$V(x) = \int_0^x g(s)ds,$$

the function V is 2π-periodic. Finally we assume that the set of zeroes of g is totally disconnected. (We have in mind as a particular case, the forced pendulum equation, when $g(x) = A \sin x$).

Theorem 3 *Under the above assumptions, the set* $\mathbb{G}$ *of regular value for* (6) *is open and dense in*

$$Y = \{y \in C_T^0 : \int_0^T y(t)dt = 0\}.$$

Proof : It suffices to apply theorem 2, with $Lx = \ddot{x}$ and $Px = \int_0^T x(t)dt$. □

The hypothesis "the set of zeros is totally disconnected" is essential in the sense that if g vanishes on an interval, then theorem 3 is not true.

Corollary 1 *For every* $y \in \mathbb{G}$, *there exists* $\varepsilon > 0$ *such that, if* $|y-f|_\infty \leq \varepsilon$, *then* $\ddot{x} + g(x) = f$ *has a* T-*periodic solution.*

Proof : By a result of [10], (6) has a T-periodic solution for every $y \in Y$. If $y \in \mathbb{G}$, the corresponding linearized problems only admit the trivial solution. It suffices then to apply the implicit function theorem. □

Corollary 2 *For every* $y \in \mathbb{G}$, *there exists a* $k_0 \geq 2$ *such that, for every prime integer* $k \geq k_0$, *there is a periodic solution of* (6) *with minimal period* kT.

Proof : By a result of [4], it suffices to prove that
(a) the T-periodic solutions of (2) are isolated,

(b) every T-periodic solution of (2) having Morse index equal to zero is nondegenerate.

If $y \in \mathcal{G}$, every T-periodic solution of (6) is nondegenerate, and, hence, in particular, isolated. □

Remarks : 1. Corollary 1 was first proved in [7] by another approach.

2. It is not known if the conclusion of corollary 1 holds for every $y \in Y$.

3 Conservative systems with periodic nonlinearity

In this section, we consider the existence of T-periodic solutions of the system

$$(P(t)x')' + Q(t)x + \nabla V(x) = y(t), \tag{7}$$

where $P : \mathbb{R} \to S(\mathbb{R}^N,\mathbb{R}^N)$ is a C^1 T-periodic mapping from $\mathbb{R}$ to the space of symmetric real matrices of order N, such that, for some $\mu > 0$ and all $(t,v) \in \mathbb{R} \times \mathbb{R}^N$,

$$(P(t)v,v) \geq \mu|v|^2,$$

$Q : \mathbb{R} \to S(\mathbb{R}^N,\mathbb{R}^N)$ is continuous and T-periodic, $V \in C^2(\mathbb{R}^N,\mathbb{R})$ and $y : \mathbb{R} \to \mathbb{R}^N$ is continuous and T-periodic.

Equations (7) includes linearly coupled pendulums equations and Josephson multipoint systems for which $(V_{1\text{-}2\text{-}3})$ are satisfied.

We denote by L the linear operator

$$C_T^2 \to C_T^0 : x \longmapsto (P(t)x')' + Q(t)x.$$

As usual we denote by Y the range of L and by Z the kernel of L. We assume that Z consists of constant functions. It is easy to verify

assumptions (L_1) and (L_2). It suffices then to apply theorem 2 in order to obtain the following result :

Theorem 4 *If* V *satisfies assumptions* $(V_{1\text{-}2\text{-}3})$, *then the set* $\mathcal{G}$ *of regular values for* (7) *is open and dense in* Y.

Let us denote by m the dimension of Z.

Corollary 3 *If* V *satisfies assumptions* $(V_{1\text{-}2\text{-}3})$, *then for every* $y \in \mathcal{G}$, *system* (7) *has at least* 2^m *geometrically distinct* T*-periodic solutions.*

Proof : By a sligthly generalized version of theorem 3 of [3], if all the T-periodic solutions of (7) are non degenerate, then there are at least 2^m geometrically distinct of them. □

Remark : Let us recall ([1], [3]) that, for every $y \in Y$, system (7) has at least (m+1) geometrically distinct periodic solutions.

Finally similar results hold for first order systems. More precisely we consider the Hamiltonian systems

$$J\dot{x} + Q(t)x + \nabla V(x) = y(t) \tag{8}$$

where $Q : \mathbb{R} \to S(\mathbb{R}^{2N},\mathbb{R}^{2N})$ is continuous and T-periodic, $V \in C^2(\mathbb{R}^{2N},\mathbb{R})$ and $y : \mathbb{R} \to \mathbb{R}^{2N}$ is continuous and T-periodic.

Let us recall that

$$J = \begin{pmatrix} 0 & -id_{\mathbb{R}^N} \\ id_{\mathbb{R}^N} & 0 \end{pmatrix}$$

is the standard symplectic matrix. We denote by L the linear operator

$$C_T^1 \to C_T^0 : x \longmapsto J\dot{x} + Q(t)x.$$

We assume that the kernel Z of L consists of constant functions and that y belongs to the range Y of L. We obtain from theorem 2 the following result :

Theorem 5 *If* V *satisfies assumptions* ($V_{1\text{-}2\text{-}3}$), *then the set* $\mathbb{G}$ *of regular values for* (8) *is open and dense in* Y.

Let us denote by m the dimension of Z as before.

Corollary 4 *If* D^2V *is bounded on* $\mathbb{R}^{2N}$, *then, for every* $y \in \mathbb{G}$, *system* (8) *has at least* 2^m *geometrically distinct* T-*periodic solutions.*

Proof : By a slightly more general version of theorem 4 of [3], if all the T-periodic solutions of (8) are non degenerate, then there are at least 2^m geometrically distinct of them. □

Let us consider the simple case when $Q(t) \equiv 0$. In this case $Y = \{y \in C^0_T : \int_0^T y(t)dt = 0\}$ and $Z = \mathbb{R}^{2N}$.

The corresponding assumptions for V are

(A_1) For every $\xi \in \mathbb{R}^{2N}$ such that $|\xi| = 1$, the set

$$\mathcal{N}(\xi) = \{x \in \mathbb{R}^{2N} : (\nabla V(x),\xi) = 0 \text{ and } D^2V(x)\xi = 0\}$$

is totally disconnected.

(A_2) V is periodic with respect to each variable.

Corollary 5 *If* V *satisfies assumptions* (A_1) *and* (A_2), *then the set* $\mathbb{G}$ *of regular values for* (8) *is open and dense in* Y.

Moreover for every $y \in \mathbb{G}$, system (8) has at least 4^N geometrically distinct T-periodic solutions. □

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