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Closed orbits of fixed energy for singular Hamiltonian systems

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CLOSED ORBITS OF FIXED ENERGY FOR SINGULAR HAMILTONIAN SYSTEMS

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§1. Introduction

This paper deals with the existence of periodic solutions of

$$q'' + V'(q) = 0 \tag{1.1}$$

such that

$$\tfrac{1}{2}|q'|^2 + V(q) = h \tag{1.2}$$

where $q \in \mathbf{R}^N$, h is a given number, $V \in C^2(\mathbf{R}^N \setminus \{0\}, \mathbf{R})$ has a singularity at $x = 0$, and V' denotes the gradient of V.

Our main results are collected in Theorems 3.6, 4.12 and 5.1.

In the former we deal with potentials which, roughly, behaves like $-\frac{1}{|x|^a}$ with $a > 2$ (referred to in the sequel as "Strong force" case) and prove the existence of solutions q of (1.1–2) such that $q(t) \neq 0 \quad \forall t \in \mathbf{R}$ (non-collisions).

In the latters we are concerned with the case in which $V(x) \approx -\frac{1}{|x|^b}$ with $0 < b < 2$, and prove the existence of solutions q which can, possibly, pass through the singularity $x = 0$ (collisions).

To have an idea of the kind of problem we can handle, let us state two specific results concerning potentials of the type

$$V(x) = -\frac{1}{|x|} + W(x)$$

when (1.1) becames the perturbed Kepler's equation

$$q'' + \frac{q}{|q|^3} + W'(q) = 0.$$

THEOREM 1.1. *Suppose* $V(x) = -\frac{1}{|x|} + W(x)$ *with* $W \in C^2(\mathbf{R}^N; \mathbf{R})$ *satisfying*

(W1) $3W'(x)\cdot x + W''(x)x\cdot x \geq 0$;

(W2) $W'(x)\cdot x > -\frac{1}{|x|}$;

(W3) $W'(x)\cdot x - W(x) \geq 0$;

(W4) $\liminf_{|x|\to\infty} [W(x) + \frac{1}{2}W'(x)\cdot x] \geq 0$;

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Then for all $h < 0$ (1.1–2) has a periodic solution.

THEOREM 1.2. *Let $h < 0$ be given and suppose $V(x) = -\frac{1}{|x|} + \varepsilon U(x)$ with U smooth in all $\mathbf{R}^N$. Then there exists $\bar{\varepsilon}$ (depending on h and $\|U\|_{C^2}$) such that $\forall|\varepsilon| < \bar{\varepsilon}$ the problem*

$$q'' + \frac{q}{|q|^3} + \varepsilon W'(x) = 0$$

$$\tfrac{1}{2}|q'(t)|^2 - \frac{1}{|q(t)|} + \varepsilon W(q(t)) = h$$

has at least one solution.

Theorems 1.1 and 1.2 follow from the much more general results contained in Theorems 4.12 and 5.1, respectively.

In the last years there has been a remarkable amount of work on existence of periodic solutions of systems with singular potentials, having a given number $T > 0$ as period, see, for ex. [**2**], [**4**], [**7**], [**8**], [**11**], [**6**], [**14**], [**18**].

On the contrary, much less is known on problem (1.1–2) where the energy is prescribed rather then the period. As far as we know, only the papers [**12**], [**5**] deal with such a problem in the large (for perturbation results see, for example [**15**]).

A short comparison with these two works is in order.

First of all we remark that [**12**] covers a rather restricted class of potentials satisfying the Strong force condition, only.

As for [**5**], it deals with the existence of solutions of (1.1–2) confined in an annulus $\mathcal{A}$ where the shape of V differs strongly from that of $-\frac{1}{|x|^b}$.

For example, neither the potentials $V(x) \approx -\frac{1}{|x|} - \frac{1}{|x|^b}$, $0 < b < 2$, (covered by Theorem 4.12 or, for $1 < b < 2$, by Theorem 1.1) nor any $V(x) = -\frac{1}{|x|} + \varepsilon U(x)$, $\varepsilon \geq 0$, (see Theorem 1.2) can be handled by [**5**].

On the other hand we do not confine the solutions in any annulus, and in the case of Theorems 4.12, 5.1, the solutions we find could be collisions.

When $V \in C^1(\mathbf{R}^N, \mathbf{R})$ an usual way to seek for solutions of (1.1–2) is to look for stationary points of

$$\tilde{f}(u) = \tfrac{1}{2}\int_0^1 |u'|^2\,dt \cdot \int_0^1 [h - V(u)]\,dt$$

on $H^1(S^1, \mathbf{R}^N\setminus\{0\})$ with $\int_0^1 |u'|^2\,dt > 0$. If this is the case, $q(t) = u(\omega t)$ where

$$\omega^2 = \frac{\int_0^1 [h - V(u)]\,dt}{\frac{1}{2}\int_0^1 |u'|^2\,dt},$$

solves (1.1–2). See, for example, [**19**].

In the present paper the idea of the proof relies on a variational principle,discussed in section 2, which amounts to find solutions of (1.1–2) as critical points at positive level of the functional

$$f(u) = \tfrac{1}{2}\int_0^1 |u'|^2\,dt \cdot \int_0^1 V'(u)\cdot u\,dt$$

on the set

$$M_h = \{\, u \in \Lambda : \int_0^1 [V(u) + \tfrac{1}{2}V'(u)\cdot u]\, dt = h \,\}.$$

where

$$\Lambda = \{\, u \in H^1(S^1; \mathbf{R}^N) \quad \text{such that} \quad u(t) \neq 0 \ \forall t \,\}.$$

Our principle is related to the preceding one by the fact that $\tilde{f}_{|M_h} = f$ and $\nabla f_{|M_h} = 0$ if and only if $\nabla \tilde{f} = 0$.

A similar approach has been used in [**3**] and, earlier, for semilinear elliptic boundary values problems, in [**13**], [**1**], [**16**] but it is new in connection with singular Hamiltonian systems.

To clarify why our approach is appropriate for our purposes and seems more suitable for a rather direct application of the Lusternik- Schnirelman (LS, for short) theory let us shortly outline the arguments of the proof.

Assuming $V(x) \approx -\frac{1}{|x|^\alpha}$, $\alpha > 0$ near $x = 0$ we distinguish between $\alpha > 2$ and $\alpha < 2$.

In the "Strong Force" case, studied in section 3, it is natural to take $h > 0$. It turns out that for such an h: (i) $M_h \neq \emptyset$ and it is a smooth manifold; (ii) $\operatorname{cat} M_h = +\infty$ (here cat denotes the LS category); (iii) f is bounded below on M_h; (iv) f satisfies the Palais-Smale (PS for short) condition on M_h; (v) M_h is complete. Then the LS theory applies yielding infinitely many critical points for f on M_h with $f(u) > 0$.

The case in which V does not satisfy the "Strong force" condition (as it is the case in theorems 1.1 and 1.2) is discussed in section 4 and requires some care. Taking $h < 0$, which is now the "natural" value of the energy, one still has that (i), (ii) and (iii) hold, but now M_h is no more complete.

To overcome such a problem (and the related lack of PS) we modify V by setting

$$V_\varepsilon(x) = V(x) - \varepsilon \frac{1}{|x|^2}.$$

A remarkable feature of our approach is that the manifold M_h^ε, corresponding to the potential V_ε, coincide with the manifold M_h corresponding to V, and one is led to seek critical points of

$$f_\varepsilon = \tfrac{1}{2} \int_0^1 |u'|^2\, dt \cdot \int_0^1 V_\varepsilon'(u)\cdot u\, dt$$

on M_h. Since V_ε satisfies the Strong Force condition, the preceding arguments yield a critical point $u_\varepsilon \in M_h$, such that $f_\varepsilon(u_\varepsilon) > 0$. A limiting procedure, based on some energy estimates, allows us to show that $u_\varepsilon \to u$ as $\varepsilon \to 0$ and this gives a solution of (1.1–2).

The hypotheses of Theorems 3.6 and 4.12 are global in nature. In the last section we state a result, Theorem 5.1 (which is related to Theorem 4.12) where such assumptions are made in $\{\, V \leq h \,\}$ only.

The same variational approach can be used to handle a class of Hamiltonian systems including the N-body problem. This results will make the object of a forthcoming paper.

Notations. In all the paper we let

$$x \cdot y = \sum_{i=1}^{N} x_i y_i \qquad \forall x, y \in \mathbf{R}^N$$

$$|x| = \sqrt{x \cdot x} \qquad \forall x \in \mathbf{R}^n$$

$$H = H^1(S^1, \mathbf{R}^N)$$

$$(u|v) = \int_0^1 u' \cdot v' + \int_0^1 u \cdot v \qquad \forall u, v \in H$$

$$\|u\|^2 = (u|u) \qquad \forall u \in H$$

$$\Omega = \mathbf{R}^N \setminus \{0\}$$

$$\Lambda = \{\, u \in H \quad \text{such that} \quad u(t) \neq 0 \ \forall t \,\}$$

§2. The Variational Principle

In this section we state the Variational principle.
Always in the sequel we assume

$$V \in C^2(\Omega; \mathbf{R}).$$

We define $f \in C^1(\Lambda, \mathbf{R})$ by

$$f(u) = \tfrac{1}{4} \int_0^1 |u'|^2 \, dt \cdot \int_0^1 V'(u) \cdot u \, dt \tag{2.1}$$

and $g \in C^1(\Lambda, \mathbf{R})$ by

$$g(u) = \int_0^1 [V(u) + \tfrac{1}{2} V'(u) \cdot u] \, dt \tag{2.2}$$

and set, for $h \in \mathbf{R}$

$$M_h = \{ u \in \Lambda \, : \, g(u) = h \}. \tag{2.3}$$

We remark that, from the Sobolev embedding theorem, it follows immediately

$$u_n \rightharpoonup \bar{u}, \quad u_n, \bar{u} \in \Lambda \qquad \Longrightarrow \qquad g(u_n) \to g(\bar{u}) \quad \text{and} \quad \nabla g(u_n) \to \nabla g(\bar{u}). \tag{2.4}$$

In the rest of this section it is understood that $M_h \neq \emptyset$. It will be shown in §3, 4 that this is actually the case for suitable values of h related to the behaviour of V at $x = 0$ and at $|x| \to \infty$.

LEMMA 2.1. *Let V satisfy*

(A1) $$3V'(x) \cdot x + V''(x) x \cdot x \neq 0 \qquad \forall x \in \Omega.$$

Then M_h is a C^1 manifold of codimension 1 in Λ. More precisely, there results:

$$(\nabla g(u) | u) \neq 0 \qquad \forall u \in M_h. \tag{2.5}$$

Moreover, if V satisfies also

(A2) $$V'(x) \cdot x > 0 \qquad \forall x \in \Omega$$

then $f(u) \geq 0$ on M_h and $f(u) = 0$, $u \in M_h$, if and only if u is a constant.

PROOF: By direct calculation one has

$$(\nabla g(u) | u) = \int_0^1 \left[\tfrac{3}{2} V'(u) \cdot u - \tfrac{1}{2} V''(u) u \cdot u \right] \, dt$$

and the first statement follows from (A1). The second one easily follows from (A2). ∎

REMARK 2.2 M_h is obviously closed with respect to Λ, but not necessarely with respect to H, as we will see for the class of potentials discussed in §4. However, if $\overline{M}_h$ denotes the closure of M_h in H, one has that $\overline{M}_h \setminus M_h \subset \partial\Lambda$. ∎

LEMMA 2.3. *Suppose (A1) and (A2) hold. Let $u \in M_h$ be a critical point of f constrained on M_h such that $f(u) > 0$. Then, setting*

$$\omega^2 = \frac{\int_0^1 V'(u)\cdot u\, dt}{\int_0^1 |u'|^2\, dt}, \tag{2.6}$$

we have that $q(t) = u(\omega t)$ is a (non-constant) periodic solution of (1.1–2).

PROOF: Let $u \in M_h$ be a critical point of f constrained on M_h. Then there exists $\lambda \in \mathbf{R}$ such that

$$\nabla f(u) = \lambda \nabla g(u) \tag{2.7}$$

and, taking the scalar product with u, one obtains

$$(\nabla f(u)|u) = \lambda(\nabla g(u)|u).$$

From this one easily deduces

$$\lambda = \tfrac{1}{2}\int_0^1 |u'|^2\, dt.$$

Inserting this value into (2.7) one obtains

$$\int_0^1 u'\cdot v'\, dt \cdot \int_0^1 V'(u)\cdot u\, dt = \int_0^1 |u'|^2\, dt \int_0^1 V'(u)\cdot v \qquad \forall v \in H$$

Note that $f(u) > 0$ implies $\int_0^1 |u'|^2\, dt > 0$. Then it follows

$$\omega^2 u'' - V'(u) = 0 \tag{2.8}$$

with ω^2 given by (2.6), and $q(t) = u(\omega t)$ solves (1.1). Moreover, (2.8) being autonomous, the conservation of energy yields

$$\tfrac{1}{2}\omega^2 |u'(t)|^2 + V(u(t)) = c. \tag{2.9}$$

Integrating, one finds

$$\tfrac{1}{2}\omega^2 \int_0^1 |u'|^2\, dt + \int_0^1 V(u)\, dt = c$$

and thus, since $u \in M_h$, $c = h$. This implies that q (which is non-constant since $\int_0^1 |u'|^2\, dt > 0$) satisfies (1.2). ∎

§3 Existence results (Strong forces)

To prove the existence of critical points of f on M_h an important rôle is played by the behaviour of V as $x \to 0$. Let us explain this fact with the "model" case

$$V(x) = -\frac{1}{|x|^\alpha}, \qquad \alpha > 0.$$

Note that $\forall \alpha \neq 2$ (A1) and (A2) hold and the variational principle applies. Here the "natural" values of the energy are: $h > 0$ if $\alpha > 2$ and $h < 0$ if $0 < \alpha < 2$. In fact if $q(t)$ is a radial, periodic solution of

$$q'' + \alpha \frac{q}{|q|^{\alpha+1}} = 0$$

the corresponding energy is

$$\begin{aligned} h &= \tfrac{1}{2}|q'(t)|^2 - \frac{1}{|q(t)|^\alpha} \\ &= \left(\frac{\alpha}{2} - 1\right) \frac{1}{|q(t)|^\alpha}. \end{aligned}$$

On M_h (nonempty by lemmas 3.3 and 4.3), the functional f takes the form

$$f(u) = \tfrac{1}{2} \int_0^1 |u'|^2\,dt \cdot \int_0^1 \frac{\alpha}{|u|^\alpha}.$$

If $u \to \bar{u} \in \overline{M}_h \backslash M_h$, one has (see remark 2.2) $\bar{u} \in \partial\Lambda$. Then it is well known (see Lemma 3.1) that

$$\int_0^1 \frac{1}{|u|^\alpha} \to \infty \qquad \text{if} \quad \alpha > 2$$

while, when $0 < \alpha < 2$, the integral above can converge to a finite value.

This model case shows that it is worthwhile to distinguish between potentials which behave (as $|x| \to 0$) like $-\frac{1}{|x|^a}$, $a > 2$ (Strong forces) or like $-\frac{1}{|x|^b}$, $0 < b < 2$ (Weak forces). The former is discussed in this section, the latter will be discussed in §4.

It is worth noticing that one of the interest of the Strong force case relies on the fact that the Weak force one will be handled perturbing V with a Strong force potential.

We start dealing with potentials satisfying (A1), (A2) and

(A3) $\exists \gamma > 2$, such that $V'(x) \cdot x \le -\gamma V(x) \qquad \forall x \in \Omega$

(A4) $\exists \beta > 2$ and $r > 0$ such that $V'(x) \cdot x \ge -\beta V(x) \qquad \forall 0 < |x| \le r$

(A5) $\limsup_{|x| \to \infty} \left[V(x) + \tfrac{1}{2} V'(x) \cdot x\right] \le 0$

We note that (A2) and (A3) imply

$$V(x) < 0 \qquad \forall x \in \Omega \tag{3.1}$$

First of all we show that (A4) implies the so colled "Strong force condition" [**10**]. We recall that in [**10**] it is proved:

LEMMA 3.1. *Suppose that V satisfies the "Strong force condition", i.e.*

$$\exists r > 0 \quad \text{and} \quad \alpha \geq 2 \qquad \text{such that} \qquad V(x) \leq -\frac{c}{|x|^\alpha} \quad \forall 0 < |x| < r.$$

Then, for any sequence u_n in Λ converging weakly and uniformly to $\bar{u} \in \partial\Lambda$ we have that

$$\int_0^1 V(u_n)\,dt \to -\infty.$$

LEMMA 3.2. *If (A4) holds then $\exists c_1 > 0$ such that*

$$V(x) \leq -\frac{c_1}{|x|^\beta} \qquad \forall 0 < |x| \leq r. \tag{3.2}$$

As a consequence we have that

$$\int_0^1 V(u_n)\,dt \to -\infty \qquad \forall u_n \rightharpoonup u \in \partial\Lambda \tag{3.3}$$

PROOF: For $|y| = r$ define $\rho_y : (0,1] \to \mathbf{R}$ by

$$\rho_y(\lambda) = -V(\lambda y) > 0.$$

From (A4) it follows

$$\rho_y'(\lambda) = -V'(\lambda y)\cdot y \leq \beta\frac{V(\lambda y)}{\lambda} = -\beta\frac{\rho_y(\lambda)}{\lambda}$$

hence

$$\rho_y(\lambda) \geq \rho_y(1)\lambda^{-\beta} \geq c_2\lambda^{-\beta} \qquad \forall 0 < \lambda \leq 1.$$

where

$$c_2 = \min\{\, -V(y) \mid |y| = r \,\}.$$

Then, letting $y = \frac{x}{|x|}r$, there results for $0 < |x| \leq r$:

$$V(x) = V\left(\frac{|x|}{r}y\right) = -\rho_y\left(\frac{|x|}{r}y\right) \leq -\frac{c_1}{|x|^\beta}$$

with $c_1 = c_2 r^\beta$.

The last statement follows from Lemma 3.1. ∎

Next we prove

LEMMA 3.3. *Suppose (A1-2-3-4-5) hold and let $h > 0$. Then:*

(1) $M_h \neq \emptyset$ *and* M_h *is complete;*
(2) $\mathrm{cat}_{M_h}(M_h) = \infty$*; more precisely* $\forall m \geq 0\ \exists X \subset M_h$, X *compact, such that* $\mathrm{cat}_{M_h}(X) \geq m$.

PROOF: Let $u \in \Lambda$ be fixed. For $a > 0$, one has:

$$g_u(a) \equiv g(au) = \int_0^1 [V(au) + \tfrac{1}{2}V'(au)\cdot au]\, dt.$$

According to (2.5) $\frac{d}{da} g_u(a) \neq 0$, hence g_u is strictly monotone. Using (A5) it immediately follows that

$$\lim_{a\to\infty} g_u(a) \leq 0.$$

Let $a \to 0^+$. Then $au(t) \to 0$ uniformly and (A4), (3.2) imply

$$g_u(a) \geq \left(1 - \frac{\beta}{2}\right) \int_0^1 V(au)\, dt \geq \left(\frac{\beta}{2} - 1\right) \frac{c_1}{a^\beta} \int_0^1 \frac{1}{|u|^\beta}\, dt$$

and $g_u(a) \to +\infty$ as $a \to 0^+$.

Then for all $h > 0$ the equation $g_u(a) = h$ has a unique solution $a(u)$ and $a(u)u \in M_h$.

Again from (2.5) it follows that a depends continuosly on u. Then M_h is a deformation retract of Λ. From [9] it is known that $\mathrm{cat}_\Lambda(\Lambda) = \infty$ and that $\forall m$ it exists $\tilde{X} \subset \Lambda$, $\tilde{X}$ compact, with $\mathrm{cat}_\Lambda(\tilde{X}) \geq m$ and thus (2) follows.

To show that M_h is complete, let us take a sequence $\{u_n\} \subset M_h$ such that $u_n \to \bar{u}$ in H (actually, it suffices $\{u_n\} \subset M_h$ such that $u_n \to \bar{u}$ weakly and uniformly in $[0,1]$). We claim that $\bar{u} \in \Lambda$. Otherwise, there is an interval $I \subset [0,1]$ and an integer $\bar{n} > 0$ such that $|u_n(t)| < r\ \forall t \in I$, $\forall n > \bar{n}$. Then, using (A4), it follows readily, for n large

$$\begin{aligned} h &= \int_0^1 \left[V(u_n) + \tfrac{1}{2}V'(u_n)\cdot u_n\right] dt \\ &= \int_{[0,1]\setminus I} \left[V(u_n) + \tfrac{1}{2}V'(u_n)\cdot u_n\right] dt + \int_I \left[V(u_n) + \tfrac{1}{2}V'(u_n)\cdot u_n\right] dt \\ &\geq \left(1 - \frac{\beta}{2}\right) \int_0^1 V(u_n)\, dt + c_1, \end{aligned}$$

in contradiction with lemma 3.2. So $\bar{u} \in \Lambda$ and (recall that $u_n \to \bar{u}$ uniformly) $g(\bar{u}) = \lim g(u_n) = 0$. Then $\bar{u} \in M_h$, as required. ∎

To investigate the PS condition, we first show:

LEMMA 3.4. *Suppose that (A1-2-3-4-5) hold and let $u_n \in M_h$ be such that*

$$f(u_n) \leq C. \tag{3.4}$$

Then both $\|u_n'\|_{L^2}$ and $\|u_n\|_{L^\infty}$ are bounded.

PROOF: Inserting the expression for f in (3.4) we find

$$\tfrac{1}{4}\int_0^1 |u_n'|^2\,dt\cdot\int_0^1 V'(u_n)\cdot u_n\,dt \leq C \tag{3.5}$$

Moreover, using (A3), we find

$$h=\int_0^1 [V(u_n)+\tfrac{1}{2}V'(u_n)\cdot u_n]\,dt \leq \left(\frac{1}{2}-\frac{1}{\gamma}\right)\int_0^1 V'(u_n)\cdot u_n\,dt$$

hence

$$\int_0^1 V'(u_n)\cdot u_n \geq k, \quad \text{where} \quad k=\frac{h}{\frac{1}{2}-\frac{1}{\gamma}}>0. \tag{3.6}$$

From (3.5) and (3.6) one deduces

$$\|u_n'\|_{L^2}\leq c_1 \tag{3.7}$$

To show that $\|u_n\|_{L^\infty}$ is bounded, we argue by contradiction. Let $u_n=\xi_n+w_n$, with

$$\xi_n=\int_0^1 u_n\,dt.$$

From (3.7) it follows that $\|w_n'\|_{L^2}\leq c_1$, and up to a subsequence, $w_n\to\bar{w}$ uniformly. As a consequence, if $\|u_n\|_{L^\infty}\to\infty$, then $|\xi_n|\to\infty$. Then, from

$$\min|u_n(t)|\geq|\xi_n|-\max|w_n(t)|\geq|\xi_n|-c_2$$

it would follow that $|u_n(t)|\to\infty$ uniformly. Using (A5) we would then find

$$\limsup_{n\to\infty}\int_0^1 [V(u_n)+\tfrac{1}{2}V'(u_n)\cdot u_n]\,dt\leq 0 \tag{3.8}$$

contradicting $g(u_n)=h>0$ ∎

LEMMA 3.5. *Let (A1–2–3–4–5) hold. Then f satisfies PS on M_h, namely $\forall\{u_n\}\subset M_h$ such that*

$$f(u_n)\leq c \tag{3.9}$$

$$\nabla f_{|M_h}(u_n) \tag{3.10}$$

$\exists u_{n_k}\to\bar{u}\in M_h$ such that $\nabla f_{|M_h}(\bar{u})=0$.

PROOF: From (3.9) and Lemma 3.4 it follows that $\|u_n\|\leq$ const. Then, up to a subsequence, $u_n\to\bar{u}$ uniformly and weakly in H, and $\bar{u}\in M_h$ (see the proof of Lemma 3.3).

There results

$$\nabla f_{|M_h}(u) = \nabla f(u_n) - \lambda_n \nabla g(u_n) \tag{3.11}$$

From (3.10) it follows

$$\nabla f(u_n) - \lambda_n \nabla g(u_n) \to 0. \tag{3.12}$$

Multiplying by u_n one has

$$(\nabla f(u_n)|u_n) - \lambda_n(\nabla g(u_n)|u_n) \to 0.$$

Since

$$(\nabla f(u)|u) - \lambda(\nabla g(u)|u) = \tfrac{1}{2}\left[\tfrac{1}{2}\int_0^1 |u'|^2\,dt - \lambda\right]\cdot \int_0^1 [3V'(u)\cdot u + V''(u)u\cdot u]\,dt$$

and since, by (2.4) and (A1),

$$\int_0^1 [3V'(u_n)\cdot u_n + V''(u_n)u_n\cdot u_n]\,dt \to \int_0^1 [3V'(\bar u)\cdot \bar u + V''(\bar u)\bar u\cdot \bar u]\,dt > 0$$

we have that

$$\tfrac{1}{2}\int_0^1 |u_n'|^2\,dt - \lambda_n \to 0. \tag{3.13}$$

Finally, from (3.11) it follows

$$\nabla f_{|M_h}(u_n) = -\left(\int_0^1 V'(u_n)\cdot u_n\,dt\right)u_n'' + \left(\tfrac{1}{2}\int_0^1 |u_n'|^2\,dt - \tfrac{3}{2}\lambda_n\right)V'(u_n)$$
$$+ \tfrac{1}{2}\left(\tfrac{1}{2}\int_0^1 |u_n'|^2\,dt - \lambda_n\right)V''(u_n)u_n.$$

Since $\int_0^1 V'(u_n)\cdot u_n\,dt \to \int_0^1 V'(\bar u)\cdot \bar u > 0$, $V'(u_n) \to V'(\bar u)$ and $V''(u_n)u_n \to V''(\bar u)\bar u$ we deduce, using (3.12), that u_n'' converges (up to a subsequence) in H, and the result follows. ∎

We can now state the main result of this section.

THEOREM 3.6. *Suppose V satisfies (A1–2–3–4–5). Then $\forall h > 0$ problem (1.1–2) has a periodic solution $q(t)$, with $q(t) \neq 0 \quad \forall t$.*

PROOF: According to the variational principle (lemma 2.3) it suffices to find critical points u of $f_{|M_h}$ with $f(u) > 0$.

These critical points will be found by using the LS theory. Let

$$\mathcal{K}_m = \{ X \subset M_h \mid \mathrm{cat}_{M_h}(X) \geq m \}$$

and

$$b_m = \inf_{X\in\mathcal{K}_m} \max_X f \tag{3.14}$$

Note that $\mathcal{K}_m \neq \emptyset \quad \forall m$ by Lemma 3.3(2). Moreover b_m is a non-decreasing sequence with $0 \leq b_m < +\infty \quad \forall m$.

Since PS holds for $f_{|M_h}$ (Lemma 3.5) then the LS theory, extended to C^1 manifold in [**17**], implies that each b_m is a critical level and, if

$$b \equiv b_m = b_{m+1} = \cdots = b_{m+k}$$

then

$$\mathrm{cat}_{M_h}(Z_b) \geq k+1,$$

where

$$Z_b = \{\, u \in M_h \mid f(u) = b, \quad \nabla f_{|M_h}(u) = 0 \,\}$$

We claim $b_3 > 0$. If not, the preceding remark with $b \equiv b_1 = b_2 = b_3$ yields

$$\mathrm{cat}_{M_h}(Z_0) \geq 3.$$

But Lemma 2.1 implies that $Z_0 = \{\, u \equiv \text{const} \,\} \cap M_h$. The arguments of Lemma 3.3 show then that

$$Z_0 \cong S^{N-1}$$

which implies $\mathrm{cat}_{M_h}(Z_0) = 2$, a contradiction. Then the level b_3 carries a critical point u of $f_{|M_h}$ such that $f(u) > 0$, and this completes the proof. ∎

EXAMPLE 3.7. If $V(x) = -\frac{1}{|x|^b}$ with $b > 2$ then (A1–2–3–4–5) hold true. Note that for $b = 2$ $V_2(x) = -\frac{1}{|x|^2}$ verifies

$$3V_2'(x)\cdot x + V_2''(x)x\cdot x = 0 \qquad \forall x \neq 0. \tag{3.15}$$

On the other hand, all the periodic solutions of

$$q'' + 2\frac{q}{|q|^4} = 0$$

have energy $h = 0$.

EXAMPLE 3.8. Let

$$V(x) = -\frac{1}{|x|^b} + W(x)$$

with $b > 2$ and $W \in C^2(\Omega, \mathbf{R})$. Then (A1) and (A5) became respectively:

$$3W'(x)\cdot x + W''(x)x\cdot x < \frac{b(b-2)}{|x|^b} \qquad \forall x \neq 0 \tag{3.16}$$

$$W(x) + \tfrac{1}{2}W'(x)\cdot x \leq 0 \qquad \text{as} \quad |x| \to \infty. \tag{3.17}$$

As for (A2) (A3) and (A4), they are verified provided

$$W'(x)\cdot x > -\frac{b}{|x|^b} \qquad \forall x \neq 0 \tag{3.18}$$

$$\exists \gamma > 2 \colon W'(x)\cdot x + \gamma W(x) \leq \frac{\gamma - b}{|x|^b} \tag{3.19}$$

$$\exists \beta \in (2,\gamma],\ r > 0\colon \qquad W'(x)\cdot x + \beta W(x) \geq \frac{\beta - b}{|x|^b} \quad \forall 0 < |x| \leq r \tag{3.20}$$

For example, if $W(x) = -\frac{1}{|x|^c}$, $c > 2$, then (3.16) to (3.20) hold. In fact, (3.16), (3.17) and (3.18) are trivially verified; to satisfy (3.19) and (3.20) it sufficies to take, respectively, $\gamma > \max\{b, c\}$ and $2 < \beta < \min\{b, c\}$.

Moreover, let us remark that that if W is smooth on all $\mathbf{R}^N$ then (3.16), (3.18–19–20) impose no restrictions near $x = 0$. This is clear for (3.16) and (3.18) because $b > 2$. So for (3.19) and (3.20) it suffices to take any β, γ satisfying $2 < \beta < b < \gamma$.

§4 Existence results (Weak forces)

We study here the case when V behaves like $-\frac{1}{|x|^a}$ with $0 < a < 2$. In particular, this will include the interesting case of perturbations of the Kepler potential $-\frac{1}{|x|}$, such as

$$V(x) = -\frac{1}{|x|} + W(x).$$

Note that in the present case the meaning of a solution of (1.1–2) must be specified, because solutions passing through the singularity $x = 0$ could arise. The following definition has been introduced in [4]

DEFINITON 4.1. *We say that $q \in H^1(S^1;\mathbf{R}^N)$ is a solution of (1.1–2) if*

(1) *The set $\{\, t \in S^1 \mid q(t) = 0 \,\}$ has zero measure;*
(2) *in the set $\{\, t \in S^1 \mid q(t) \neq 0 \,\}$ q is of class C^2 and solves (1.1–2).*

In addition to (A1) and (A2) we suppose

(A3′) $\exists 0 < \alpha < 2$ such that $V'(x)\cdot x \geq -\alpha V(x) \quad \forall x \in \Omega$

(A4′) $\exists 0 < \delta < 2$ and $r > 0$ such that $V'(x)\cdot x \leq -\delta V(x) \quad \forall 0 < |x| \leq r$

(A5′) $\liminf\limits_{|x|\to\infty} \left[V(x) + \frac{1}{2}V'(x)\cdot x\right] \geq 0$

We will follow here a procedure similar to that of §3. First, as in lemma 3.1 we have:

LEMMA 4.2. *If (A3′–4′) hold then $\exists c_2 > 0$ such that*

$$V(x) \leq -\frac{c_2}{|x|^\alpha} \qquad \forall 0 < |x| \leq r.$$

PROOF: It suffices to repeat the arguments of lemma 3.1 using (A3′) and taking into account that (A4′) yields $\min\{-V(y) : |y| = r\} < 0$. ∎

We explicitely remark that (3.1) and (3.3) do not necessarely hold in the present situation.

The next Lemma substitute Lemma 3.3.

LEMMA 4.3. *Suppose (A1–2–3′–4′–5′) hold and let $h < 0$. Then:*

(1) $M_h \neq \emptyset$;
(2) $\operatorname{cat}_{M_h}(M_h) = \infty$; *more precisely* $\forall m \geq 0 \quad \exists X \subset M_h$, *$X$ compact, such that* $\operatorname{cat}_{M_h}(X) \geq m$.

PROOF: From (A1) it still follows that

$$g_u(a) \equiv g(au) = \int_0^1 [V(au) + \tfrac{1}{2}V'(au)\cdot au]\, dt$$

trictly monotone, and from (A5′) one deduces that

$$\liminf_{a\to\infty} g_u(a) \geq 0.$$

'hen, using (A3′) and Lemma 4.2 we deduce that, for each $u \in \Lambda$,

$$g_u(a) \leq \left(1-\frac{\delta}{2}\right)\int_0^1 V(au)\,dt \leq -\left(1-\frac{\delta}{2}\right)\frac{c_2}{a^\alpha}\int_0^1 \frac{1}{|u|^\alpha}\,dt$$

$g_u(a) \to -\infty$ as $a \to 0^+$. Therefore $\forall h < 0$, the equation $g_u(a) = h$ has a unique :tion and $M_h \neq \emptyset$.

'he remainder of the proof is the same as that of Lemma 3.3. ∎

ı the present situation, as we have already remarked, M_h is no more complete, so LS theory cannot be directly applied. To be able do deal with such a situation we lify V setting

$$V_\varepsilon(x) = V(x) - \frac{\varepsilon}{|x|^2}, \qquad \varepsilon > 0$$

:EMARK 4.4. Let

$$g_\varepsilon(u) = \int_0^1 [V_\varepsilon(u) + \tfrac{1}{2}V_\varepsilon'(u)\cdot u] \qquad \text{and} \qquad M_{h,\varepsilon} = \{\, u \in \Lambda \,:\, g_\varepsilon(u) = h\,\}$$

'ording to example 3.7 there resut

$$g_\varepsilon(u) = g(u)$$

hence $M_{h,\varepsilon} = M_h$. Moreover

$$V_\varepsilon'(x)\cdot x = V'(x)\cdot x + 2\frac{\varepsilon}{|x|^2} \geq V'(x)\cdot x > 0$$

›refore Lemma 2.3 applies and, $\forall h < 0$, the critical points of f_ε on M_h such that ı) > 0 give rise to periodic solutions of

$$q'' + V'(q) + 2\varepsilon\frac{q}{|q|^4} = 0 \tag{4.1}$$

h energy

$$\tfrac{1}{2}|q'(t)|^2 + V(q(t)) - \frac{\varepsilon}{|q|^2} = h. \tag{4.2}$$

In order to find critical points of f_ε on M_h we state some lemmas which are the ınterpart of Lemmas 3.4, 3.5. We always suppose (A1–2–3′–4′–5′).

LEMMA 4.5. *If $u_n \in M_h$ is such that*

$$f_\epsilon(u_n) \le C \tag{4.3}$$

then $\|u_n'\|_{L^2}$ and $\|u_n\|_{L^\infty}$ are bounded.

PROOF: From (4.3) it follows

$$\begin{aligned} C \ge f_\epsilon(u_n) &= \tfrac{1}{4}\int_0^1 |u_n'|^2\,dt \cdot \int_0^1 \left[V'(u_n)\cdot u_n + 2\frac{\epsilon}{|u_n|^2}\right] dt \\ &\ge \tfrac{1}{4}\int_0^1 |u_n'|^2\,dt \cdot \int_0^1 V'(u_n)\cdot u_n\,dt. \end{aligned} \tag{4.4}$$

If $u_n \in M_h$ then (A3′) implies

$$h = \int_0^1 [V(u_n) + \tfrac{1}{2}V'(u_n)\cdot u_n]\,dt \ge \left(\frac{1}{2} - \frac{1}{\alpha}\right)\int_0^1 V'(u_n)\cdot u_n\,dt,$$

hence

$$\int_0^1 V'(u_n)\cdot u_n \ge k := \frac{h}{\frac{1}{2} - \frac{1}{\alpha}} > 0.$$

This and (4.4) yield

$$\int_0^1 |u_n'|^2\,dt \le c_1 := \frac{4c}{k}. \tag{4.5}$$

From (4.5) and (A5′) it then follows, as in Lemma 3.4, that $\|u_n\|_{L^\infty} \le c_2$. ∎

LEMMA 4.6. *f_ϵ satisfies PS on M_h.*

PROOF: Let $u_n \in M_h$ be a PS-sequence. By Lemma 4.5 one has

$$\|u_n\| \le C$$

hence $u_n \to \bar{u}$ uniformly and weakly in H. We claim that $\bar{u} \in M_h$. Indeed, in view of (2.4), it siffices to show that $\bar{u} \in \Lambda$. We shall prove this by contradiction. First, let $\bar{u} \equiv 0$. Then $u_n \to 0$ uniformly and (A4') and Lemma 4.2 imply, for n large,

$$\begin{aligned} h &= \int_0^1 [V(u_n) + \tfrac{1}{2}V'(u_n)\cdot u_n]\,dt \le \left(1 - \frac{\delta}{2}\right)\int_0^1 V(u_n)\,dt \\ &\le -c_2\left(1 - \frac{\delta}{2}\right)\int_0^1 \frac{1}{|u_n|^\alpha}\,dt. \end{aligned}$$

Since the last term tends to $-\infty$, we have a contradiction.

Next, let $\bar{u} \in \partial\Lambda$ with $\bar{u} \not\equiv 0$ (hence $\bar{u} \not\equiv$ const.). There results

$$f_\epsilon(u_n) = \frac{1}{2}\int_0^1 |u_n'|^2 \cdot \int_0^1 [h - V_\epsilon(u_n)].$$

Since

$$V_\varepsilon(x) = V(x) - \frac{\varepsilon}{|x|^2} \le -\frac{\varepsilon}{|x|^2},$$

then V_ε satisfies the SF condition and

$$\int_0^1 [h - V_\varepsilon(u_n)] \to \infty.$$

Moreover, there results

$$0 < \int_0^1 |\bar{u}'|^2 \le \liminf \int_0^1 |u_n'|^2$$

and we reach a contradiction, proving that $\bar{u} \in M_h$.

The rest of the proof follows as in Lemma 3.5. ∎

We explicitely remark that the arguments of lemma 4.6 actually show that the sublevels $\{ u \in M_h \mid f_\varepsilon(u) < c \}$ are complete.

The preceding result allows us to apply the LS theory to f_ε on M_h; repeating the arguments of Theorem 3.6 we find

LEMMA 4.7. *$\forall \varepsilon > 0 \quad \exists u_\varepsilon \in M_h$ such that $\nabla f_{\varepsilon|M_h}(u_\varepsilon) = 0$ satisfying*

$$f(u_\varepsilon) = b_m = \inf_{X \in \mathcal{K}_m} \max_X f_\varepsilon, \qquad m \ge 3$$

and $u_\varepsilon \ne const$. Moreover, setting

$$\omega_\varepsilon^2 = \frac{\int_0^1 V_\varepsilon'(u_\varepsilon) \cdot u_\varepsilon \, dt}{\int_0^1 |u_\varepsilon'|^2 \, dt} \tag{4.6}$$

and $y_\varepsilon(t) = u_\varepsilon(\omega_\varepsilon t)$, there results

$$y_\varepsilon''(t) + V_\varepsilon'(y_\varepsilon(t)) = 0 \tag{4.7}$$

and

$$\tfrac{1}{2}\omega_\varepsilon^2 |u_\varepsilon'(t)|^2 + V_\varepsilon(u_\varepsilon(t)) = h. \tag{4.8}$$

REMARK 4.8. It will be convenient to take u_ε in such a way that

$$f_\varepsilon(u_\varepsilon) = \inf_{X \in \mathcal{K}_m} \max_X f_\varepsilon$$

where $m \ge 3$ is fixed independently from ε. In particular, in the following lemma we will take $m = 3$. ∎

In the sequel.our plan is to show that u_ε converges to some u^* which gives rise to a solution y^* of (1.1–2). For this, some estimates are in order.

LEMMA 4.9. *$\exists k > 0$ such that $\|u_\varepsilon\| \le k$ and $u_\varepsilon \to u^*$ uniformly.*

PROOF: As anticipated in Remark 4.8, we have that

$$b_\varepsilon := f_\varepsilon(u_\varepsilon) = \inf_{\mathrm{cat}_{M_h}(A)\ge 3} \max_{u\in A} f_\varepsilon(u).$$

Since $V_\varepsilon(x) = V(x) - \frac{\varepsilon}{|x|^2} \ge V(x) - \frac{1}{|x|^2} \quad \forall \varepsilon \le 1$, then there results

$$f_\varepsilon(u) \le f_1(u) \qquad \forall 0 < \varepsilon \le 1,\ \forall u \in \Lambda.$$

Thus

$$f_\varepsilon(u_\varepsilon) = b_\varepsilon \le b := \inf_{\mathrm{cat}_{M_h}(A)\ge 3} \max_{u\in A} f_1 \qquad \forall 0 < \varepsilon \le 1 \tag{4.9}$$

and the result follows from Lemma 4.5. ∎

LEMMA 4.10. *There results*

(1) *$V(u^*(t)) \not\equiv h$;*
(2) *$u^*(t) \not\equiv 0$.*

PROOF: (1) If not, $u^*(t) \neq 0$ for all t and one has

$$V(u_\varepsilon(t)) \to V(u^*(t)) = h$$
$$V'(u_\varepsilon(t))\cdot u_\varepsilon(t) \to V'(u^*(t))\cdot u^*(t)$$

uniformly. Therefore

$$h = g(u_\varepsilon) = \int_0^1 [V(u_\varepsilon) + \tfrac{1}{2}V'(u_\varepsilon)\cdot u_\varepsilon]\,dt \to \int_0^1 [V(u^*) + \tfrac{1}{2}V'(u^*)\cdot u^*]\,dt$$
$$= h + \tfrac{1}{2}\int_0^1 V'(u^*)\cdot u^*\,dt$$

Hence $\int_0^1 V'(u^*)\cdot u^* = 0$, in contradiction with (A2).

(2) If not, $u^* \equiv 0$ and $u_\varepsilon \to 0$ uniformly. Then, using (A4′) and the fact that $u_\varepsilon \in M_h$, it follows (for ε small enough)

$$h = \int_0^1 [V(u_\varepsilon) + \tfrac{1}{2}V'(u_\varepsilon)\cdot u_\varepsilon]\,dt \le \left(1 - \frac{\delta}{2}\right)\int_0^1 V(u_\varepsilon)$$

namely

$$\int_0^1 V(u_\varepsilon) \ge \frac{h}{1 - \frac{\delta}{2}}.$$

On the other side, since $u_\varepsilon \to 0$ uniformly, from lemma 4.1 we infer

$$\int_0^1 V(u_\varepsilon) \to -\infty \qquad (\varepsilon \to 0^+)$$

a contradiction. ∎

LEMMA 4.11. *$\exists \delta, \Delta > 0$ such that*

$$\delta \leq \omega_\varepsilon \leq \Delta.$$

PROOF: From Lemma 4.10 we deduce that there exists a closed interval I such that I has positive measure and

$$u^*(t) \neq 0, \quad V(u^*(t)) \neq h \qquad \forall t \in I.$$

Integrating (4.8) in I, one finds

$$\tfrac{1}{2}\omega_\varepsilon^2 \int_I |u_\varepsilon'|^2\,dt + \int_I V_\varepsilon(u_\varepsilon)\,dt = h|I| \tag{4.10}$$

Since

$$\int_I |u_\varepsilon'|^2\,dt \leq \int_0^1 |u_\varepsilon'|^2\,dt \leq C$$

then (4.10) implies

$$\tfrac{1}{2}\omega_\varepsilon^2 \geq \frac{\int_I [h - V_\varepsilon(u_\varepsilon)]\,dt}{C}.$$

But on I one has that $u_\varepsilon \to u^*$ uniformly and that $u^* \neq 0$, so that

$$\int_I [h - V_\varepsilon(u_\varepsilon)]\,dt \to \int_I [h - V(u^*)]\,dt \qquad (\varepsilon \to 0^+).$$

From (4.8) $h - V_\varepsilon(u_\varepsilon) \geq 0$ and thus, from the definition of I, one has

$$\int_I [h - V(u^*)]\,dt > 0.$$

This shows that $\omega_\varepsilon \geq \delta > 0$.

To prove $\omega_\varepsilon \leq \Delta$, we start using (4.8) and (4.9) to find

$$\tfrac{1}{2}\omega_\varepsilon^2 \int_0^1 |u_\varepsilon'|^2\,dt = \int_0^1 [h - V_\varepsilon(u_\varepsilon)] = \frac{f_\varepsilon(u_\varepsilon)}{\frac{1}{2}\int_0^1 |u_\varepsilon'|^2\,dt} \leq \frac{b}{\frac{1}{2}\int_0^1 |u_\varepsilon'|^2\,dt}.$$

Then

$$\tfrac{1}{4}\left(\tfrac{1}{2}\int_0^1 |u_\varepsilon'|^2\,dt\right)^2 \leq \frac{b}{\omega_\varepsilon^2}.$$

If $\omega_\varepsilon \to \infty$, it follows that $\int_0^1 |u_\varepsilon'|^2\,dt \to 0$ and hence both u_ε and y_ε converges uniformly to some constant $\xi \in \mathbf{R}^N$. From Lemma 4.10 $\xi \neq 0$ and $V(\xi) \neq h$. Using now (4.7) we have (since $V_\varepsilon'(y_\varepsilon)$ converges uniformly to $V'(\xi)$) that y_ε converges in C^2 to ξ. Finally, passing to the limit into (4.8), we find $V(\xi) = h$, a contradiction. ∎

We are now in position to state the main result of this section

THEOREM 4.12. *Suppose (A1-2-3'-4'-5') hold. Then $\forall h < 0$ (1.1-2) has a non-costant periodic solution.*

PROOF: We shall show that u^* gives rise to a solution of (1.1-2), in the sense of definition 4.1.

Taking into account the preceding lemmas, this follows in a rather standard way. We report, for the reader's convenience, the complete proof.

Let

$$J = \{\, t \in [0,1] \mid u^*(t) = 0 \,\}.$$

From (4.10), with J replacing I and using lemmas 4.8 and 4.11 we deduce

$$\begin{aligned} \int_J V_\epsilon(u_\epsilon)\, dt &= |J| h - \tfrac{1}{2}\omega_\epsilon^2 \int_J |u_\epsilon'|^2\, dt \\ &\geq |J| h - \tfrac{1}{2}\delta^2 k^2. \end{aligned} \tag{4.11}$$

But $u_\epsilon \to 0$ uniformly on J and hence, if J has positive measure, from Lemma 4.1 we have that

$$\int_I V_\epsilon(u_\epsilon)\, dt \to -\infty,$$

in contradiction with (4.11). Thus J has zero measure.

Let $K_n \subset [0,1]\setminus J$ be an increasing sequence of compact sets with

$$\cup_{n\geq 1} K_n = [0,1]\setminus J,$$

and set

$$K_n^* = \{\, u^*(t) \mid t \in K_n \,\}.$$

Each $K_n^* \subset \Omega$ is compact and has a neighborhood $\mathcal{N}_n$ such that $\overline{\mathcal{N}}_n \subset \Omega$. Then $V_\epsilon \to V$ in $C^1(\mathcal{N}_n, \mathbf{R})$ and therefore

$$V_\epsilon'(u_\epsilon(t)) \to V'(u^*(t)) \qquad \text{uniformly in} \quad K_n.$$

Since u_ϵ solves

$$\omega_\epsilon^2 u_\epsilon'' + V_\epsilon'(u_\epsilon) = 0$$

and $\omega_\epsilon \to \omega^* \neq 0$ (Lemma 4.11), it follows that

$$u_\epsilon \to u^* \qquad \text{in} \quad C^2(K_n, \mathbf{R}^N)$$

and

$$\omega^* u^{*\prime\prime} + V'(u^*) = 0 \qquad \text{on} \quad K_n.$$

Since $\cup K_n = [0,1]\setminus J$, then

$$\omega^* u^{*\prime\prime} + V'(u^*) = 0 \qquad \forall t \in [0,1]\setminus J$$

and $y^*(t) = u^*(\omega^* t)$ solves

$$y^{*\prime\prime} + V'(y^*) = 0 \qquad \forall t \in [0,1]\backslash J.$$

The energy conservation (1.2) follows directly from (4.8). ∎

EXAMPLE 4.13. Any $V(x) = -\frac{1}{|x|^a}$ with $0 < a < 1$ satisfies (A1–2–3′–4′–5′).

To enlight the signification of Theorem 4.11 we discuss below a specific example concerning a perturbation of the Kepler potential.

EXAMPLE 4.14. Let us take

$$V(x) = -\frac{1}{|x|} + W(x)$$

with $W \in C^2(\Omega, \mathbf{R})$. In this case (A1) and (A5′) became, respectively

$$3W'(x)\cdot x + W''(x)x\cdot x > -\frac{1}{|x|} \qquad \forall x \in \Omega \tag{4.12}$$

$$\liminf_{|x|\to\infty} \left[W(x) + \tfrac{1}{2}W'(x)\cdot x\right] \geq 0 \tag{4.13}$$

Similarly (A2) and (A3') became, respectively

$$W'(x)\cdot x > -\frac{1}{|x|} \qquad \forall x \in \Omega \tag{4.14}$$

$$\exists 0 < \alpha < 2 \quad \text{such that} \quad W'(x)\cdot x + \alpha W(x) \geq \frac{\alpha - 1}{|x|} \qquad \forall x \in \Omega \tag{4.15}$$

while (A4′) gives

$$\exists \alpha \leq \delta < 2 \quad \text{and} \quad r > 0 \quad \text{such that} \quad W'(x)\cdot x + \delta W(x) \leq \frac{\delta - 1}{|x|} \qquad \forall 0 < |x| \leq r \tag{4.16}$$

For example any $W(x) = -\frac{1}{|x|^a}$ with $0 < a < 2$ verifies the above conditions (take α, δ such that $0 < \alpha < a < \delta < 2$ and $0 < \alpha < 1 < \delta < 2$). ∎

The case discussed in example 4.13 allows us to derive Theorem 1.1.

PROOF OF THEOREM 1.1: It sufficies to note that (W1), (W2) and (W4) imply (4.12), (4.14) and (4.13) respectively. To satisfy (4.15) we take $\alpha = 1$ and use (W3). Lastly, since W is smooth at $x = 0$, then $W'(x)\cdot x + \delta W(x)$ is bounded in any neighborhood of the origin for any δ, so that (4.16) holds taking $\delta > 1$. ∎

§5. Other existence results

From the conservation of energy it follows that any solution q of (1.1–2) is such that

$$V(q(t)) \leq h \qquad \forall t$$

Therefore it is natural to expect that assumptions (A1–2–3–4) or (A1–2–3′–4′) need to be verified only in

$$\{ x \in \Omega \mid V(x) \leq h \} := \Omega_h. \tag{5.1}$$

We will discuss only the "Weak force" case; in the "Strong force" one, $V < 0$ and $h > 0$ imply $\Omega_h = \Omega$.

Let us denote by D_h the connected component of Ω_h such that $0 \in \bar{D}_h$; and let $\partial D_h = \{ x \in D_h \mid V(x) = h \}$.

THEOREM 5.1. *Let $h < 0$ be given. Suppose that $\bar{D}_h$ is compact and $V: \Omega \to \mathbf{R}$ satisfy (A4′) and*

($A1_h$) $3V'(x)\cdot x + V''(x)x\cdot x > 0 \qquad \forall x \in D_h$;

($A2_h$) $V'(x)\cdot x > 0 \qquad \forall x \in D_h$;

($A3'_h$) $\exists 0 < \alpha' < 2$ *such that* $V'(x)\cdot x \geq -\alpha' V(x) \quad \forall x \in D_h$;

($A6_h$) $V \in C^4$ *in a neighborhood of* ∂D_h *and* $\theta = \max_{\xi \in \partial D_h} [V''(\xi)\xi\cdot\xi] < 0$.

Then (1.1–2) has, corresponding to such a value of h, a periodic solution.

PROOF: Since $V'(\xi)\cdot\xi > 0$ on ∂D_h, then ∂D_h is star-shaped with respect to $x = 0$. Set

$$G_h = \Omega \setminus D_h.$$

For every $x \in G_h$ there exists a unique $\xi \in \partial D_h$ and $s > 1$ such that

$$x = s\xi.$$

We claim that there exist functions $A, B, S \in C^2(\partial D_h)$ such that, letting

$$\Phi(x) = -\frac{A(\xi)}{|s - S(\xi)|} + B(\xi)$$

the modified potential

$$\tilde{V}(x) = \begin{cases} V(x) & x \in D_h \\ \Phi(x) & x \in G_h \end{cases}$$

is of class C^2 in Ω.

To see this, it suffices to take

$$A(\xi) = \frac{(V''(\xi)\xi\cdot\xi)^2}{4V'(\xi)\cdot\xi}$$

$$B(\xi) = h - \tfrac{1}{2}V''(\xi)\xi\cdot\xi$$

$$S(\xi) = 1 - \frac{V''(\xi)\xi\cdot\xi}{2V'(\xi)\cdot\xi}$$

Let us remark that $(A2_h)$ and $(A5_h)$ imply

$$A(\xi) > 0$$
$$B(\xi) > h - \theta > h$$
$$S(\xi) < 1.$$

Lastly we rescale $\tilde{V}$ setting

$$\hat{V}(x) = \tilde{V}(x) + L$$

where $L = \max(\theta - h, 0) \geq 0$. Since $\hat{V}(x) = V(x) + L$ in D_h and $L \geq 0$, it follows immediately that $\hat{V}$ satisfies (A1–2) and (A3′) in D_h. As for (A4′), one has that, for $\hat{\delta} > \delta$ there results

$$V'(x)\cdot x \leq -\delta V(x) = -\hat{\delta} V(x) + (\hat{\delta} - \delta) V(x).$$

Since $V(x) \to -\infty$ as $x \to 0$, taking r small enough one deduces

$$V'(x)\cdot x \leq -\hat{\delta} V(x) - \hat{\delta} L = -\hat{\delta}(V(x) + L),$$

and (A4′) follows.

Next, we take $x = s\xi \in G_h$. For such an x there results

$$\begin{aligned} \hat{V}'(x)\cdot x &= \frac{d}{d\lambda}\tilde{V}'(\lambda x)_{|\lambda=1} \\ &= \frac{d}{d\lambda}\Phi(\lambda x)_{|\lambda=1} \\ &= \frac{A(\xi)}{|s - S(\xi)|} \end{aligned} \tag{5.1}$$

and, similarly

$$\hat{V}''(x)x\cdot x = -\frac{sA(\xi)}{|s - S(\xi)|^2} - \frac{A(\xi)}{|s - S(\xi)|}.$$

Hence

$$3\hat{V}'(x)\cdot x + \hat{V}''(x)x\cdot x = \frac{A(\xi)}{|s - S(\xi)|} > 0$$

and (A1) holds.

Since $A(\xi) > 0$, (A2) follows form (5.1), and since $\bar{D}_h$ is compact and $B(\xi) \geq h - \theta$, then

$$\begin{aligned} \liminf_{|x|\to\infty}[\hat{V}(x) + \tfrac{1}{2}\hat{V}'(x)\cdot x] &= \liminf_{s\to\infty}\left[-\tfrac{1}{2}\frac{A(\xi)}{|s - S(\xi)|} + B(\xi) + L\right] \\ &\geq h - \theta + L \geq 0, \end{aligned}$$

and also (A5′) holds.

Finally we prove that (A3′) holds in G_h. It is enough to notice that

$$\hat{V}'(x)\cdot x = \frac{A(\xi)}{|s - S(\xi)|} \geq \frac{A(\xi)}{|s - S(\xi)|} - (B(\xi) + L) = -\hat{V}(x) \qquad \forall x \in G_h.$$

We are now in position to apply Theorem 4.12, from which it follows the existence of a periodic solution of

$$\begin{cases} q'' + \hat{V}'(q) = 0 \\ \frac{1}{2}|q'|^2 + \hat{V}(q) = e \end{cases} \tag{5.2}$$

for all $e < 0$. Since $L < -h$, then $e = h + L < 0$ is an admissible value of the energy. For such a choice of e (5.2) becames

$$\begin{cases} q'' + \tilde{V}'(q) = 0 \\ \frac{1}{2}|q'|^2 + \tilde{V}(q) = h \end{cases}$$

and hence $\tilde{V}(q(t)) \leq h$. For $x = s\xi \in G_h$ one has that

$$\tilde{V}(x) = \Phi(s\xi) > \Phi(\xi) = h.$$

therefore $q(t) \in D_h$, and $\tilde{V}(q) = V(q)$ and q satisfies (1.1-2). ∎

As application of theorem 5.1 we can prove Theorem 1.2.

PROOF OF THEOREM 1.2: We first notice that D_h will be compact provided ε is small enough. Hence there exist constants m, m', m'', M, M', M'' such that

$$m \leq U(x) \leq M$$
$$m' \leq U'(x)\cdot x \leq M'$$
$$m'' \leq U''(x)x\cdot x \leq M''$$

for all $x \in \bar{D}_h$. It is then immediate to check that, for $|\varepsilon|$ small enough, $(A1_h\text{-}2_h\text{-}3'_h\text{-}6_h)$ are verified. ∎

REMARK 5.2. We notice that $\bar{\varepsilon}$ of theorem 1.2 can be explicitely estimated in terms of h and of the C^2 norm of U. For example, if $\|U\|_{L^\infty} = \|U'\|_{L^\infty} = \|U''\|_{L^\infty} = 1$, it is not difficult to see that $\bar{\varepsilon}$ can be taken to be $\min\{\frac{|h|}{2}, \frac{|h|^3}{4(3|h|+2)}\}$.

The following example is related to Theorem 1.1 and can be obtained as a stright application of Theorem 5.1:

EXAMPLE 5.3. Let $h < 0$ be given and suppose $V(x) = -\frac{1}{|x|} + W(x)$ with W smooth in Ω and such that

(1) $V(x) \to -\infty$ as $|x| \to 0$ and $W(x) \geq 0 \quad \forall x$ with $|x| = \frac{1}{|h|}$;

(2) $3W'(x)\cdot x + W''(x)x\cdot x > -\frac{1}{|x|} \quad \forall |x| \leq \frac{1}{|h|}$;

(3) $W'(x)\cdot x > -\frac{1}{|x|} \quad \forall |x| \leq \frac{1}{|h|}$;

(4) $W''(x)x\cdot x < \frac{2}{|x|} \quad \forall |x| \leq \frac{1}{|h|}$;

Then (1.1-2) has at least one periodic solution.

We point out that (1) is used only to show that $\{ x \mid V(x) \leq h \} \subset \{ x \mid |x| \leq \frac{1}{|h|} \}$.

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