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SECOND COLLEGE ON VARIATIONAL PROBLEMS IN ANALYSIS

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Rearrangements in variational problems and applications

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These are preliminary lecture notes, intended only for distribution to participants

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Example Loss of compactness in a variational problem on an unbounded domain

Consider the problem: (P) Minimise $\int_{-\infty}^{\infty} u^2 + (u')^2$

$$\int_{-\infty}^{\infty} u^3 = 1$$
$$u \in W_0^{1,2}(\mathbb{R})$$

If $u \in W_0^{1,2}(\mathbb{R})$ is any function then $u_t(x) = u(x-t)$ satisfies

$u_t \to 0$ weakly in $W_0^{1,2}(\mathbb{R})$ as $t \to \infty$, and $\| \ \|_{1,2}$ and $\| \ \|_3$ are unchanged.

It is now not difficult to show that if $\{u^{(n)}\}_{n=1}^{\infty}$ is any bounded sequence, then a sequence $\{t(n)\}_{n=1}^{\infty}$ can be found with $u^{(n)}_{t(n)} \to 0$ weakly as $n \to \infty$.

Hence, any minimising sequence for problem (P) can be replaced by a minimising sequence that converges weakly to 0.

There is thus no hope of proving that *every* minimising sequence has a subsequence that converges to a minimiser.

Possible Remedy Seek a procedure that replaces minimising sequences by minimising sequences that have better convergence properties.

Rearrangement procedures are effective in some examples.

μ_n will denote n-dimensional Lebesgue measure.

§1 Rearrangement Procedures and Rearrangement Inequalities.

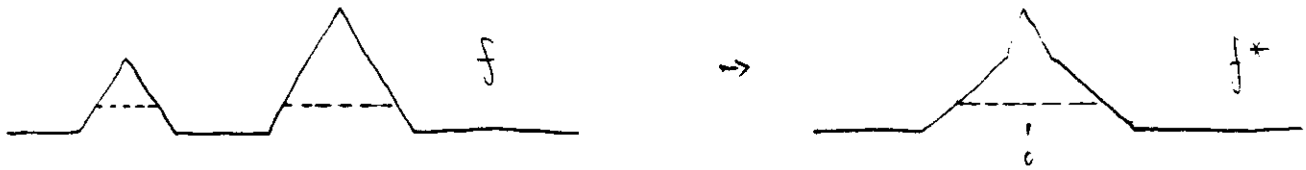
In this section *functions* f are real, non-negative, μ_n-measurable, s.t. $\mu_n(f^{-1}[\alpha,\infty)) < \infty$ $\forall \alpha > 0$

Definition Two non-negative meas. functions f and g on $\mathbb{R}^n$ are called *rearrangements* of one another if the sets

$$f^{-1}(\alpha,\infty) \quad , \quad g^{-1}[\alpha,\infty)$$

have the same n-dimensional Lebesgue measure

<u>Example 1</u> Symmetric decreasing rearrangement of a function of a real variable



The (essentially unique) symmetric decreasing rearrangement of f is constructed as follows:

Define the <u>distribution function</u> of f by
$$F(\alpha) = \mu_n f^{-1}[\alpha, \infty) \qquad \alpha > 0.$$

Let
$$f^*(s) = \begin{cases} \max\{\alpha > 0 \mid F(\alpha) \geq 2|s|\} & \text{if there are any such } \alpha \\ 0 & \text{otherwise.} \end{cases}$$

Thus, each set $f^{-1}[\alpha, \infty)$ is replaced by an interval of the same measure, symmetric about 0.

<u>Example 2</u> Steiner symmetrisation of a function about a plane H: $\underset{\sim}{x} \cdot \hat{u} = 0$.

Choose cartesian coordinates $x_1, \dots, x_n$ in $\mathbb{R}^n$ such that H is the plane $x_n = 0$; define
$$f^*(x_1, \dots, x_n) = f(x_1, \dots, x_{n-1}, \cdot\,)^*$$
i.e. symmetric decreasing rearrangement in the last variable.

Thus the restriction of f to each line perpendicular to H is rearranged as a function symmetric decreasing about H. To see that f^* is then a rearrangement of f, apply Fubini's Theorem:
$$\mu\{\underset{\sim}{x} \mid f^*(\underset{\sim}{x}) \geq \alpha\} = \int_{\mathbb{R}^{n-1}} \mu_1\{x_n \mid f^*(x_1, \dots, x_n) \geq \alpha\}\, dx_1 \cdots dx_{n-1}$$
$$= \int_{\mathbb{R}^{n-1}} \mu_1\{x_n \mid f(x_1, \dots, x_n) \geq \alpha\}\, dx_1 \cdots dx_{n-1}$$
$$= \mu\{\underset{\sim}{x} \mid f(\underset{\sim}{x}) \geq \alpha\}.$$

Example 3 Schwartz symmetrisation. f^* is the (essentially unique) rearrangement of f as a decreasing function of $|\underset{\sim}{x}|$. Thus each set $f^{-1}[\alpha,\infty)$ is replaced by a ball of the same n-dimensional measure.

$$f^*(\underset{\sim}{x}) = \max\{\alpha \mid F(\alpha) \geq |\underset{\sim}{x}|^n \omega_n\} \qquad \omega_n = \text{vol. of unit ball.}$$

1. Theorem If f, g are functions on $\mathbb{R}^n$ and f is a rearrangement of g then $\|f\|_p = \|g\|_p$ for $1 \leq p \leq \infty$.

Proof. Case $1 \leq p < \infty$: Notice that

$$\mu\{x \mid f(x) \geq \alpha\} = \mu\{x \mid g(x) \geq \alpha\} \qquad \forall \alpha > 0$$
$$\iff \mu\{x \mid f(x) \geq \beta^{1/p}\} = \mu\{x \mid g(x) \geq \beta^{1/p}\} \qquad \forall \beta > 0$$
$$\iff \mu\{x \mid f(x)^p \geq \beta\} = \mu\{x \mid g(x)^p \geq \beta\} \qquad \forall \beta > 0$$
$$\iff f^p \text{ is a rearrangement of } g^p.$$

So it is enough to look at the case $p = 1$.

Now

$$\|f\|_1 = \int_{\mathbb{R}^n} f(x)\,dx = \int_{\mathbb{R}^n} \int_0^{f(x)} dt\,dx$$

$$= \int_0^\infty \int_{f(x) \geq t} dx\,dt = \int_0^\infty \mu_n f^{-1}[t,\infty)\,dt$$

$$= \int_0^\infty \mu_n g^{-1}[t,\infty)\,dt$$

$$= \cdots = \|g\|_1.$$

So $\|f\|_1 = \|g\|_1$.

case $p = \infty$: If $\alpha > \|f\|_\infty$ then $\mu f^{-1}[\alpha,\infty) = 0$, so $\mu g^{-1}[\alpha,\infty) = 0$, so $\alpha > \|g\|_\infty$. Hence $\|f\|_\infty \geq \|g\|_\infty$.

Similarly $\|f\|_\infty \leq \|g\|_\infty$

Remark: Moreover $\int \varphi \circ f = \int \varphi \circ g$ for any non-negative Borel function φ.

2. Theorem Let f, g be functions on $\mathbb{R}$. Let $*$ denote symmetric decreasing rearrangement. Then

$$\int_{-\infty}^\infty fg \leq \int_{-\infty}^\infty f^* g^*.$$

Proof.

Simple case: f and g are indicator functions of sets A, B;

$$f(x) = 1_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}.$$

Then $1_A^* = 1_{[-\alpha,\alpha]}$ $\quad \alpha = \frac{1}{2}\mu(A)$, $\quad 1_B^* = 1_{[-\beta,\beta]}$ $\quad \beta = \frac{1}{2}\mu(B)$.

Now

$$\int_{-\infty}^{\infty} 1_A 1_B = \int_{-\infty}^{\infty} 1_{A\cap B} = \mu(A \cap B) \le \min\{\mu(A), \mu(B)\}$$

$$= \min\{2\alpha, 2\beta\} = \mu\left([-\alpha,\alpha] \cap [-\beta,\beta]\right) = \int_{-\infty}^{\infty} 1_{[-\alpha,\alpha]} 1_{[-\beta,\beta]}$$

$$= \int_{-\infty}^{\infty} 1_A^* 1_B^*.$$

General case

$$\int_{-\infty}^{\infty} f(x)g(x)\,dx = \int_{-\infty}^{\infty} \int_0^{f(x)} dt \int_0^{g(x)} ds\,dx$$

$$= \int_0^{\infty}\int_0^{\infty} \int_{\substack{f(x)\ge t \\ g(x)\ge s}} 1\,dx \quad ds\,dt$$

$$= \int_0^{\infty}\int_0^{\infty} \int_{-\infty}^{\infty} 1_{\mathcal{F}(t)} 1_{\mathcal{G}(s)}\,dx\,ds\,dt \qquad \mathcal{F}(t) = f^{-1}[t,\infty) \text{ etc}$$

$$\le \int_0^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty} 1^*_{\mathcal{F}(t)} 1^*_{\mathcal{G}(s)}\,dx\,ds\,dt$$

$$= \int_0^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty} 1_{\mathcal{F}^*(t)} 1_{\mathcal{G}^*(t)}\,dx\,ds\,dt$$

$\mathcal{F}^*(t)$ is the symmetric interval of length $\mu\mathcal{F}(t)$; identical with $f^{*-1}[t,\infty)$

$$= \int_0^{\infty}\int_0^{\infty} \int_{\substack{f^*(x)\ge t \\ g^*(x)\ge s}} 1\,dx\,ds\,dt$$

$$= \int_{-\infty}^{\infty} f^*(x)g^*(x)\,dx$$

Corollaries The same inequality when $*$ represents Steiner- or Schwarz symmetrizations of functions on $\mathbb{R}^N$ follows by exactly the same method.

3. Theorem Let f, g be functions on $\mathbb{R}^n$, * Steiner- or Schwarz-symmetrisation. Then $\|f^* - g^*\|_2 \le \|f - g\|_2$.

Proof.
$$\|f^* - g^*\|_2^2 = \|f^*\|_2^2 + \|g^*\|_2^2 - 2\int_{\mathbb{R}^n} f^* g^*$$
$$= \|f\|_2^2 + \|g\|_2^2 - 2\int_{\mathbb{R}^n} f^* g^* \qquad (\text{Th } 1)$$
$$\le \|f\|_2^2 + \|g\|_2^2 - 2\int_{\mathbb{R}^n} fg \qquad (\text{Th } 2)$$
$$= \|f - g\|_2^2 .$$

Remark $\|f^* - g^*\|_p \le \|f - g\|_p$ for $1 \le p \le \infty$ in fact.
Thus * is a continuous map from L^p into L^p.

4. Theorem Riesz's Inequality. If f, g, h are functions on $\mathbb{R}^n$ and * is Steiner- or Schwarz symmetrisation then
$$\int_{\mathbb{R}^n} f(x) g(x-y) h(y)\, dx\, dy \le \int_{\mathbb{R}^n} f^*(x) g^*(x-y) h^*(y)\, dx\, dy$$

Proof of special case where one of f, g, h is already symmetric.
Since f, g, h can be permuted by a change of variables, assume $g = g^*$.

Case $n = 1$

1st step Let $f = 1_A$, $g = 1_B$, $h = 1_C$, where A, B, C are measurable sets of finite measure
$$\mu_1(A) = 2\alpha$$
$$B \overset{*}{=} [-\beta, \beta] \qquad (g = g^*)$$
$$\mu_1(C) = 2\gamma$$

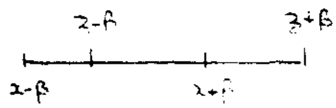
Simplifying assumption to be removed later: A is a finite union of intervals.
Notice that the integral in question has the form
$$\int_{-\infty}^{\infty} f\, g_* h \qquad (* = \text{convolution}).$$

Properties of $g_* h$

(i) $\int_{-\infty}^{\infty} g_* h = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x-y)h(y)\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u)h(y)\,du\,dy = 4\beta\gamma$

(ii) For $x < z < x+\beta$

$$g_*h(z) - g_*h(x) = \int_{z-\beta}^{z+\beta} h(y)\,dy - \int_{x-\beta}^{x+\beta} h(y)\,dy$$



$$= \int_{x+\beta}^{z+\beta} h - \int_{x-\beta}^{z-\beta} h$$

(cancelling overlap)

Since $0 \le h \le 1$ we conclude: $|g_*h(z) - g_*h(x)| \le z - x$.
Repeated applications show the restriction $x < z < x+\beta$ is not necessary. Thus:

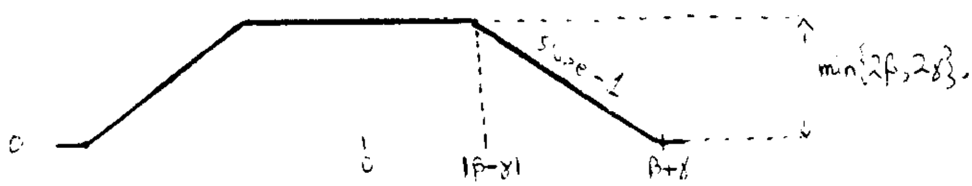
g_*h is Lipschitz rank 1 i.e. $\|(g_*h)'\|_\infty \le 1$.

(iii) $g_*h \to 0$ at $\pm\infty$

(iv) $0 \le g_*h(x) = \int_{-\infty}^{\infty} 1_B(x-y)1_C(y)\,dy = \int 1_{-x+B}\, 1_C \le \min\{2\beta, 2\gamma\}$.

Symmetric case: If $C = [-\gamma, \gamma]$ then g_*h looks like this

symmetric decreasing.



Write K for the set of functions φ that satisfy

(i) $\int \varphi \le 4\beta\gamma$

(ii) $\|\varphi'\|_\infty \le 1$ (weak derivative)

(iii) $\varphi \to 0$ at $\pm\infty$

(iv) $0 \le \varphi \le \min\{2\beta, 2\gamma\}$.

Now consider the maximisation of $\int_A \varphi$ subject to

$\varphi \in K$, $\mu_1(A) = 2\alpha$, A finite union of intervals.

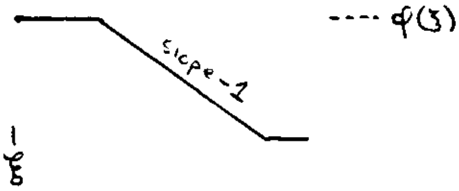
Note

(a) For a fixed φ, by expanding those intervals of A at whose ends φ takes large values, at the expense of the other intervals, we can ensure φ takes the same value at the ends of all intervals.

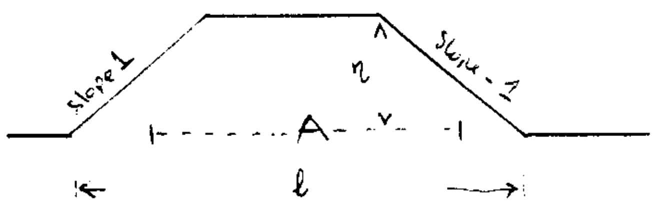
(b) After applying (a), we can then permute the intervals of A and $\mathbb{R} \setminus A$, and the corresponding restrictions of φ, to ensure

- A is a single interval
- $\varphi \in K$ (continuity is not destroyed)

(c) Suppose φ attains its supremum on the interval A at ξ. The value of $\int_A \varphi$ will be increased if φ is modified to the right of ξ to have the form



while maintaining $\int_\xi^\infty \varphi$. Applying a similar procedure to the left of ξ shows that we need only consider functions of the form.



(d) To maximise the integral over A, we should minimise $\ell = 4\beta\gamma\eta^{-1} + \eta$ subject to $0 \le \eta \le \min\{2\beta, 2\gamma\}$ — this is achieved by $\eta = \min\{2\beta, 2\gamma\}$, which gives $1_{[-\beta,\beta]} * 1_{[-\gamma,\gamma]}$.

We have now shown

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 1_A(x)\, 1_{[-\beta,\beta]}(x-y)\, 1_C(y)\,dy\,dx$$

$$\le \int_{-\infty}^{\infty} 1_{[-\alpha,\alpha]}(x)\, 1_{[-\beta,\beta]}(x-y)\, 1_{[-\gamma,\gamma]}(y)\,dy\,dx$$

when A is a finite union of intervals.

For general A, 1_A is the limit in L^2 of a sequence $\{1_{A_n}\}_{n=1}^{\infty}$ where each A_n is a finite union of intervals; the continuity of $*$ on L^2 gives the desired inequality.

Completion of 1-dim case. For general f, g, h with $g^* = g$ we now have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(x-y)h(y)\,dy\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{f(x)} ds \int_0^{g(x-y)} dt \int_0^{h(y)} du\, dy\,dx$$

$$= \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \iint_{\substack{f(x)\geq s\\ g(x-y)\geq t\\ h(y)\geq u}} dx\,dy\;ds\,dt\,du$$

$$= \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \iint 1_{\Phi(s)}(x)\; 1_{\Gamma(t)}(x-y)\; 1_{\Theta(y)}\; dx\,dy\;ds\,dt\,du \qquad \Phi(s) = f^{-1}[s,\infty) \text{ etc}$$

$$\leq \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \iint 1_{\Phi(s)^*}(x)\; 1_{\Gamma(t)}(x-y)\; 1_{\Theta(y)^*}\; dx\,dy\;ds\,dt\,du$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^*(x)g(x-y)h^*(y)\,dy\,dx.$$

General n:

First case: $f = 1_A$, $g = 1_B$, $h = 1_C$ where A, C are sets of finite measure and B is a ball centre 0. We can even assume A and C are bounded – can approximate 1_A and 1_C in L^2 and use continuity of * to get the unbounded case.

Claim: If E is a bounded measurable set, and S is the ball centre 0 with the same measure as E, then $\exists$ hyperplanes $H(1), H(2), \ldots\ldots$ through the origin, such that the successive Steiner symmetrisations $E_1 = E^{*H(1)}$, $E_2 = E_1^{*H(2)}$ ...... satisfy $\| 1_{E_k} - 1_S \|_1 \to 0$ as $k \to \infty$.

Given the claim proceed as follows. Let A^*, C^* be balls centre 0 with the same volumes as A, C. Perform, by the claim, finitely many Steiner symmetrisations of A, in hyperplanes through 0, to obtain A_1 with $\| 1_{A_1} - 1_{A^*} \|_1 \leq 1/2$. Perform the same symmetrisations on C to obtain C_1 – then $\| 1_{C_1} - 1_{C^*} \|_1 \leq \| 1_C - 1_{C^*} \|_1$, because

$$\begin{aligned}\| 1_{C_1} - 1_{C^*} \|_1 &= 2\{\mu_n(C_1) - \mu_n(C_1 \cap C^*)\} \\ &= 2\{\mu_n(C) - \textstyle\int 1_{C_1} 1_{C^*}\} \\ &\leq 2\{\mu_n(C) - \textstyle\int 1_C 1_{C^*}\} \qquad (\text{theorem } 2) \\ &= \| 1_C - 1_{C^*} \|\end{aligned}$$

Now find finitely many symmetrisations that turn C_1 into C_2 satisfying

$\| 1_{C_2} - 1_{C^*} \| \le \frac{1}{2}$, and perform the same operation on A_1 to obtain A_2, proceeding in this manner, sequences $\{A_k\}_{k=1}^{\infty}$, $\{C_k\}_{k=1}^{\infty}$ with $1_{A_k} \to 1_{A^*}$, $1_{C_k} \to 1_{C^*}$ in L^1. By the case $n=1$ we have

$$\iint 1_A(x)\, 1_B(x-y)\, 1_C(y)\, dx\,dy \le \iint 1_{A_k}(x)\, 1_B(x-y)\, 1_{C_k}(y)\, dx\,dy.$$

Letting $k \to \infty$ in the right hand side gives the desired inequality.

The result for general f, g, h $(g = g^*)$ follows by the same interchange of integration used in 1 dimension.

For a proof of the Claim, see the Appendix:

H.J. Brascamp, Elliott H. Lieb & J.M. Luttinger : A general rearrangement inequality for multiple integrals : J. Functional Analysis 17, 227 - 237, (1974)

<u>5. Theorem</u> Let $*$ denote Steiner or Schwarz symmetrisation. If $u \in W^{1,2}(\mathbb{R}^n)$ Then $u^* \in W^{1,2}(\mathbb{R}^n)$ and $\| \nabla u^* \|_2 \le \| \nabla u \|_2$.

<u>Proof</u> For $t > 0$ let $G_t(x) = (4\pi t)^{-n/2} \exp(-x^2/4t)$. Define

$$I_t(u) = t^{-1} \left\{ \int_{\mathbb{R}^n} u^2 - \iint_{\mathbb{R}^n \mathbb{R}^n} u(x)\, G_t(x-y)\, u(y)\, dx\,dy \right\}$$

Riesz's inequality gives $I_t(u^*) \le I_t(u)$

We obtain the result by letting $t \to 0$. Let $u \in L^2$. Then

$$\| \nabla u \|_2^2 = \int_{\mathbb{R}^n} k^2 |\hat{u}(k)|^2 dk, \qquad \hat{} = \text{Fourier transform},$$

the integral being infinite if $u \notin W^{1,2}$. Now

$$I_t(u) = t^{-1} \{ \|u\|_2^2 - \langle u, G_t * u \rangle \}$$

$$= t^{-1} \{ \| \hat{u} \|_2^2 - \langle \hat{u}, \widehat{G_t * u} \rangle \}$$

$$= t^{-1} \{ \| \hat{u} \|_2^2 - \langle \hat{u}, \hat{G}_t \hat{u} \rangle \}$$

$$= t^{-1} \int_{\mathbb{R}^n} |\hat{u}|^2 [1 - \hat{G}_t]$$

$$= \int_{\mathbb{R}^n} |\hat{u}(k)|^2 t^{-1} [1 - e^{-k^2 t}]\, dk$$

Suppose $u \in W^{1,2}$. Now $1 - e^{-s} \leq s$ by convexity, so $0 \leq t^{-1}[1 - e^{-k^2 t}] \leq k^2$.
The dominated convergence theorem yields

$$I_t(u) \to \int k^2 |\hat{u}(k)|^2 dk = \|\nabla u\|_2^2 .$$

Suppose $u \in L^2 \setminus W^{1,2}$. Now $e^s \geq 1+s$ so $1 - e^{-s} \geq 1 - (1+s)^{-1} = s/(1+s)$,
so $t^{-1}[1 - e^{-k^2 t}] \geq k^2/(1+k^2 t)$, so by monotone convergence

$$\lim_{t \to 0} I_t(u) \geq \int k^2 |\hat{u}(k)|^2 dk = \infty .$$

Now for $u \in W^{1,2}$, $I_t(u) \to \|\nabla u\|_2^2$ and $I_t(u^*) \leq I_t(u)$ so
$I_t(u^*) \not\to \infty$ so $u^* \in W^{1,2}$ and $\|\nabla u^*\|_2 \leq \|\nabla u\|_2$

<u>Remark</u> Also true for $W^{1,p}$, $1 \leq p < \infty$; different proof!
The proof given above is due to Elliott H. Lieb.