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SECOND COLLEGE ON VARIATIONAL PROBLEMS IN ANALYSIS

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Rearrangements in variational problems and applications (II)

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These are preliminary lecture notes, intended only for distribution to participants

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We now state some special cases of the results of:

P. L. Lions "Symétrie et compacité dans les espaces de Sobolev"
J. Functional Analysis 49 (1982), 315–334.

6. Theorem Let $n \geq 1$, $s_n = 2N/(N-2)$ if $n \geq 3$, $s_n = \infty$ if $n = 1, 2$.

(i) Let C be the closed convex cone of all Schwarz-symmetric functions in $W^{1,2}(\mathbb{R}^n)$. Then C is compactly embedded in $L^p(\mathbb{R}^n)$ for $2 < p < s_n$.

(ii) Let E be the closed linear subspace of all spherically symmetric functions in $W^{1,2}(\mathbb{R}^n)$. If $n \geq 2$ then E is compactly embedded in $L^p(\mathbb{R}^n)$ for $2 < p < s_n$.

(iii) Let F be the closed convex cone of functions $u \in W^{1,2}(\mathbb{R}^n)$ such that (a) $u(\underline{x}) = u\left(\sqrt{x_1^2 + \cdots + x_{n-1}^2}, x_n\right)$,

(b) u is steiner-symmetric in the hyperplane $x_n = 0$.

If $n \geq 3$ then F is compactly embedded in $L^p(\mathbb{R}^n)$ for $2 < p < s_n$.

(iv) Let $\Omega = U \times \mathbb{R} \subset \mathbb{R}^n$ where U is a bounded domain in $\mathbb{R}^{n-1}$, and let L be the closed convex cone of functions in $W_0^{1,2}(\Omega)$ that are Steiner-symmetric in the plane of U. Then L is compactly embedded in $L^p(\Omega)$ for $2 < p < s_n$.

(Proof of one simple case given in Addendum on page 18)

Example 1

$\Omega = \{(x, y) \in \mathbb{R}^2 \mid 0 < y < 1\}$. Fix $-\infty < \lambda < \pi^2$. Then

$$\|u\| = \left(\|\nabla u\|_2^2 - \lambda \|u\|_2^2 \right)^{1/2}$$

is an equivalent norm on $H_0^1(\Omega)$.

Seek nontrivial solution in $H_0^1(\Omega)$ of

$$-\Delta u - \lambda u = q(x)|u|^{\sigma-2}u \qquad 2 < \sigma < \infty.$$

where q is a symmetric decreasing L^∞ function, $q \not\equiv 0$, $q \geq 0$

Variational formulation

$$\mathcal{P} \qquad \underset{u \in H_0^1(\Omega)}{\text{minimise}} \ \tfrac{1}{2}\|u\|^2.$$

$$\tfrac{1}{\sigma}\int_\Omega q(x)|u|^\sigma = 1$$

with $F(u) = \frac{1}{\sigma}\int_\Omega q(x)|u|^\sigma$.

Observe

- $F(\alpha u) = \alpha^\sigma F(u)$ for $\alpha > 0$; $q \not\equiv 0$ ensures $\exists\, u$ with $F(u) > 0$; now $F(\alpha u) = 1$ for some α. The constraint set is therefore nonempty.
- $F(|u|) = F(u)$ and $\| \, |u| \, \| = \|u\|$. Therefore problem $\mathcal{P}$ has a minimising sequence of non-negative functions.
- If $*$ denotes Steiner symmetrisation in $x = 0$ then for non-neg. $u \in H_0^1(\Omega)$:

$$u^* \in H_0^1(\Omega)$$
$$\|u^*\| \leq \|u\|$$
$$F(u^*) = \frac{1}{\sigma}\int_\Omega q(x)|u^*|^\sigma = \frac{1}{\sigma}\int_\Omega q^*(x)(|u|^\sigma)^*$$
$$\geq \frac{1}{\sigma}\int_\Omega q(x)|u|^\sigma = F(u).$$

Suppose $F(u) = 1$; we may choose $0 < \alpha \leq 1$ such that $F(\alpha u^*) = 1$.
Then $\|\alpha u^*\| = \alpha \|u^*\| \leq \|u^*\| \leq \|u\|$.
It follows: problem $\mathcal{P}$ has a minimising sequence of Steiner-symmetric functions.

Existence of the Minimiser Let $\{u_n\}_{n=1}^\infty$ be a minimising sequence of Steiner-symmetric functions, for problem $\mathcal{P}$. Then $\{u_n\}_{n=1}^\infty$ bounded ; by discarding a subsequence we can suppose $u_n \to u_0$, say, weakly in $H_0^1(\Omega)$.
Notice that u_0 is Steiner symmetric [the Steiner-symmetric functions form a closed convex set in $H_0^1(\Omega)$, which is therefore weakly closed].

- By the compact embedding of Steiner symmetric H_0^1-functions in L^σ, we now have $u_n \to u_0$ strongly in L^σ. Hence $F(u_0) = 1$
- By weak lower semicontinuity of $\|\ \|$, we have
$$\tfrac{1}{2}\|u_0\|^2 \leq \lim_{n\to\infty} \tfrac{1}{2}\|u_n\|^2 = \inf \mathcal{P}$$
Thus u_0 is the desired minimiser; observe u_0 is Steiner-symmetric and $u_0 \not\equiv 0$ (since $F(u_0) = 1$).

Now

$$-\Delta u_0 - \lambda u_0 = \xi \, |u_0|^{\sigma-2} u_0$$

for some $\xi \in \mathbb{R}$, by the Lagrange multiplier rule.

<u>We show $\xi > 0$</u>

Multiply through by u_0 and integrate by parts to get

$$\int |\nabla u_0|^2 - \lambda \int u_0^2 = \xi \int q(x) |u_0|^\sigma$$

ie

$$2 \, \| u_0 \|^2 = \sigma \, \xi F(u_0) = \sigma \xi .$$

Therefore $\xi = 2\sigma^{-1} \| u_0 \|^2 > 0$.

Now write $u_0 = \alpha v$ where $\alpha > 0$ is to be determined. Then

$$\alpha (-\Delta v - \lambda v) = \xi \alpha^{\sigma-1} q(x) |v|^{\sigma-2} v .$$

If we choose α so $\alpha^{\sigma-2} \xi = 1$ then

$$-\Delta v - \lambda v = q(x) \, |v|^{\sigma-2} v ,$$

moreover v is nontrivial, nonnegative, and steiner-symmetric

<u>Example 2</u> $\Omega = \mathbb{R} \times (0,1)$, $-\infty < \lambda < \pi^2$, $f : \mathbb{R} \to \mathbb{R}$, $f(0) = 0$. $q(x)$ symmetric decreasing, $q \geq 0$ $q \not\equiv 0$, $q \in L^\infty$. We seek a nontrivial solution of

$$-\Delta u - \lambda u = q(x) f(u)$$
$$u \in W_0^{1,2}(\Omega).$$

We make the following assumptions concerning f.

$f(0) = 0$, f is increasing, nonconstant and continuous

$f(t) = o(t)$ as $t \to 0$

$f(t) = o(t^{\sigma-1})$ as $t \to \infty$ for some $\sigma > 2$

$\theta F(t) \leq t f(t)$ for all $t > 0$, where $2 < \theta < \sigma$ and $F(s) = \int_0^s f$

We solve the problem by finding a non-zero critical point of

$$\Phi(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \lambda u^2 - \int_\Omega q(x) F(u) , \quad u \in H_0^1(\Omega)$$

using the following special form of the Mountain Pass Lemma due to H. Hofer.

Hofer's MPL Let Φ be a C^1 functional on a Hilbert space H, let $C \subset H$ be a closed convex set, and suppose Φ has the representation $\Phi(u) = \frac{1}{2}\|u\|^2 - J(u)$ with $J'(C) \subset C$. Let $e_0, e_1 \in C$,

$$\Gamma = \{ h \in C([0,1], C) \mid h(0) = e_0 \ \& \ h(1) = e_1 \},$$

$$\gamma = \inf_{h \in \Gamma} \max_{0 \le t \le 1} \Phi(h(t)).$$

Suppose $\gamma > \max\{\Phi(e_0), \Phi(e_1)\}$, and Φ satisfies

$(PS)_C$: Every sequence $\{u_n\}_{n=1}^{\infty}$ in C such that $\Phi'(u_n) \to 0$ and $\{\Phi(u_n)\}_{n=1}^{\infty}$ is bounded, has a convergent subsequence.

Then there exists $u \in C$ s.t. $\Phi(u) = \gamma$ and $\Phi'(u) = 0$.

Proof: Usual MPL proof, but using the deformations preserving C constructed in Hofer, Math. Annalen 261, 493-514 (1982).

For our problem we take

$$H = W_0^{1,2}(\Omega) \qquad \langle u, v \rangle = \int_\Omega \nabla u . \nabla v - \lambda \int_\Omega uv$$

$$J(u) = \int_\Omega r(x) F(u)$$

$C = \{ u \in W_0^{1,2}(\Omega) \mid u \ge 0,\ u \text{ steiner-symmetric in line } x = c \}$

$e_0 = 0$

e_1 any element of C with $\Phi(e_1) < 0$.

Since the solution we construct is non-negative, we can alter $f(s)$ for $s<0$ to suit our convenience – let us take $f(s) = 0$ for $s < 0$.

Growth of F The assumption $\theta F(s) \ge s f(s)$ ensures

$$\frac{d}{ds}\left(F(s) s^{-\theta} \right) = s^{-\theta}\left(f(s) - \frac{\theta}{s} F(s) \right) \ge 0$$

so $F(s) s^{-\theta}$ is increasing; since $f \not\equiv 0$ we therefore have $F(s) \ge a_1 s^{\theta}$ for $s \ge a_2$, where a_1, a_2 are some positive numbers.

Verification that e_1 can be chosen. Since $q \not\equiv 0$, and q is Steiner symmetric, we can choose b_1, b_2 positive such that $q(x) \geq b_2$ for $|x| \leq b_1$. Let $\bar{u}$ be any nontrivial function in C; then positive c, and a subset $A \subset (-b_1, b_1) \times (0,1)$ having positive measure, can be chosen so $\bar{u} > c$ on A. Then for $\alpha > 0$

$$J(\alpha\bar{u}) \geq \int_A r(x) F(\alpha\bar{u}) \geq \int_A b_2 a_1 (\alpha c)^\theta \quad \text{if } \alpha c \geq a_2$$
$$= b_2 a_1 c^\theta \alpha^\theta \text{ for } \alpha > a_2/c.$$

It follows that $\Phi(\alpha\bar{u}) \to -\infty$ as $\alpha \to \infty$. So $e_1 \in C$ can be chosen with $\Phi(e_1) < 0$. Note $\Phi(0) = 0$ so $e_1 \neq 0$.

Behaviour of Φ near 0. From the assumptions $f(s) = o(s)$ as $s \to 0$ and $F(s) = O(s^\sigma)$ as $s \to \infty$ we deduce:

$$\forall \varepsilon > 0 \ \exists A_\varepsilon > 0 \text{ s.t. } 0 \leq F(s) \leq \varepsilon s^2 + A_\varepsilon |s|^\sigma \quad \text{for all real } s.$$

It follows that

$$J(u) = o(\|u\|^2) \quad \text{as } u \to 0.$$

Hence

$$\Phi(u) \geq \tfrac{1}{4}\|u\|^2 \quad \text{for all } u \text{ sufficiently close to } 0.$$

If $0 < \rho < \|e_1\|$ is chosen so $\Phi(u) \geq \frac{1}{4}\rho^2$ whenever $\|u\| = \rho$, then

$$\gamma \geq \tfrac{1}{4}\rho^2 > 0 = \max\{\Phi(e_0), \Phi(e_1)\}.$$

Verification that $J'(C) \subset C$. Consider $u \in C$. We have

$$\langle J'(u), v \rangle = \int_\Omega q(x) f(u) v, \qquad u, v \in W_0^{1,2}(\Omega).$$

It follows that $w = J'(u)$ is the unique minimiser of the strictly convex functional

$$E(v) = \tfrac{1}{2}\|v\|^2 - \int_\Omega q(x) f(u) v, \qquad v \in W_0^{1,2}(\Omega).$$

Now

$$\| |w| \| = \|w\|$$
$$\int q(x) f(u) |w| \geq \int q(x) f(u) w$$

hence

$$E(|w|) \leq E(w).$$

Thus $w = |w|$, ie $w \geq 0$.

Also

$$\|w^*\| \le \|w\|$$

$$\int q(x)f(u)w^* \ge \int q(x)f(u)w$$

since $q(x)f(u)$ is steiner symmetric, hence

$$E(w^*) \le E(w).$$

Therefore $w = w^*$, so $w \in C$. Hence $J'(C) \subset C$.

Verification of $(PS)_c$

Consider a sequence $\{u_n\}_{n=1}^{\infty}$ in C satisfying

$$-M \le \Phi(u_n) \le M \qquad \forall n,$$

$$\|\Phi'(u)\| = \varepsilon_n \to 0 \quad \text{as } n \to \infty,$$

where $M > 0$ is fixed. Thus

$$-M \le \frac{1}{2}\|u_n\|^2 - \int_\Omega q(x)F(u_n) \le M \tag{1}$$

$$-\varepsilon_n\|v\| \le \langle u_n, v\rangle - \int_\Omega q(x)f(u_n)v \le \varepsilon_n\|v\| \quad \forall v \in W_0^{1,2}(\Omega). \tag{2}$$

Putting $v = u_n$ in (2) gives

$$-\varepsilon_n\|u_n\| \le \|u_n\|^2 - \int_\Omega q(x)f(u_n)v_n \le \varepsilon_n\|u_n\|. \tag{3}$$

From $(1)\times\theta - (3)$ we obtain

$$\left(\frac{\theta}{2}-1\right)\|u_n\|^2 \le \theta M + \varepsilon_n\|u_n\| + \int_\Omega q(x)(\theta F(u_n) - f(u_n)u_n)$$

$$\le \theta M + \varepsilon_n\|u_n\|.$$

Hence $\{u_n\}_{n=1}^{\infty}$ is bounded in $W_0^{1,2}(\Omega)$.

Passing to a subsequence, we shall assume $u_n \rightharpoonup u_0$, say, weakly in $W_0^{1,2}(\Omega)$. Fix $2 < p < \sigma$. Then by compactness of C in $L^p(\Omega)$, we can, by further passing to a subsequence, assume $u_n \to u_0$, and $u_n \to u_0$ a.e. in Ω. The properties

$t \mapsto f(t)t$ continuous

$f(t)t = o(t^2)$ as $t \to 0$

$f(t)t = o(t^\sigma)$ as $|t| \to \infty$

$\{u_n\}_{n=1}^{\infty}$ bounded in $L^2(\Omega)$ and in $L^\sigma(\Omega)$

$\|u_n - u_0\|_p \to 0$, $u_0 \in L^2(\Omega) \cap L^\sigma(\Omega)$

are sufficient (see Lemma 7 following) to ensure

$$\int_\Omega q(x)f(u_n)u_n \to \int_\Omega q(x)f(u_0)u_0$$

In conjunction with (3) this yields

$$\|u_n\|^2 \to \int_\Omega q(x) f(u_0) u_0 .$$

Putting $v = u_0$ in (2) yields

$$\|u_0\|^2 = \lim_{n\to\infty} \int_\Omega q(x) f(u_n) u_0$$

$$\geq \int_\Omega \liminf_{n\to\infty} q(x) f(u_n) u_0 \qquad \text{(Fatou's Lemma)}$$

$$= \int_\Omega q(x) f(u_0) u_0 .$$

Thus

$$\|u_0\| \geq \lim_{n\to\infty} \|u_n\|.$$

In conjunction with the weak convergence of u_n to u_0 in $W_0^{1,2}(\Omega)$, this ensures $u_n \to u_0$ strongly in $W_0^{1,2}(\Omega)$. This completes the verification of (PS).

We have now proved the existence of a nontrivial, non-negative, Steiner-symmetric solution of

$$-\Delta u - \lambda u = q(x) f(u), \quad u \in W_0^{1,2}(\Omega).$$

7. Lemma Let $1 \leq \alpha < \beta < \gamma < \infty$, $g \in C(\mathbb{R}, \mathbb{R})$, $g(t) = o(|t|^\alpha)$ as $t \to 0$, $g(t) = o(|t|^\gamma)$ as $|t| \to \infty$. If $\{u_k\}_{k=1}^\infty$ is a a sequence bounded in $L^\alpha(\mathbb{R}^n)$ and $L^\gamma(\mathbb{R}^n)$, convergent strongly in $L^\beta(\mathbb{R}^n)$ to $u_0 \in L^\alpha(\mathbb{R}^n) \cap L^\gamma(\mathbb{R}^n)$, then $g(u_k) \to g(u_0)$ in $L^1(\mathbb{R}^n)$.

Proof [P. L. Lions, Symmétrie et compacité dans les espaces de Sobolev...]
Fix $\varepsilon > 0$. Then there is a $K > 0$ s.t.

$$|g(t)| \leq \varepsilon |t|^\alpha + \varepsilon |t|^\gamma + K |t|^\beta \quad \forall t \in \mathbb{R}.$$

We can also choose a (large) ball B, such that

$$\int_{\mathbb{R}^n \setminus B} |u_k|^\beta + |u_0|^\beta \leq \varepsilon \qquad \text{for all } k.$$

There is an $M > 0$ s.t. $\|u_k\|_\alpha \leq M$, $\|u_k\|_\gamma \leq M$ $\forall k \geq 0$. Note M and K are independent of ε. We now have

$$\int_{\mathbb{R}^n} |g(u_k) - g(u_0)| \le \int_B |g(u_k) - g(u_0)| + \int_{\mathbb{R}^n \setminus B} |g(u_k)| + |g(u_0)|$$

$$\le \int_B |g(u_k) - g(u_0)| + \int_{\mathbb{R}^n \setminus B} \varepsilon|u_k|^\alpha + \varepsilon|u_k|^\gamma + K|u_k|^\beta + \varepsilon|u_0|^\alpha + \varepsilon|u_0|^\gamma + K|u_0|^\beta$$

$$\le \int_B |g(u_k) - g(u_0)| + \varepsilon\int_{\mathbb{R}^n} |u_k|^\alpha + |u_k|^\gamma + |u_0|^\alpha + |u_0|^\gamma + K\int_{\mathbb{R}^n \setminus B} |u_k|^\beta + |u_0|^\beta$$

$$\le \int_B |g(u_k) - g(u_0)| + \varepsilon(2M^\alpha + 2M^\gamma) + \varepsilon K$$

It will therefore suffice to show $\int_B |g(u_k) - g(u_0)| \to 0$.

From the assumption $g(t) = o(|t|^\gamma)$ as $|t| \to \infty$ we can choose $N > C$ such that $|g(t)| \le \varepsilon |t|^\gamma$ for $|t| \ge N$. Now take

$$\tilde{g}(t) = \begin{cases} g(N) & \text{if } t > N \\ g(t) & \text{if } |t| \le N \\ g(-N) & \text{if } t < -N \end{cases}.$$

Then

$$\int_B |g(u_k) - g(u_0)| \le \int_B |\tilde{g}(u_k) - \tilde{g}(u_0)| + \varepsilon\int_B |u_k|^\gamma + |u_0|^\gamma$$

$$\le \int_B |\tilde{g}(u_k) - \tilde{g}(u_0)| + 2M^\gamma \varepsilon$$

$$< \varepsilon + 2M^\gamma\varepsilon \text{ for all large } k \text{ by Dominated Convergence Th.}$$

<u>ADDENDUM</u> The symmetric decreasing non-negative functions in $W^{1,2}(\mathbb{R})$ are compactly embedded in $L^p(\mathbb{R})$, $2 < p < \infty$.

<u>Proof</u> If v is symmetric decreasing then for $x > 0$ $\|v\|_2 \ge 2\int_0^x v^2 \ge 2x\, v(x)^2$ so $v(x) \le (2x)^{-1/2} \|v\|_2$.

Let $\{v_n\}_{n=1}^\infty$ be bounded in $W^{1,2}(\mathbb{R})$, $v_n^* = v_n$. Replacing $\{v_n\}_{n=1}^\infty$ by a subsequence, we can suppose $v_n \to v_0$ weakly in $W^{1,2}(\mathbb{R})$, $v_0^* = v_0$.

Let $2 < p < \infty$. Let $\varepsilon > 0$. For any $R > 0$

$$\|v_n - v_0\|_{L^p(\mathbb{R})} \le 2\|v_n - v_0\|_{L^p(0,R)} + 2\|v_n\|_{L^p(R,\infty)} + 2\|v_0\|_{L^p(R,\infty)}$$

$$\le 2\|v_n - v_0\|_{L^p(0,R)} + 2\left(\int_R^\infty \left((2x)^{-\frac{1}{2}}\|v_n\|_2\right)^{p-2} v_n^2\right)^{1/p} + 2\left(\int_R^\infty \left((2x)^{-\frac{1}{2}}\|v_0\|_2\right)^{p-2} v_0^2\right)^{\frac{1}{p}}$$

$$\le 2\|v_n - v_0\|_{L^p(0,R)} + 2(2R)^{-\frac{p-2}{2p}} \|v_n\|_2 + 2(2R)^{-\frac{p-2}{2p}} \|v_0\|_2$$

$$\le 2\|v_n - v_0\|_{L^p(0,R)} + \varepsilon \quad \text{for a large choice of } R \text{ independent of } n$$

$$< 2\varepsilon \quad \text{for large } n \text{ by compact embedding on a bounded domain.}$$