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SECOND COLLEGE ON VARIATIONAL PROBLEMS IN ANALYSIS

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Direct methods in the calculus of variations

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These are preliminary lecture notes, intended only for distribution to participants

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– INTRODUCTION –

One often encounters problems of the type

$$(1) \qquad \min \{ F(u) : u \in A \} ,$$

where F is a functional and A is a class of functions. Also, very commonly the functional F has the form of an integral :

$$F(u) = \int_\Omega f(x, u(x), Du(x), \dots)\, dx \quad .$$

Confining ourselves to the case $f(x,u,Du)$, if A is made of sufficiently regular functions then problem (1) makes sense.

When also f is sufficiently regular, one may try to solve problem (1) via the <u>Euler equation</u> :

assume

- $A \subset C^1(\Omega; \mathbb{R}^N) \quad , \quad \Omega \subset \mathbb{R}^n ,$

and

$$u \in A , \quad u - v \in C^1_0 \Rightarrow v \in A .$$

Then

$$u \text{ minimum point} \Rightarrow F(u) \le F(u + t\varphi) \quad \forall \varphi \in C^1_0 \ \forall t$$

$$\Rightarrow \int_\Omega \frac{f(x, u+t\varphi, Du+tD\varphi) - f(x,u,Du)}{t}\, dx \ge 0 \quad \forall t > 0 \quad \forall \varphi \in C^1_0 .$$

If f is, say, C^1, then

$$\dots \Rightarrow \int_\Omega \left[\frac{\partial f}{\partial u^i}(x,u,Du)\, \varphi^i + \frac{\partial f}{\partial \xi^i_\alpha}(x,u,Du)\, D_\alpha \varphi^i \right] dx = 0 \quad \forall \varphi \in C^1_0 .$$

From this weak form, if A and f are even more regular, one gets

$$\left(\frac{\partial f}{\partial u^i} - \frac{d}{dx_\alpha} \frac{\partial f}{\partial \xi^i_\alpha} \right)(x, u, Du) = 0 \qquad \forall i \quad ;$$

this is the (strong form of the) Euler equation.
From the minimality of u, one also deduces

$$\frac{d^2}{dt^2} F(u + t\varphi)\Big|_{t=0} \geq 0 \qquad \forall \varphi \in C^1_0 \ ,$$

which yields

$$\int_\Omega \left[\frac{\partial^2 f}{\partial \xi^i_\alpha \partial \xi^j_\beta} D_\alpha \varphi^i D_\beta \varphi^j + 2 \frac{\partial^2 f}{\partial \xi^i_\alpha \partial u^j} D_\alpha \varphi^i \cdot \varphi^j + \frac{\partial^2 f}{\partial u^i \partial u^j} \varphi^i \varphi^j \right] dx \geq 0 \ .$$

It is conceivable that, if we take a small φ with large $D\varphi$, the only piece that matters is the first one: by an accurate choice of φ one gets

$$\frac{\partial^2 f}{\partial \xi^i_\alpha \partial \xi^j_\beta} \lambda^i \lambda^j \eta_\alpha \eta_\beta \geq 0 \qquad \forall \lambda \in \mathbb{R}^N \ \forall \eta \in \mathbb{R}^n \ .$$

If $N = 1$ (or $n = 1$), this amounts to say that f is convex along $(x, u(x), Du(x))$.

The direct methods, instead, rely as little as possible on the differentiability properties of f and u, while the main tool is the well-known result

Theorem (0) : if F is τ-lower semicontinuous ~~on~~ on the compact space (X, τ) then F attains its minimum on X.

Thus, we are led to the investigation of semicontinuity properties of a functional F: if we prove e.g. that F is l.s.c. and some sublevel of F

$$\{ u : F(u) \leq k \}$$

is (not empty and) compact, then F has a minimum. Unfortunately, proving that a subset of C^1 is compact in some strong topology is virtually impossible : thus, we will have to enlarge the class A to some more suitable space, where compactness is not so scarce. A natural ambient for a functional of the type

$$\int_\Omega f(x, u, Du)\, dx$$

will be some Sobolev space $W^{1,p}(\Omega; \mathbb{R}^N)$.
Let us assume ~~for the~~ henceforth

$$(2) \qquad |f(x, s, \xi)| \leq c\,(1 + |\xi|^p) \quad , \; p > 1 \, ,$$

so that F is well defined on $W^{1,p}$. But if we wanted a regular minimum point for F, we are now confronted with another problem : the minimum point that we will (hopefully) find will be in a larger space than A, and

we will later have to show that it belongs to A instead. Also, it is easily seen that if (2) holds and f is continuous, then F is lsc ~~in the~~ (better continuous) in the strong topology of $W^{1,p}$: this does not solve our problems, because compactness in the strong topology of $W^{1,p}$ means (essentially) having bounded second derivatives, and there is no reason to assume that the minimum point of F, where only first derivatives appear, posses-ses second derivatives. Instead, it is very easy to have some compactness in the weak topology of $W^{1,p}$. Alas, this makes more dif-ficult to prove semicontinuity.
Our path will thus be the following:

1) prove that, under suitable assumptions, F is lsc in the weak topology of $W^{1,p}$;

2) prove that it has compact sublevels (this depends very much on the particular functional, but in general it is easy);

3) prove that the minimum point which we get from 1) and 2) via theorem (0) is indeed more regular.

This existence — regularity scheme is the core of the direct method in the calculus of variations.

– SCALAR CASE –

It seems (from the study of the second variation of F) that convexity plays some rôle in the quest for minima. Indeed, the following result provides some insight into the link between convexity and semicontinuity.

<u>Theorem</u> (1) : Let $f : \mathbb{R}^{nN} \to [0, +\infty]$ be convex and lsc; then $F(u) = \int_\Omega f(Du(x))\,dx$ is l.s.c on the space $W^{1,1}_{loc}(\mathbb{R}^n; \mathbb{R}^N)$ with respect to the convergence in the sense of distributions.

This result is due to SERRIN, and may be found in [1]. We remark that, although f does not depend on x, u, the topology is extremely weak!

<u>Sketch of the proof</u> : weakly convergent sequences in L^p jump wildly around the limit (see e.g. $\sin(kx)$, which converges to zero). Their averages, on the contrary, behave very well. The idea is to substitute a weakly cv. sequence with ~~it~~ suitable averages. We remark that averaging combines well with convexity, since Jensen's inequality reads

$$f\left(\int v\, d\mu\right) \leq \int f(v)\, d\mu$$

for every convex f and every probability measure μ (i.e., ~~an average~~ f of the average is less or equal to the average of f). If we take a mollifier ϱ, i.e. a positive C_0^∞ function such that $\int \varrho\, dx = 1$, and ~~set~~ with support in $B(0,1)$, and set

$$\varrho_\varepsilon(x) = \frac{1}{\varepsilon^n}\, \varrho\left(\frac{x}{\varepsilon}\right),$$

we have:

1) $(v * \varrho_\varepsilon)(x) = \int v(y)\, \varrho_\varepsilon(x-y)\, dy$ is of class C^∞ for every $v \in L^1_{loc}$, and converges a.e. to v as $\varepsilon \to 0$

2) $v_\varepsilon(x) = (v * \varrho_\varepsilon)(x)$ is an average of v with respect to the probability $\mu_{x,\varepsilon}(E) = \int_E \varrho_\varepsilon(x-y)\, dy$.

Then

(3) $$f(v_\varepsilon(x)) \le [f(v) * \varrho_\varepsilon](x).$$

Since f is positive, $\int_E f(v) \le \liminf_{\varepsilon \to 0} \int_E f(v_\varepsilon)$. Applying this with $v = Du$,

$$\int_E f(Du)\, dx \le \liminf_{\varepsilon \to 0} \int_E f(Du * \varrho_\varepsilon)\, dx.$$

Now, if $u_\ell \to u$ in $\mathcal{D}'$, it is easily seen that

$$Du_\ell * \varrho_\varepsilon \to Du * \varrho_\varepsilon \quad \text{a.e., for every } \varepsilon.$$

Then, applying also (3),

$$\int_E f(Du)\,dx \le \liminf_{\varepsilon \to 0} \int_E f(Du * \varrho_\varepsilon)\,dx \le \liminf_{\varepsilon \to 0} \liminf_{\ell \to \infty} \int_E f(Du_\ell * \varrho_\varepsilon)\,dx$$

$$\le \liminf_{\varepsilon \to 0} \liminf_{\ell \to \infty} \int_E [f(Du_\ell) * \varrho_\varepsilon]\,dx.$$

This is true for any E. Take $E = \Omega' \subset\subset \Omega$ such that $\mathrm{dist}(\Omega', \partial\Omega) \ge \varepsilon_0 > 0$. Then the integral $\int_{\Omega'} [f(Du_k) * \rho_\varepsilon]\, dx$ makes an average of $f(Du_k)$ in a ball all contained in Ω, then integrates it over Ω'. It is all too conceivable (and it may be easily proved by a change of variables) that since f is positive this integral does not exceed $\int_\Omega f(Du_k)\, dx$: thus,

$$\int_{\Omega'} f(Du)\, dx \leq \liminf_{k \to \infty} \int_\Omega f(Du_k)\, dx .$$

Taking the supremum as $\Omega' \nearrow \Omega$ one has the required semicontinuity, by Beppo Levi's theorem.

This is a very special example: indeed, adding x and u makes matters worse, especially if (as is natural) we only require that f be measurable in x, while continuous in u. A crucial tool in all such situations is a result by Scorza - Dragoni:

Theorem (2): let $f : \Omega \times \overset{\text{Borel set}}{B} \to \mathbb{R}$ be

- measurable in $x \in \Omega$ for all $y \in B \subset \mathbb{R}^m$, and
- continuous in $y \in B$ for a.e. $x \in \Omega \subset \mathbb{R}^n$.

For every $\delta > 0$ there exists a ~~compact~~ closed set K_δ such that $K_\delta \subset \Omega$, meas$(\Omega \setminus K_\delta) < \delta$, and

$$f|_{K_\delta \times B} \text{ is continuous.}$$

Remark: if Ω is bounded and B compact, then $f|_{K_\delta \times B}$ is uniformly continuous

The proof (which may be found in [2]) is based on Severini - Egorov's theorem, and does not concern us.
To show how this tool is used in the semicontinuity theory, we prove another semicontinuity result:

<u>Theorem</u> (3): let $f : \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^{nN} \longrightarrow \mathbb{R}$ satisfy:
-) f is measurable in x, continuous in u, convex in ξ
-) $f \geq 0$,

and let $p \geq 1$, $q > 1$. Then the functional
$$G(u,v) = \int_\Omega f(x,u,v)\,dx$$
is e.s.c. in the topology (strong L^p)$\times$(weak L^q).

This theorem immediately gives a semicontinuity result in the weak topology of $W^{1,q}$, $q > 1$.

<u>Sketch of the proof</u>: ~~by~~ ~~a~~ it is well known that if f depends on ξ alone, then it may be approximated by an increasing sequence of piecewise affine functions. It is shown in [3] that this is true also if f depends on some parameters. Since the supremum of a family of semicontinuous functionals is semicontinuous, we may prove semicontinuity just for the approximating sequence. Then we may also assume (see [3])

H1) $\quad f(x,u,\xi) = 0 \qquad \text{if } \ |x| + |u| \geq C_0$

H2) $\quad |f(x,u,\xi)| \leq C_0(1 + |\xi|)$

H3) $\quad |f(x,u,\xi_1) - f(x,u,\xi_2)| \leq C_0(|\xi_1 - \xi_2|)$.

Let $u_\ell \to u$ in L^p, $v_\ell \rightharpoonup v$ in L^q. Then $\|v_\ell\|_q \le c$, and

$$\int_{|v_\ell| \ge C_1} |v_\ell|\,dx = \frac{1}{C_1^{q-1}} \int_{|v_\ell|\ge C_1} C_1^{q-1} |v_\ell|\,dx \le \frac{\|v_\ell\|_q^q}{C_1^{q-1}} \le \frac{c}{C_1^{q-1}} \quad ;$$

also, $u_\ell \to u$ in measure, thus $\lim_{\ell\to 0} \text{meas}\{|u_\ell - u| > \varepsilon\} = 0 \ \forall \varepsilon$.
We have obtained

$$(4) \begin{cases} \int_{|v_\ell| \ge C_1} |v_\ell|\,dx \le \frac{c}{C_1^{q-1}} \\ \lim_\ell \text{meas}\{|u_\ell - u| > \varepsilon\} = 0 \end{cases} .$$

We cut v_ℓ where it is too big:

$$v_\ell^{C_1} = \begin{cases} v_\ell & \text{if } |v_\ell| \le C_1 \\ 0 & \text{if } |v_\ell| > C_1 \end{cases} .$$

We remark that this is a hard step if we have to keep the relation $v = Du$!!
Now,

$$G(u_\ell, v_\ell) - G(u,v) = I + II + III + IV \quad , \text{ where}$$

$$\begin{aligned} I &= G(u_\ell, v_\ell) - G(u_\ell, v_\ell^C) \\ II &= G(u_\ell, v_\ell^C) - G(u, v_\ell^C) \\ III &= G(u, v_\ell^C) - G(u, v_\ell) \\ IV &= G(u, v_\ell) - G(u,v) . \end{aligned}$$

Now, $I + III$ may be dealt with by H3) and (4):

$$I + III \le 2C_0 \int_{|v_\ell| > C_1} |v_\ell|\,dx \le \frac{c\,C_0}{C_1^{q-1}} \longrightarrow 0 \text{ as } C_1 \to \infty$$

The interesting step is to get rid of II : by Scorza-Dragoni's theorem, we find the set K_δ, and by H2)

$$\int_{\Omega\setminus K_\delta} |f(x,u_\varepsilon,v_\varepsilon^{c_1}) - f(x,u,v_\varepsilon^{c_1})| \le 2C_0 \int_{\Omega\setminus K_\delta} (1+|v_\varepsilon|)$$

$$\le 2\,C_0\,\delta + 2C_0\Big(\int_{\Omega\setminus K_\delta} |v_\varepsilon|^q\Big)^{1/q} \cdot \big[\mathrm{meas}(\Omega\setminus K_\delta)\big]^{1-1/q},$$

and this vanishes as $\delta\to 0$ (independent of C_1^*). Moreover, if we set

$$B_\varepsilon = \{x\in K_\delta :\ |u_\varepsilon - u| < \varepsilon\}$$

we get:

- → $K_\delta\setminus B_\varepsilon$ is small and is dealt with as $\Omega\setminus K_\delta$
- → on B_ε the function f is uniformly continuous, thus the integral is small provided ε (alone) is small.

Thus, also II is small if ε, δ are sufficiently close to zero.

As for IV, showing that $\liminf_\varepsilon \mathrm{IV} \ge 0$ is easy (although it seems a bit of a trick): we must show that the functional

$$v \mapsto \int f(x,u(x),v)\,dx = \tilde{G}(v)$$

is weakly lsc in L^q, but since f is convex, so is $\tilde{G}$: thus, weak lsc of $\tilde{G}$ means weak closure of the <u>convex</u> set $\mathrm{epi}(\tilde{G})$, which is equivalent to its strong closure, and therefore to <u>strong lsc</u> of $\tilde{G}$. But this is easy by H3).

We may summarize the use of Scorza-Dragoni's theorem as follows: if the function φ is <u>more than</u> L^1, we may throw away $\int |\varphi|\, dx$ provided K is small: this is the case for the set where f is not continuous.

In general, we have the following semicontinuity result, which is the finest available in the case $N=1$:

<u>Theorem</u> (4): Let $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^n \to \mathbb{R}$ satisfy

- f is measurable in x, continuous in (u, ξ)
- $0 \le f(x,u,\xi) \le c(1+|\xi|^p)$

Then

$$F(u) = \int f(x,u,Du)\,dx \text{ is lsc in weak-}W^{1,p}$$

$$\Updownarrow$$

$$f \text{ is convex in } \xi .$$

This theorem shows also that convexity is a <u>necessary</u> condition for semicontinuity!!
It may be found in [4]. Of course, the interesting part is this necessity: thus we will have proved that if $N=1$

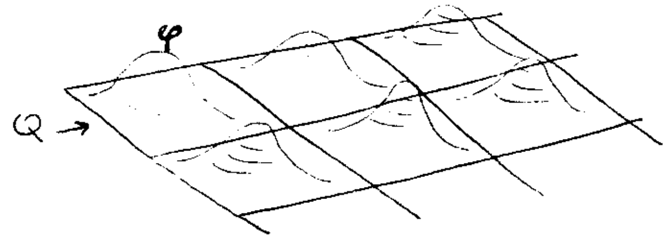
$$\begin{tikzcd}
 & G(u,v) \text{ lsc} \arrow[d, Rightarrow] \\
f(x,s,\xi) \text{ convex} \arrow[ur, Rightarrow] & F(u) \text{ lsc} \arrow[l, Rightarrow]
\end{tikzcd}$$

We assume for simplicity that f depends only on ξ. Then in the general case $N \geq 1$ we have

<u>Theorem</u> (5): if $0 \leq f(\xi) \leq c(1+|\xi|^p)$ and $F(u)$ is lsc in the weak topology of $W^{1,p}(\Omega;\mathbb{R}^N)$ then

$$(5) \qquad \int_\Omega f(\xi + D\varphi(x))\,dx \geq \int_\Omega f(\xi)\,dx \quad \forall \xi \in \mathbb{R}^{nN}\ \forall \varphi \in C_0^1(\mathbb{R};\mathbb{R}^N)$$

Sketch of the proof: take any $\varphi \in C_0^1$, and let Q be a cube containing Ω. Define φ on all of Q by taking $\varphi = 0$ outside Ω. Then define φ on all of $\mathbb{R}^n$ by periodicity.



Set $\varphi_\ell(x) = \frac{1}{\ell}\varphi(\ell x)$:
then $\varphi_\ell \to 0$ in $L^\infty(\mathbb{R}^n;\mathbb{R}^N)$, and also $D\varphi_\ell = D\varphi(\ell x)$ is bounded in L^∞; therefore we ~~as~~ may assume

$$\varphi_\ell \longrightarrow 0 \qquad w^*\text{-}W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^N),$$

and in particular $\varphi_\ell \to 0 \quad w\text{-}W^{1,p}(\Omega;\mathbb{R}^N)$.

Take

$$u = \xi \cdot x, \qquad u_\ell = u + \varphi_\ell.$$

Then

$$u_\ell \to u \quad w\text{-}W^{1,p}(\Omega;\mathbb{R}^N),$$

so that by the semicontinuity of F

$$F(u) \leq \liminf_{\ell\to\infty} F(u_\ell),$$

which means

$$\int_\Omega f(\xi) \le \liminf_h \int_\Omega f(\xi + D\varphi_h(x))\,dx$$

$$= \liminf_h \int_\Omega f(\xi + D\varphi(hx))\,dx .$$

Since $f \ge 0$, the integral over Ω is less than that over Q, and by a simple change of variables

$$\int_\Omega f(\xi) \le \liminf_h \frac{1}{h^n} \int_{hQ} f(\xi + D\varphi(y))\,dy .$$

But as φ is Q-periodic, the last integral equals

$$h^n \int_Q f(\xi + D\varphi(y))\,dy ,$$

and the result is proved.

To prove the "necessity of convexity" part of theorem 4, we shall insert some special functions φ into inequality (5): assume

$$N = 1$$

and take any two vectors $\alpha, \beta \in \mathbb{R}^n$ and any $\lambda \in (0,1)$; define

$$\xi = \lambda\alpha + (1-\lambda)\beta$$

$$\alpha_1 = \alpha - \xi$$
$$\beta_1 = \beta - \xi \quad .$$

First we remark that (5) holds also for every

$\varphi \in W_0^{1,p}(\Omega;\mathbb{R}^N)$: indeed, for any such φ, take a sequence $(\varphi_k) \subset C_0^1(\Omega;\mathbb{R}^N)$ such that $\varphi_k \to \varphi$ in $W^{1,p}$; by the continuity of f and the growth condition $0 \le f(\xi) \le c(1+|\xi|^p)$, we may apply Lebesgue's theorem in ~~ξ~~

$$\int_\Omega f(\xi)\,dx \le \liminf_k \int_\Omega f(\xi + D\varphi_k(x))\,dx$$

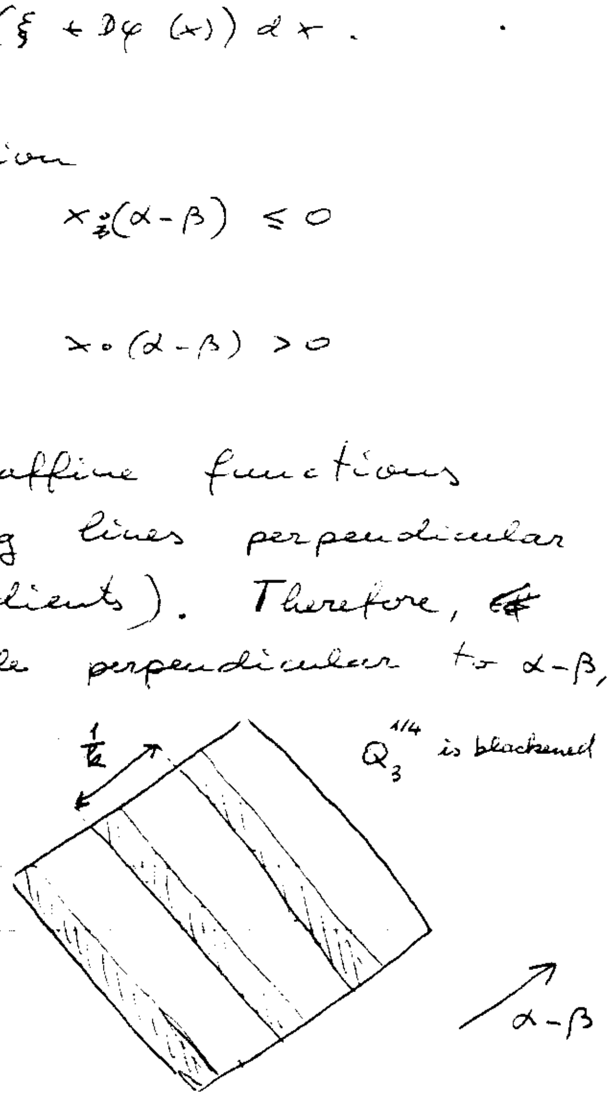
to have

$$\int_\Omega f(\xi)\,dx \quad \le \quad \int_\Omega f(\xi + D\varphi(x))\,dx\,.$$

Next, remark that the function

$$\psi(x) = \begin{cases} \alpha_1 \cdot x & \text{if} \quad x \cdot (\alpha-\beta) \le 0 \\ \beta_1 \cdot x & \text{if} \quad x \cdot (\alpha-\beta) > 0 \end{cases}$$

is continuous (i.e., piecewise affine functions ~~weld~~ weld to each other along lines perpendicular to the difference of their gradients). Therefore, ~~if~~ we take a unit cube Q with one side perpendicular to $\alpha-\beta$, we slice it (orthogonally to $\alpha-\beta$), ~~as~~ into k slices of ~~thick~~ thickness $\frac{1}{k}$, we divide each slice in two slices, one with thickness $\frac{\lambda}{k}$, the other $\frac{1-\lambda}{k}$, and we call Q_k^λ the union of the λ-stripes, $Q_k^{1-\lambda}$ the rest: Q_k^λ and $Q_k^{1-\lambda}$ are thus intertwined.



We define a $W^{1,\infty}$ function ψ_k on Q by set=

ting

$$D\psi_k = \alpha_1 \quad \text{on } Q_k^{\lambda}$$

$$D\psi_k = \beta_1 \quad \text{on } Q_k^{1-\lambda}$$

$$\psi_k = 0 \quad \text{on the side of } Q \text{ where } (\alpha-\beta)\cdot x \text{ is minimum.}$$

It is clear that $\psi_k \to 0$ in $w^*\text{-}W^{1,\infty}(Q)$ as $k \to \infty$. Moreover, if we take a C_0^1 function φ_0 on Q which has value 1 in a cube Q' concentric with Q, but with side $1-2\varepsilon$, and such that

$$0 \le \varphi_0 \le 1, \qquad |D\varphi_0| \le \frac{2}{\varepsilon},$$

and we define

$$\varphi_k = \varphi_0 \cdot \psi_k,$$

we have

$$\varphi_k \in W_0^{1,\infty}(Q)$$

$$\varphi_k \to 0 \quad \text{in } L^\infty(Q)$$

$$D\varphi_k = D\varphi_0 \cdot \psi_k + \varphi_0 \cdot D\psi_k,$$

$$\cancel{|D\varphi_k - D\psi_k| \le \frac{2}{\varepsilon}\|\psi_k\|_{L^\infty} + \|D\psi_k\|_{L^\infty}}$$

Then by (5)

$$\int_Q f(\xi)\,dx \le \int_Q f(\xi + D\varphi_k(x))\,dx \le \int_Q f(\xi + D\psi_k(x))\,dx + \int_{Q\setminus Q'} c\,(1 + |D\varphi_k|^p + |D\psi_k|^p).$$

But the measure of $Q \setminus Q'$ is small as $\varepsilon \to 0$, and

$$\limsup_k \|D\varphi_k\|_\infty \le \frac{2}{\varepsilon} \lim_k \|\psi_k\|_\infty + \limsup_k \|D\psi_k\|_\infty$$

$$= \limsup_k \|D\psi_k\|_\infty ,$$

therefore

$$(6) \qquad \int_Q f(\xi)\,dx \le \limsup_k \int_Q f(\xi + D\psi_k(x))\,dx + o(\varepsilon).$$

But on Q_k^λ

$$\xi + D\psi_k = \xi + \alpha_1 = \alpha ,$$

on $Q_k^{1-\lambda}$

$$\xi + D\psi_k = \xi + \beta_1 = \beta ,$$

$\operatorname{meas}(Q_k^\lambda) = \lambda$, $\operatorname{meas}(Q_k^{1-\lambda}) = 1-\lambda$, $\operatorname{meas}(Q) = 1$, thus (6) becomes

$$f(\xi) \le \lambda f(\alpha) + (1-\lambda) f(\beta) ,$$

or

$$f(\lambda\alpha + (1-\lambda)\beta) \le \lambda f(\alpha) + (1-\lambda) f(\beta) ,$$

that is, f is convex.

If we try to repeat this proof in the case $n > 1$, $N > 1$, we stop at the point where we have to join two different affine functions: this is possible along a line ONLY if the differences of ALL the N components of the gradient are parallel to the same vector, i.e., only if the two gradients differ by a matrix of rank one.

Thus, whereas in the vector-valued case $N > 1$ convexity is still a sufficient condition for semicontinuity, we haven't been able to prove that it is also necessary: we have only found that condition (5) is necessary.
This condition, which we write down again here,

$$\int_\Omega f(\xi)\,dx \le \int_\Omega f(\xi + D\varphi(x))\,dx \quad \forall \xi \in R^{nN} \quad \forall \varphi \in W_0^{1,p}(\Omega; R^N),$$

is called quasi-convexity.
It states only that

among all functions with given <u>linear</u> boundary value $\xi \cdot x$, the infimum of $F(u)$ is attained by the linear function $\xi \cdot x$ itself.

— VECTOR - VALUED CASE —

We start by criticizing the rôle of convexity in the existence theorems for vector-valued functions ($N>1$).

First, a functional that must (in almost all cases) have a minimum is the area functional. This may be written either in cartesian or in parametric form. The latter is ($u: \mathbb{R}^2 \to \mathbb{R}^3$)

$$\int_\Omega \sqrt{\left|\det \frac{\partial(u^1,u^2)}{\partial(x,y)}\right|^2 + \left|\det \frac{\partial(u^1,u^3)}{\partial(x,y)}\right|^2 + \left|\det \frac{\partial(u^2,u^3)}{\partial(x,y)}\right|^2}\; dx\,dy ,$$

and the function

$$f(\xi) = \sqrt{(\xi_{11}\xi_{22}-\xi_{12}\xi_{21})^2 + (\xi_{11}\xi_{32}-\xi_{12}\xi_{31})^2 + (\xi_{21}\xi_{32}-\xi_{22}\xi_{31})^2}$$

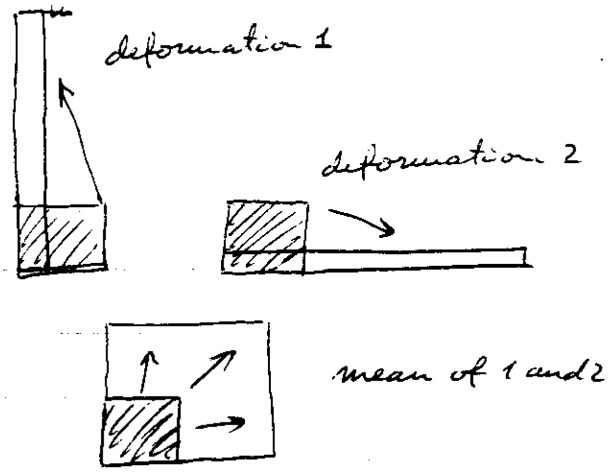
is NOT convex, as one may verify with

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \lambda = 1/2 \quad :$$

indeed, $f(\alpha) = 0 = f(\beta)$, while $f(\lambda\alpha + (1-\lambda)\beta) = \frac{1}{4}$.
Still, it is clear that the least area among the surfaces which are linear at the boundary is attained by the ~~linear~~ flat surface, i.e., this f is quasiconvex.

Again, the functionals of elasticity must be semicontinuous, but they are not convex!
To see this, first think of the effort needed to

take a square sheet of some rubber and to make it four times as long in one directions, and one fourth in the other (the area remains unchanged), then think of the energy needed to realize the mean of this deformation $\begin{pmatrix} 4 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$ and the symmetric one $\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 4 \end{pmatrix}$: this mean is $\begin{pmatrix} 17/8 & 0 \\ 0 & 17/8 \end{pmatrix}$, the area is more than quadrupled and surely the energy is much more than for the other two deformations:



$$E\left(\frac{\text{def } 1 + \text{def } 2}{2}\right) \gg \frac{1}{2} E(\text{def } 1) + \frac{1}{2} E(\text{def } 2)$$

Still, if you take a rubber cube and deform its boundary to fit a given rectangle, you expect that the inside deforms linearly too.

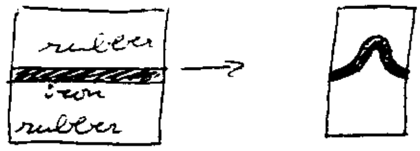
If a function $f(x,\xi)$ is convex in ξ, then also the functional is convex:

$$F(\lambda u + (1-\lambda)v) = \int f(x, \lambda Du + (1-\lambda)Dv)\,dx$$
$$\leq \int \lambda f(x, Du) + (1-\lambda) f(x, Dv)\,dx = \lambda F(u) + (1-\lambda)F(v).$$

The case is different for quasiconvex functions: in the elastic energy setting, for example, when f depends on x it is not to be expected that the minimum energy for linear deformations

$144 : 4 \times 20 + 45$

of the boundary is attained by a linear deformation: think of squeezing a rubber cube with a metal bar inserted:



Thus, for a function depending on (x, u, ξ), we say that it is quasiconvex if

$$\int_{\operatorname{spt}\varphi} f(x_0, u_0, \xi_0 + D\varphi(y))\,dy \geq \int_{\operatorname{spt}\varphi} f(x_0, u_0, \xi_0)\,dy$$

$$\forall x_0 \quad \forall u_0 \quad \forall \xi_0 \qquad \forall \varphi \in W_0^{1,\infty}.$$

With this definition, we have:

<u>Theorem</u> (6): let $f : \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$ be

- measurable in x, continuous in (u, ξ) ~~quasiconvex~~
- $0 \leq f(x, u, \xi) \leq C(1 + |\xi|^p)$.

Then

$$F(u) \text{ is lsc in } w\text{-}W^{1,p}(\mathbb{R}^n; \mathbb{R}^N)$$

$$\Updownarrow$$

$$f \text{ is quasiconvex}$$

We remark that the downward arrow is essentially theorem (5). This result may be found in [5], and in less general versions in papers by MORREY, MEYERS, BALL. The proof will be only in the case $p = +\infty$, where f need only be bounded locally.

The typical example of a quasiconvex function is the following: take an $n \times N$ matrix ξ, and denote by $M(\xi)$ the vector composed by all the determinants of the submatrices of ξ (they are many...), let f be a convex function of the appropriate number of variables and set

$$g(\xi) = f(M(\xi)).$$

Then g is quasiconvex.

<u>Proof of theorem (6), upward arrow</u>: within the proof of theorem (4) (see page 15) we proved that quasiconvexity implies

$$f(\xi)\cdot \mathrm{meas}(\Omega) \leq \liminf_{l\to\infty} \int_\Omega f(\xi + D\varphi_l(x))\,dx$$

for all ξ and all $\varphi_l \rightharpoonup 0$ w^*-$W^{1,\infty}(\Omega;\mathbb{R}^N)$.
This is sort of a semicontinuity inequality, except that it holds only for $u = \xi\cdot x$.
Take for simplicity $\Omega = (0,1)^n$, and divide it into $2^{n\nu}$ cubes with side $2^{-\nu}$. Call Q_l^ν, $l = 1 \dots 2^{n\nu}$, these cubes.
For any function u, define

$$(Du)_l^\nu = ⨍_{Q_l^\nu} Du \qquad\qquad (Du)^\nu = \sum_{l=1}^{2^{n\nu}} (Du)_l^\nu \cdot \mathbf{1}_{Q_l^\nu},$$

a step-function whose value on each of the cubes is the average of Du on the same cube.

Take a sequence $\varphi_k \to 0$ in w^*-$W^{1,\infty}$, ~~and~~ so that we must prove

$$\int_\Omega f(Du)\,dx \le \liminf_k \int_\Omega f(Du + D\varphi_k)\,dx .$$

Now

$$\int_\Omega f(Du + D\varphi_k) - f(Du)\,dx = \int_\Omega f(Du + D\varphi_k) - f((Du)^\nu + D\varphi_k)) +$$

$$+ \int_\Omega f((Du)^\nu + D\varphi_k) - f((Du)^\nu) + $$

$$+ \int_\Omega f((Du)^\nu) - f(Du) .$$

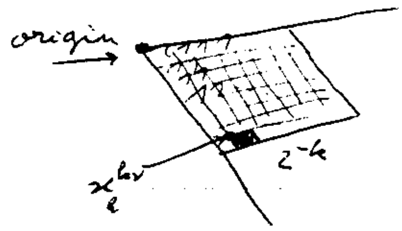
The first integral ~~goes to~~ is small as $\nu \to \infty$ ~~with ν~~, indepen= dent of k: indeed, ~~since~~ f is uniformly continuous on bounded sets, and $(Du)^\nu \to Du$ a.e. The same is true for the last ~~second~~ integral. As for the second, once we fix ν and we let $k \to \infty$ in each of the cubes Q_ℓ^ν, we have

$$\liminf_{k\to\infty} \int_{Q_\ell^\nu} f((Du)_\ell^\nu + D\varphi_k) - f((Du)_\ell^\nu) \ge 0,$$

and the proof is achieved.

Now we see how to adapt the proof of theorem (5) to the case with x, u; for simplicity, we assume f to be straight= forward continuous (Thus avoiding the use of Scorza-Dra= goni's theorem) and $p = +\infty$.
Take a very small cube, say $2^{-k} Q$, where Q is the unit cube, and fix one of its corners x_0 as "origin".

Divide it into a bunch of extra-small cubes of side $2^{-k\nu}$, and take the functions



$$\varphi_k^\nu(x) = \begin{cases} 2^{-k\nu}\varphi(2^{k\nu}x) & \text{in } 2^{-k}Q \\ 0 & \text{outside}. \end{cases}$$

For fixed k, denote by x_k^ν the corner of the cube $Q_\ell^{k\nu}$ closest to the "origin", so that $Q_\ell^{k\nu} = x_\ell^\nu + 2^{-k\nu}Q$. Also, set $u_\ell^\nu = u(x_\ell^\nu)$, where u is the ~~linear~~ affine function with gradient ξ_0 and value u_0 at x_0. Then

$$F(u+\varphi_\ell) - F(u) = \int_{2^{-k}Q} f(x,(u+\varphi_\ell)(x),(Du+D\varphi_\ell)(x)) - f(x,u(x),Du(x))\,dx$$

$$= \sum_\ell \int_{Q_\ell^{k\nu}} f(x,(u+\varphi_\ell)(x),(Du+D\varphi_k)(x)) - f(x_\ell^\nu, u_\ell^\nu, \xi_0 + D\varphi_\ell(x))\,dx$$

$$+ \sum_\ell \int_{Q_\ell^{k\nu}} f(x_\ell^\nu, u_\ell^\nu, \xi_0 + D\varphi(2^{k\nu}x))\,dx \quad - \int_{2^{-k}Q} f(x,u(x),\xi_0)\,dx$$

The first integral is small as $\nu \to \infty$, since f is uniformly continuous and u is continuous.
The second may be rewritten as

$$\frac{1}{2^{nk\nu}} \sum_\ell \int_Q f(x_\ell^\nu, u_\ell^\nu, \xi_0 + D\varphi(y))\,dy$$

This is a Cauchy sum over $2^{-k}Q$ of the continuous function

$$x \mapsto \int_Q f(x,u(x),\xi_0 + D\varphi(y))\,dy$$

therefore it converges as $\nu \to \infty$ to

$$\int_{2^{-k}Q} \int_Q f(x, u(x), \xi_0 + D\varphi(y))\,dy \quad dx \;.$$

By the semicontinuity of F we get

$$\liminf_k \int_{2^{-k}Q} \int_Q f(x,u(x), \xi_0 + D\varphi(y))\,dy\,dx \geq \int_{2^{-k}Q} f(x,u(x),\xi)\,dx \,,$$

therefore, multiplying by the measure of $2^{-k}Q$

$$\liminf_k \quad 2^{nk} \int_{2^{-k}Q} \Big[\int_Q f(x,u(x),\xi_0+D\varphi(y))dy - f(x,u(x),\xi_0) \Big] dx \geq 0 \;.$$

The functions inside the square brackets is continuous, so that as $k \to \infty$ this mean value converges to its value at x_0 : thus

$$\int_Q f(x_0, u_0, \xi_0 + D\varphi(y))dy \geq f(x_0, u_0, \xi_0) \;,$$

as we had to prove.

An important thing must be remarked: for quasiconvex functions we do not have any approximation result from below with nicer functions. Thus, when dealing with w-$W^{1,p}$ semicontinuity and a growth of the order $|\xi|^p$, we are confronted (compare page 9-10) with the problem of discarding $|v_k|^p$ (which is in L^1) on some small set. This is not at all obvious. The way that may be used to overcome this problem is this approximation lemma (see [6]) :

<u>Lemma</u> (7) : let $\Omega \subset \mathbb{R}^M$ be a bounded open set with Lipschitz continuous boundary, $p \geq 1$, $N \geq 1$; there exists a constant $c = c(n, M, p, \Omega)$ such that for all $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ and all $K > 0$ there exists $\tilde{u} \in W^{1,\infty}(\Omega; \mathbb{R}^N)$ such that

$$\|\tilde{u}\|_{1,\infty} \leq K , \quad \text{meas}\, \{ u \neq \tilde{u} \} \leq c \cdot \frac{\|u\|^p_{W^{1,p}}}{K^p}$$

— REGULARITY —

Having exhausted the problem of semicontinuity (and therefore of existence, as it is enough to have $f(x,s,\xi) \geq |\xi|^p$ to obtain the required compactness) we investigate the problem of regularity of the solution.

The most up-to-date result in this field is the following (see [7]):

Theorem (8): let ~~...~~ $f : \mathbb{R}^{nN} \to \mathbb{R}$ be a locally Lipschitz continuous function satisfying

$$|f(\xi)| \leq c(1+|\xi|^2) \quad , \quad |Df(\xi)| \leq c(1+|\xi|)$$

and let $\xi_0 \in R^{nN}$ be such that for some $\gamma > 0$

$$f \in C^2 \text{ in a neighbourhood of } \xi_0$$

$$(7) \qquad \int_{spt\varphi} f(\xi_0 + D\varphi(y))\,dy \geq \int_{spt\varphi} (f(\xi_0) + \gamma|D\varphi(y)|^2)\,dy \quad \forall \varphi \in C_0^1 .$$

Then if u is a minimizer of $\int f(Dv)\,dx$ and

$$\lim_{r \to 0} \fint_{B_r(x_0)} |Du - \xi_0|^2\,dx = 0$$

there is a neighbourhood of x_0 in which $u \in C^{1,\alpha}$ for all $\alpha < 1$.

The actual theorem provides an extension to the case of growth $p \geq 2$, and with f depending on (x,u) too.

The proof is not very easy to follow, as it is quite technical.

Step 1 : inequality (7) holds also for a generic ξ sufficiently close to ξ_0.

Pf: since $f \in C^2$ around ξ_0, this is not too surprising. The constant γ changes slightly.

Step 2 : The rescaled function

$$f_{\xi,t}(\eta) = \frac{f(\xi+t\eta) - f(\xi) - tDf(\xi)\eta}{t}$$

satisfies $|f_{\xi,t}(\eta)| \le c|\eta|^2$, $|Df_{\xi,t}(\eta)| \le c|\eta|$
and $\int f_{\xi,t}(D\varphi)dx \ge \gamma \int |D\varphi|^2 dx$.

Step 3 (CRUCIAL) Let
$|g(\eta)| \le c_1|\eta|^2$, $|Dg(\eta)| \le c_1|\eta|$, $\int g(D\varphi) \ge \gamma \int |D\varphi|^2$, and let u be a minimizer of $\int g(Dv)$. Then for a suitable $c = c(c_1, \gamma)$

$$\boxed{\fint_{B_{r/2}} |Du|^2 \le \frac{c}{r^2} \fint_{B_r} |u-(u)_r|^2}$$

This is called CACCIOPPOLI INEQUALITY

Pf: fix $\frac{r}{2} < t < s < r$, take a cut-off function ζ from B_t to B_s, and set

$$\varphi_1 = [u-(u)_r]\zeta, \qquad \varphi_2 = [u-(u)_r](1-\zeta).$$

Then

$$\gamma \int_{B_s} |D\varphi_1|^2 \le \int_{B_s} g(D\varphi_1) = \int_{B_s} g(Du - D\varphi_2).$$

$$= \int_{B_s} g(Du) + \int_{B_s} [g(Du - D\varphi_2) - g(Du)].$$

By the linear growth of Dg, and since $Du - D\varphi_2 = Du$ in B_t,

$$\int_{B_s} [g(Du - D\varphi_2) - g(Du)]dx \le c \int_{B_s \setminus B_t} (|Du| + |D\varphi_2|)|D\varphi_2|$$

$$\le c \int_{B_s \setminus B_t} |Du|^2 + |D\varphi_2|^2 .$$

By the minimality of u,

$$\int_{B_s} g(Du) \le \int_{B_s} g(Du - D\varphi_1) = \int_{B_s \setminus B_t} g(D\varphi_2) \le c \int_{B_s \setminus B_t} |D\varphi_2|^2 .$$

Then

$$\int_{B_t} |Du|^2 \le \int_{B_s} |D\varphi_1|^2 \le \frac{c}{\gamma} \int_{B_s \setminus B_t} |Du|^2 + |D\varphi_2|^2$$

$$\le \bar{c} \int_{B_s \setminus B_t} |Du|^2 + \frac{c}{(s-t)^2} \int_{B_s \setminus B_t} |u - (u)_r|^2 dx .$$

What we do now is known as "hole-filling technique": we add to both sides $\bar{c} \int_{B_t} |Du|^2$, so that

$$\int_{B_t} |Du|^2 \le \frac{\bar{c}}{\bar{c}+1} \int_{B_s} |Du|^2 + \frac{c}{(s-t)^2} \int_{B_r} |u - (u)_r|^2 dx .$$

Setting $\psi(\sigma) = \int_{B_\sigma} |Du|^2$, we have thus

$$\psi(t) \le \vartheta \psi(s) + \frac{K}{(s-t)^2}, \quad \forall \frac{r}{2} < t < s < r .$$

with $\vartheta < 1$ and $K = c \int_{B_r} |u - (u)_r|^2 dy$.

Then (miracle) ∄ an algebraic lemma says that

$$\psi\left(\frac{r}{2}\right) \le c\,\frac{K}{r^2},$$

that is,

$$\int_{B_{r/2}} |Du|^2\,dx \le \frac{c}{r^2}\int_{B_r} |u-(u)_r|^2\,dx.$$

We remark that step 3 says (via Sobolev-Poincaré inequality) that

$$\int_{B_{r/2}} |Du|^2\,dx \le c\left(\fint_{B_r} |Du|^\alpha\right)^{1/\alpha} \quad \text{with } \alpha < 2\,!$$

From this, via Gehring's theorem, we could deduce that $u \in W^{1,2+\varepsilon}$ for a particular $\varepsilon > 0$.

Step 4 (DECAY ESTIMATE). Assume u is a minimizer of $\int f(Dv)$, and set

$$U(x_0,r) = \fint_{B_r(x_0)} |Du-(Du)_r|^2\,dx.$$

Then there is a constant C such that for all τ there is $\varepsilon(\tau)>0$ such that

$$U(x_0,r) < \varepsilon \Rightarrow U(x_0,\tau r) \le C\tau^2\, U(x_0,r)$$

Before proving step 4, we state its consequences:

Step 5: $U(x_0,\tau^k r) \le c\,\tau^{2k\alpha}\,U(x_0,r)$ provided $C\tau^2 < \tau^{2\alpha}$.

This is not difficult (although technical)

Step 6 Du is Hölder continuous in some ball around x_0

This follows from step 5, which means that if $\rho < R$ then

$$\fint_{B_\rho} |Du - (Du)_\rho|^2 \leq C\left(\frac{\rho}{R}\right)^{2\alpha} \fint_{B_R} |Du - (Du)_R|^2 .$$

This implies (via a theorem by Campanato) $Du \in C^{0,\alpha}$.

Proof of step 4: by contradiction, assume that for a sequence of balls $B_{r_\ell}(x_\ell)$

$$U(x_\ell, \tau r_\ell) > C\tau^2 U(x_\ell, r_\ell) \quad \text{but} \quad U(x_\ell, r_\ell) = \lambda_\ell^2 \to 0 .$$

Part 1 RESCALING

we define for each ℓ

$$v_\ell(z) = \frac{u(x_\ell + r_\ell z) - (u)_{x_\ell, r_\ell} - r_\ell (Du)_{x_\ell, r_\ell} z}{\lambda_\ell r_\ell} ,$$

so that

$$(v_\ell)_{B_1(0)} = 0 , \qquad \fint_{B_1(0)} |Dv_\ell|^2 = 1 .$$

Then $v_\ell \rightharpoonup v$ weakly in $B_1(0)$

Since u was a minimizer, we have

$$\int_{B_1} \left[f(\lambda_\ell Dv_\ell + (Du)_\ell + t D\varphi) - f(\lambda_\ell Dv_\ell + (Du)_\ell) \right] dz \geq 0 ,$$

Therefore (as we did to obtain Euler's equation) after some trouble, and considering that

$\lambda_\varepsilon Dv_\varepsilon \to 0$ a.e., we get

$$\int_{B_1} D^2 f(A)\, Dv\, D\varphi\, dz = 0 \quad \forall \varphi ,$$

for a suitable A close to ξ_0. Then v is the solution of an elliptic system with constant coefficients, and thus

$$\fint_{B_{2\tau}} |Dv - (Dv)_{2\tau}|^2 \le c_0 \tau^2 .$$

Part 2 reconstruction of $U(x_\varepsilon, \tau r_\varepsilon)$

We remark that v_ε (more precisely $w_\varepsilon = v_\varepsilon - \dfrac{(Du)_{\tau r_\varepsilon} - (Du)_{r_\varepsilon}}{\lambda_\varepsilon}$) minimizes

$$\int_{B_1} f_{(Du)_{\tau r_\varepsilon}, \lambda_\varepsilon} (D\psi)\, dz ,$$

therefore we may apply Caccioppoli inequality to have

$$\begin{aligned} U(x_\varepsilon, \tau r_\varepsilon) &= \fint_{\tau r_\varepsilon} |Du - (Du)_{\tau r_\varepsilon}|^2 \\ &= \lambda_\varepsilon^2 \fint_\tau |Dv_\varepsilon - (Dv_\varepsilon)_\tau|^2 = \lambda_\varepsilon^2 \fint_\tau |Dw_\varepsilon|^2 \\ &\le \lambda_\varepsilon^2 \tau^{-2} \fint_{2\tau} |w_\varepsilon - (w_\varepsilon)_{2\tau}|^2 \\ &= \frac{\lambda_\varepsilon^2}{\tau^2} \fint_{2\tau} |v_\varepsilon - (v_\varepsilon)_{2\tau} - (Dv_\varepsilon)_\tau z|^2 . \end{aligned}$$

As $\ell \to \infty$,

$$\limsup \frac{U(x_\ell, \tau r_\ell)}{\lambda_\ell^2} \leq c \fint_{2\tau} \frac{|v - (v)_{2\tau} - (Dv)_\tau z|^2}{\tau^2}$$

$$\leq c \fint_{2\tau} |Dv - (Dv)_\tau|^2$$

$$\leq c \fint_{2\tau} |Dv - (Dv)_{2\tau}|^2 + |(Dv)_{2\tau} - (Dv)_\tau|^2 .$$

$$\leq c C_0 \tau^2 + c \left| \fint_\tau [Dv - (Dv)_{2\tau}] \right|^2$$

$$\leq c C_0 \tau^2 + c \fint_\tau |Dv - (Dv)_{2\tau}|^2$$

$$\leq c C_0 \tau^2 + c \fint_{2\tau} |Dv - (Dv)_{2\tau}|^2$$

$$\leq c_1 \tau^2 ,$$

which contradicts the assumption for $C' > c_1$.

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