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ADRIATICO CONFERENCE ON FOURIER OPTICS AND HOLOGRAPHY

WAVE PROPAGATION AND FOURIER OPTICS

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1. INTRODUCTION.

In this lecture I will present some basic theory of diffraction and image formation that will be needed in this Workshop. The following topics will be reviewed:

- propagation of a wave field through space, based on the wave equation.
- diffraction from an aperture in the Kirchhoff approximation. An indication will be given when this approximation is reliable. In particular, the Fresnel and Fraunhofer diffraction will be treated.
- image formation by a thin lens. Here the concept of resolving power will be discussed.
- image formation by a general optical system and the role of coherence.
- basic principles of holography.

2. WAVE PROPAGATION AND DIFFRACTION.

An important ingredient in the study of image formation is the propagation of waves through a medium charaterized by a constant (both in space and time) refractive index. We start our discussion with the wave equation, which can straightforwardly be derived from Maxwell's equations:

$$\nabla^2 \mathbf{e} - \frac{1}{c^2}\frac{\partial^2 \mathbf{e}}{\partial t^2} = 0\,, \tag{2.1}$$

where $\mathbf{e}(\mathbf{r},t)$ is the electric field. For the other components of the electromagnetic field the same equation holds. In some situations we are allowed to consider the corresponding scalar equation:

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{2.2}$$

For instance, when an electromagnetic wave which is polarised along the grooves of a grating is diffracted from this grating, the problem can be formulated in terms of a single scalar quantity. Also in other cases this simplified approach often gives a good description of the phenomena.

In the rest of this paper we will assume that the waves are strictly monochromatic, an assumption that can be fulfilled, for instance, when using light from mode-locked lasers. In that case the field $u(\mathbf{r},t)$ can be represented by:

$$u(\mathbf{r},t) = u(\mathbf{r})\exp(i\omega t) + \text{c.c.}, \tag{2.3}$$

where c.c. stands for "complex conjugate". The differential equation for $u(\mathbf{r})$ follows directly from eq.(2.2):

$$(\nabla^2 + k_o^2)\, u(\mathbf{r}) = 0, \text{ where } k_o = \omega/c \tag{2.4}$$

Eq.(2.4) is known as Helmholtz' equation, while k_o is the wave number: $k=2\pi/\lambda$, where λ is the wavelength.

a. Propagation through a homogeneous medium.

We will first study how the field $u(\mathbf{r})$ propagates through space when its values are given on a surface (plane) $z=z_o$: $u(x,y,z=z_o)$ is known.

To this end, we first Fourier transform this function with respect to the variables x and y:

$$u(\mathbf{r},z_o) = (2\pi)^{-1}\int d\mathbf{k}_t\, U(\mathbf{k}_t,z_o)\, e^{i\mathbf{k}_t\cdot\mathbf{r}}, \tag{2.5}$$

where $\mathbf{r}=(x,y)$ and $\mathbf{k}_t=(k_x,k_y)$ are the two-component transversal coordinates and spatial frequencies, respectively. As $u(\mathbf{r},z)$ has to satisfy (2.4) the exponential factor $\exp(i\mathbf{k}_t\cdot\mathbf{r})$ has to be completed to a solution in three-dimensional space of (2.4): $\exp(i\mathbf{k}_t\cdot\mathbf{r})$ has to be replaced by $\exp[i(\mathbf{k}_t\cdot\mathbf{r}+k_z z)]$, where on account of Helmholtz' equation, k_z has to be determined from:

$$k_t^2 + k_z^2 = k_o^2 \tag{2.6}$$

So we find for the solution:

$$u(\mathbf{r},z) = (2\pi)^{-1}\int d\mathbf{k}_t\, U(\mathbf{k}_t,z_o)\, e^{i[\mathbf{k}_t\cdot\mathbf{r}+k_z(z-z_o)]}, \tag{2.7}$$

$$\text{where } k_z = \pm\left[k_o^2 - k_x^2 - k_y^2\right]^{1/2}. \tag{2.8}$$

As $U(\mathbf{k}_t,z_o)$ may be constructed from the given distribution by inverting the Fourier transform (2.5), the field $u(\mathbf{r},z)$ can be calculated everywhere in space.

At this stage we can introduce the concept of *transfer function*. For this we need the Fourier transform $U(\mathbf{k}_t, z_o)$ of the initial distribution $u(\mathbf{r},z_o)$. $U(\mathbf{k}_t,z_o)$ is called the *spatial frequency spectrum* of the field at $z=z_o$. After propagation to the plane z we find from (2.7) that the spatial frequency spectrum of the field at z, $u(\mathbf{r},z)$, is given by:

$$U(\mathbf{k}_t,z) = U(\mathbf{k}_t,z_o)\exp\left[ik_z(z-z_o)\right] \tag{2.9}$$

The generic notation for a transfer function is given by:

$$U_{out}(\mathbf{k}) = H(\mathbf{k})\, U_{in}(\mathbf{k}), \qquad (2.10)$$

where U_{in} and U_{out} denote the spatial frequency spectrum in the input and output plane, respectively. $H(\mathbf{k})$ is called the transfer function. Consequently, in our example of eq. (2.9) the transfer function is given by:

$$H(\mathbf{k}_t) = \exp\left[ik_z(z-z_0)\right], \text{ where } k_z = +\left[k_0^2 - k_x^2 - k_y^2\right]^{1/2}. \qquad (2.11)$$

It should be noted that the expression under the square root could become negative, because k_x and k_y vary between $-\infty$ and $+\infty$. In that case k_z becomes purely imaginary and leads to plane waves whose amplitudes decrease exponentially in the direction of propagation: such waves are called *evanescent waves*. They do not play a significant role in conventional microscopy. Therefore, we will ignore them. Plane waves for which k_z is a real number are called *homogeneous waves*. In our presentation we will only consider homogeneous waves.

It is instructive to relate the ingoing field $u(\mathbf{r},z_0)$ and the outgoing field $u(\mathbf{r},z)$: combining (2.7) and the expression for $U(\mathbf{k}_t,z_0)$ in terms of $U(\mathbf{r},z_0)$:

$$U(\mathbf{k}_t,z_0) = (2\pi)^{-1}\int d\mathbf{r}\, u(\mathbf{r},z_0)\, e^{-i\mathbf{k}_t\cdot\mathbf{r}},$$

we find after a straightforward calculation:

$$u(\mathbf{r},z) = \int h(\mathbf{r}-\mathbf{r}')\, u(\mathbf{r}',z_0)\, d\mathbf{r}', \qquad (2.12)$$

where the transfer function in the spatial domain, $h(\mathbf{r}-\mathbf{r}')$, is given by:

$$h(\mathbf{r}-\mathbf{r}') = (2\pi)^{-2}\int d\mathbf{k}_t \exp\left[i\mathbf{k}_t\cdot(\mathbf{r}-\mathbf{r}')+ik_z(z-z_0)\right], \text{ where } k_z = +\left[k_0^2 - k_t^2\right]^{1/2}. \qquad (2.13)$$

Notice that eq. (2.13) has the form of a convolution.

We will now consider a related problem:

b.Diffraction from an aperture in a screen.

We consider an aperture in a flat opaque screen: fig.2.1. We choose the normal on the screen as the z axis. An incident wave impinging upon a screen with aperture A will give rise to a field distribution on the right hand side of the screen. In principle, we have to solve a rigorous boundary–value problem, but often we can get away with a simplified approach due to Kirchhoff: we assume that the field "on the right hand side" of the screen is zero on the opaque part and is equal to the undisturbed field inside the aperture. If the z coordinate of the plane of the screen is $z=z_0$ we now can solve for the field to the right of the screen (which we suppose for simplicity to be a

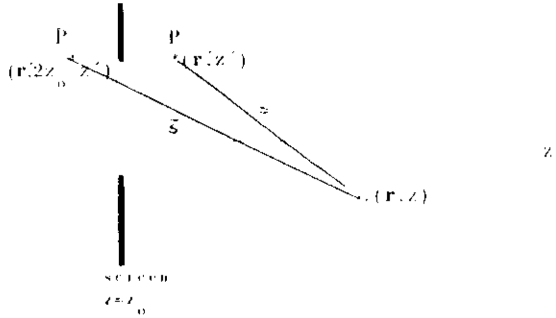

Fig. 2.1 The geometry of diffraction of a wave from an aperture A.

a perfect conductor):

$$u(\mathbf{r},z) = \int_A \frac{\partial G(\mathbf{r},z,\mathbf{r}_0,z_0)}{\partial n'}\, u(\mathbf{r}_0,z_0)\, dx_0\, dy_0 \qquad (2.14)$$

where A denotes the transparant aperture in the screen. $G(\mathbf{r},z,\mathbf{r}',z')$ is the appropriate Green's function: this function can be considered as the response of the system to a unit delta point source located at $(\mathbf{r},z)$ in the point $(\mathbf{r}',z')$. According to the Kirchhoff-Sommerfeld construction we find (G has to be zero for $\mathbf{r}'$ on the screen):

$$G(\mathbf{r},z,\mathbf{r}',z') = (4\pi)^{-1}\left[\frac{e^{iks}}{s} - \frac{e^{ik\bar{s}}}{\bar{s}}\right], \qquad (2.15)$$

where s is the distance between the point $(\mathbf{r},z)$ and a point $(\mathbf{r}',z')$ and $\bar{s}$ is the distance between $(\mathbf{r},z)$ and the mirror image $(\mathbf{r}',2z_0-z')$ of $(\mathbf{r}',z')$ with respect to the screen:

$$s=\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{1/2};\ \bar{s}=\left[(x-x')^2+(y-y')^2+(z-2z_0+z')^2\right]^{1/2} \qquad (2.16)$$

$\partial/\partial n$ denotes the derivative in the direction of the outward normal, in our case $\partial/\partial n = -\partial/\partial z$.

The treatment presented here is not mathematically rigorous, because we *assume* that the field distribution in the aperture is the same as when the aperture and screen were absent. The assumption seems acceptable if the dimensions of the aperture are large with respect to the wavelength of the light.

Combining (2.14) and (2.15) we find the basic formula:

$$u(\mathbf{r},z) = \frac{1}{i\lambda}\int d\mathbf{r}_o\, u(\mathbf{r}_o,z_o)\,\frac{e^{ik\rho}}{\rho}\cos\theta. \qquad (2.17)$$

where $\rho = [|\mathbf{r}-\mathbf{r}_o|^2 + (z-z_o)^2]^{1/2}$. θ is the angle between the normal $\mathbf{n}$ and the vector joining the points $(\mathbf{r}_o,z_o)$ and $(\mathbf{r},z)$. Eq. (2.17) is known as the *Huygens Fresnel formula*. As we will exclusively deal with paraxial image formation, which means that the angles of the normals on the wavefronts make small angles with the optic axis, we can safely approximate $\cos(\mathbf{n},\mathbf{r}-\mathbf{r}_o)$ by unity. The cosine factor in (2.17) is known as the ***obliquity factor*** and has been subject of much study and controversy.

In the paraxial approximation, for large values of z, we may approximate $|\mathbf{r}-\mathbf{r}_o|$ by:

$$\rho \cong |z-z_o| + \frac{1}{2}\,\frac{|\mathbf{r}-\mathbf{r}_o|^2}{|z-z_o|} + \ldots\ldots \qquad (2.18)$$

Eq. (2.17) then reduces to:

$$u(\mathbf{r},z) = \frac{e^{ik|z-z_o|}}{i\lambda|z-z_o|}\int d\mathbf{r}_o \exp\left[\frac{ik}{2|z-z_o|}|\mathbf{r}-\mathbf{r}_o|^2\right] u(\mathbf{r},z_o) \quad (2.19)$$

Notice, that the integral in (2.19) has the form of **a convolution**. The approximation (2.19) is known as the ***Fresnel approximation***. An even more **drastic** approximation is obtained when neglecting the terms quadratic in $\mathbf{r}_o$ in the evaluation of $(r-r_o)^2$:

$$(\mathbf{r}-\mathbf{r}_o)^2 = r^2 - 2\mathbf{r}.\mathbf{r}_o + r_o^2 \cong r^2 - 2\,\mathbf{r}.\mathbf{r}_o \qquad (2.20)$$

This approximation can only be trusted in the "deep" **far field** and is known as the ***Fraunhofer approximation***. The corresponding far-field amplitude is given by:

$$u(\mathbf{r},z) = \frac{\exp(ik|z-z_o|)}{i\lambda|z-z_o|}\exp\left(\frac{ikr^2}{2|z-z_o|}\right)\int_A d\mathbf{r}_o\, u(\mathbf{r},z_o)\exp\left[\frac{-ik}{|z-z_o|}\mathbf{r}.\mathbf{r}_o\right] \qquad (2.21)$$

The factor exp(ikz) is a constant phase factor in a **plane z=constant** and therefore has no observable consequences in that plane. We will henceforth omit this factor. Apart from the phase curvature factor $\exp(ikr^2/2|z-z_o|)$, $u(\mathbf{r},z)$ and $u(\mathbf{r},z_o)$ are ***related by a Fourier transform***. Notice, that the Fraunhofer approximation no longer has the form of a convolution like in eq. (2.19). We will see that under certain circumstances the phase curvature factor is compensated by other factors. As an example, it can easily be checked that an aperture that is illuminated by a spherical wave which converges to a point P on a screen which is parallel to the plane of the aperture, leads to a field distribution around P which is the Fourier transform of the distribution in the

aperture. This situation often arises in imaging.

3.IMAGE FORMATION BY THIN LENSES.

Let us consider a thin lens as discussed in elementary treatments of geometrical optical image fromation (Fig.3.1). The lens is specified by its thickness function $d(\mathbf{r})$, where $\mathbf{r}$ is the transversal coordinate. The assumption of a thin lens is used when relating the field distribution in the input plane $u_{in}(\mathbf{r})$ and the distribution in the output plane $u_{out}(\mathbf{r})$:

$$u_{out}(\mathbf{r}) = t(\mathbf{r})\, u_{in}(\mathbf{r}), \qquad (3.1)$$

where $t(\mathbf{r})$ is the ***optical*** thickness of the lens: $t(\mathbf{r}) = \exp[ik(n-1)d(\mathbf{r})]$ if $d(\mathbf{r})$ denotes the *geometrical* thickness of the lens and n the refractive index of the material of the lens. Note that the amplitude transmission function $t(\mathbf{r})$ is assumed ***not*** to depend upon the incident distribution: of course, such an assumption cannot hold for a thick lens,

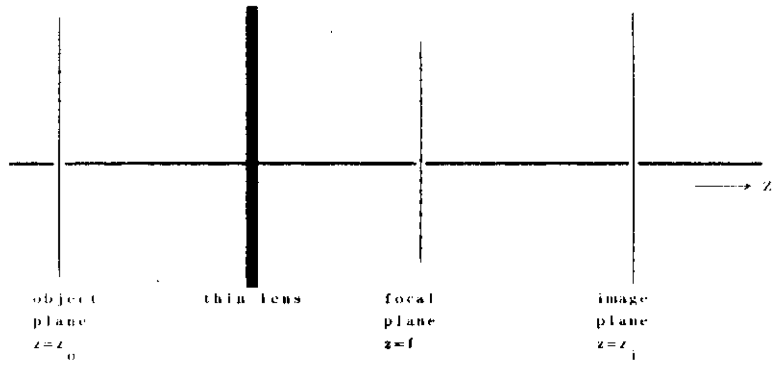

Fig. 3.1 A thin lens

where $t(\mathbf{r})$ depends on the angle of incidence of the incident field.

We consider a double convex lens of which the front face has a radius of curvature R_1 and the back face a radius R_2. If d_o is the thickness of the lens in the centre, it is directly shown that

$$d(\mathbf{r}) = d_o - R_1 + (R_1^2 - r^2)^{1/2} + R_2 - (R_2^2 - r^2)^{1/2} \qquad (3.2)$$

In the paraxial approximation ($r \ll R_1, R_2$) this reduces to:

$$d(\mathbf{r}) \approx d_o - \frac{1}{2} r^2 \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \tag{3.3}$$

Remembering from elementary geometrical optics the relation for the focal length f in terms of the radii of curvature of the front and back surfaces of the lens:

$$\frac{1}{f} = (n-1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right), \tag{3.4}$$

we directly find the relation:

$$u_{out}(\mathbf{r}) = \exp(iknd_o) \exp\left(- \frac{ik}{2f} r^2\right) u_{in}(\mathbf{r}) \tag{3.5}$$

We did not specify the z coordinates belonging to the input and output planes. This is not necessary, because these planes are supposed to coincide in the thin lens approximation: their physical separation has already been accounted for by $d(\mathbf{r})$.

It is of particular interest to calculate the field distribution in the back focal plane of a thin lens. Combining the results (3.5) and (2.21) we find:

$$u_f(\mathbf{r}_f) = \frac{\exp(ikr_f^2/2f)}{i\lambda f} \int_A u_{out}(\mathbf{r}) \exp\left[\frac{-ik}{f} \mathbf{r}.\mathbf{r}_f\right] d\mathbf{r}$$

$$= \frac{e^{inkd_o}}{i\lambda f} \int_A u_{in}(\mathbf{r}) \exp\left[\frac{-ik}{f} \mathbf{r}.\mathbf{r}_f\right] d\mathbf{r} \tag{3.6}$$

Here $\mathbf{r}_f$ denotes the transversal coordinate in the back focal plane z=f. We see that the quadratic phase factor (the phase curvature) has dropped out.

We see that apart from an irrelevant constant phase factor $\exp(iknd_o)$ and a constant factor $(i\lambda f)^{-1}$ ***the field distribution in the back focal plane is the Fourier transform of the incident field***. In the following we will omit constant phase factors.

4.IMAGE FORMATION IN GENERAL.

We will see that the simple considerations of the preceding section can be extended to more realistic imaging systems. We will study the schematic setup sketched in Fig. 4.1. We proceed heuristically: if the imaging system is ideal it must image a point source in the object plane into a point image in the image plane. The imaging process is described as follows: the system is characterised by an entrance and an exit pupil. These apertures do not have to correspond to real apertures: they can be effective apertures that can be calculated from the physical diaphragms inside the optical system. We suppose that the wave propagation through the optical system.

denoted by "Optics" in Fig.4.1, can be described in terms of geometrical optics. The exit pupil is defined as the geometrical optical image of the entrance pupil. The entrance pupil can be chosen ***ad libitum***.

Let us now insert the wave character of the fields: the unit point source at the position $\mathbf{r}_o'$ in the object plane is represented by the δ-function $\delta(\mathbf{r}_o - \mathbf{r}_o')$. The

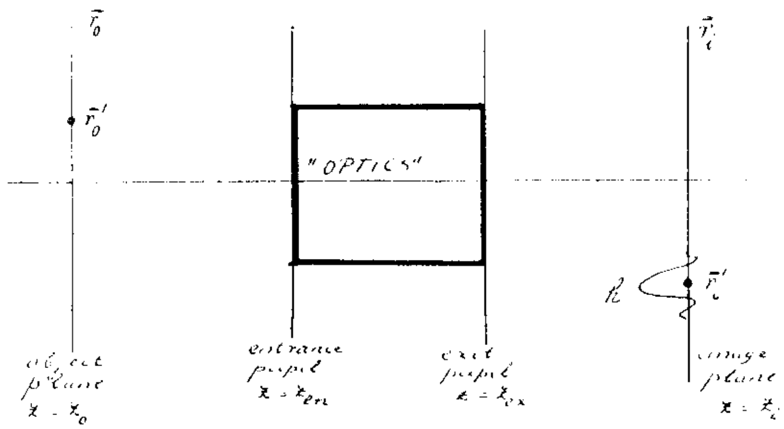

Fig.4.1: a general imaging process

outgoing wave produced by the system is a spherical wave emanating from the exit pupil. This wave gives rise to a diffraction pattern $h(\mathbf{r}_i - \mathbf{r}_i')$ around the geometrical image $\mathbf{r}_i'$ of the point source in $\mathbf{r}_o'$. If M denotes the lateral magnification, including the sign, we have $\mathbf{r}_i' = M\, \mathbf{r}_o'$. By choosing the scale in the image plane such that the image of the unit of length in the object plane is imaged into the unit of length in the image plane, we can write $\mathbf{r}_i' = \mathbf{r}_o'$. The diffraction pattern $h(\mathbf{r}_i - \mathbf{r}_o')$ is supposed to be invariant under a shift of the object point source in the object plane. If an imaging system satisfies such a condition, it is said to be *isoplanatic*.

As the imaging process is a *linear process*, the superposition principle tells us that the field distribution $u_i(\mathbf{r}_i)$ in the image plane for a general field distribution in the object plane $u_o(\mathbf{r}_o)$ is given by:

$$u_i(\mathbf{r}_i) = \int_A h(\mathbf{r}_i - \mathbf{r}_o)\, u_o(\mathbf{r}_o)\, d\mathbf{r}_o \tag{4.1}$$

The convolution in (4.1) is a consequence of the isoplanacy (also referred to as spatial stationarity). The transfer function $h(\mathbf{r})$ is a good characterisation of the performance of the optical system. (4.1) becomes very simple after a Fourier transformation:

$$F_I(\mathbf{k}) = H(\mathbf{k})\, U_O(\mathbf{k}) \qquad (4.2)$$

$H(\mathbf{k})$ specifies how strongly the spatial frequencies $\mathbf{k}$ are transmitted by the system. Ideally $H(\mathbf{k})$ should be equal to unity for the range of spatial frequencies that are transmitted by the system. Remember that this range is determined by the extent of the (effective) entrance pupil: if we model the object as a diffraction grating, an incident wave will be diffracted into different orders. If the first order diffracted beam is not transmitted through the entrance pupil, the optical system will not "see" this grating. We say that the grating is not resolved by the system. More generally, if we have an arbitrary object, this object can be written as a superposition of different gratings. This is what we do when we Fourier transform the wave field at the object. Each spatial frequency $\mathbf{k}$ corresponds to a grating with a grating constant (k_x^{-1}, k_y^{-1}). From this picture it is clear that not all spatial frequencies are transmitted: those plane waves which have been diffracted over angles larger than the acceptance angle of the entrance pupil are not transmitted: for those spatial frequencies $H(\mathbf{k})$ equals zero. The finite extent of the entrance pupil leads to a finite ***resolving power*** of the system, which is defined as the maximal transmitted spatial frequency.

There are two reasons why in actual situations the assumptions made in this section are not satisfied:

(i) The system has aberrations, e.g. spherical aberration. The system may still be isoplanatic, but the magnitude of $H(\mathbf{k})$ will become less than unity.

(ii) the system is not fully isoplanatic: some aberrations violate isoplanacy: the shape of the diffraction pattern of a point source depends on the position of the source.

In the latter case we may apply our considerations to areas of the object plane for which the isoplanacy condition is reasonably well satisfied. These areas are called "isoplanatic patches".

Up till now we have assumed the light to be ***fully coherent***, because we assumed the amplitude and phase of the light to be fully determined in every point in space. In practice, this situation can be approximately realised by using laser light, but the deviations from full coherence are so significant that we have to investigate the influence of partial coherence of the incident illumination on the performance of optical systems.

5. SPATIAL AND TEMPORAL COHERENCE.

In actual situations the light field is a stochastic quantity. This is due to the fact that the times of emission of photons by the atoms is random, while the different atoms of the light source radiate independently. The fact that an excited state of an atom has a finite lifetime τ has as a consequence that the emitted spectral line has a finite width of the order of τ^{-1}. Also the spatial extent of the source manifests itself in the statistical properties of its light. I will discuss two extreme cases:

a. Temporal coherence.

Let us assume a pointlike light source that emits pulses of light. The light therefore has a certain spectral width. The pulses are directed towards an opaque screen with two apertures O_1 and O_2 (Fig. 5.1). O_1 and O_2 become secondary sources which generate an excitation at the point P in the plane of observation. If the distances O_1P and O_2P differ by an amount δ, where δ/c (c is the velocity of light) is larger than the pulse

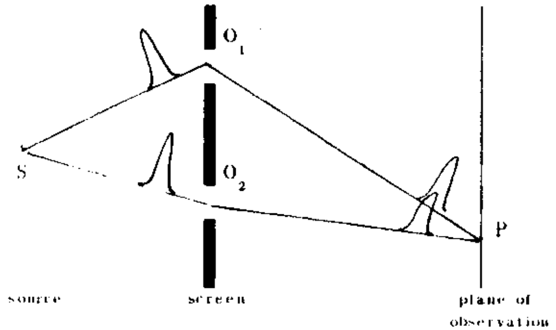

Fig. 5.1 Illustrating temporal coherence

duration τ then the pulses will not "see" each other in the point P of the plane of observation. The resulting intensity at P will be the sum of the intensities of the pulses emitted by O_1 and O_2. There will be no interference. If $\delta/c<\tau$ then interference will occur. This is expressed more quantitatively in terms of the visibility of the interference pattern from which we can determine the degree of temporal coherence of the light. We will not dwell further on this subject which is so important for spectroscopy.

The other extreme case is:

b. Spatial coherence.

Here we have a similar situation as in Fig.5.1 but now the source has a finite extent (Fig. 5.2). The source is assumed to be fully monochromatic and to be composed of atoms that radiate independently of each other. Let us consider two atoms S_1 and S_2 which radiate independently light of a frequency ω. Each atom will give rise to an interference pattern in the plane of observation. The patterns due to S_1 and S_2 are shifted with respect to each other. If the number of atoms becomes larger, the resulting interference pattern will be washed out almost completely, depending on the distance between O_1 and O_2 as we will see soon.

The calculation of the interference pattern can be done as follows: let u_1 and u_2 denote the fields generated by the secondary sources in O_1 and O_2, respectively at the point of observation. u_1 and u_2 are stochastic quantities, because they are the amplitudes resulting from the emissions of many atoms in the source. The resulting interference pattern is an average over the collection of radiating atoms. In this way we find for the observed intensity at P:

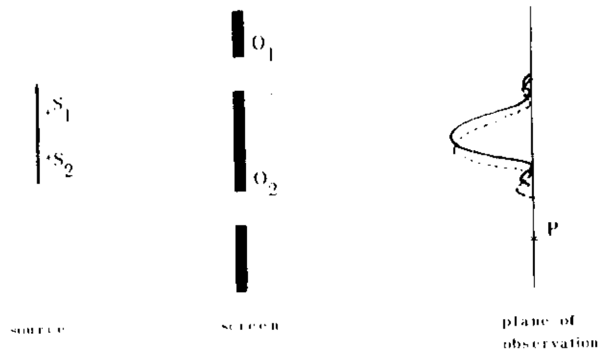

Fig. 5.2 Illustrating temporal coherence

$$
\begin{aligned}
I(P) &= \langle |u_1(P) + u_2(P)|^2\rangle \\
&= \langle |u_1(P)|^2\rangle + \langle |u_2(P)|^2\rangle + \langle u_1^*(P)u_2(P)\rangle + \langle u_1(P)u_2^*(P)\rangle
\end{aligned}
\tag{5.1}
$$

The angular brackets $\langle\,\rangle$ denote averaging over the source. The first two terms on the r.h.s. of eq. (5.1) are the intensities generated when only O_1 *or* O_2 are present. The last two terms describe the interference pattern as a function of the position of the point of observation P. They give information on the light source. The *degree of spatial coherence* $\gamma(\mathbf{r}_1,\mathbf{r}_2)$ between two points $\mathbf{r}_1$ and $\mathbf{r}_2$ in space measures the spatial correlation of the light field in the points $\mathbf{r}_1$ and $\mathbf{r}_2$ and is defined by:

$$
\gamma(\mathbf{r}_1,\mathbf{r}_2) = \frac{\langle u(\mathbf{r}_1)u^*(\mathbf{r}_2)\rangle}{\left[\, I(\mathbf{r}_1)I(\mathbf{r}_2)\right]^{1/2}} \tag{5.2}
$$

where $I(\mathbf{r}_1)$ and $I(\mathbf{r}_2)$ are the intensities at $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively. The interference pattern in P is described in terms of the degree of spatial coherence in the plane of the apertures O_1 and O_2 located at $\mathbf{r}_1$ and $\mathbf{r}_2$.

We will now discuss a famous relation between the characteristics of a source and the degree of spatial coherence in the far field of the source which is expressed by the *Van Cittert - Zernike theorem.*

We consider a planar source S which is described by its correlation function $\Gamma_S(\mathbf{r}_S,\mathbf{r}_S')$, where $\mathbf{r}_S$ and $\mathbf{r}_S'$ are any two points on S:

$$
\Gamma_S(\mathbf{r}_S,\mathbf{r}_S') = \langle u_S(\mathbf{r}_S)u_S^*(\mathbf{r}_S')\rangle \tag{5.3}
$$

We assume further the source to be statistically homogeneous, by which we mean that the coherence function only depends upon the relative position $\mathbf{r}_S-\mathbf{r}_S'$ of the two points $\mathbf{r}_S$ and $\mathbf{r}_S'$ and that the intensity across the source does not change "appreciably" over distances for which Γ is different from zero. This means that $\Gamma_S(\mathbf{r}_S,\mathbf{r}_S')$ can approximately be written in the form:

$$
\Gamma_S(\mathbf{r}_S,\mathbf{r}_S') = \gamma(\mathbf{r}_S-\mathbf{r}_S')\, I\left(\frac{\mathbf{r}_S+\mathbf{r}_S'}{2}\right) \tag{5.4}
$$

We now calculate the far field of S in the Fraunhofer region:

$$
u(\mathbf{r}) = \int_S d\mathbf{r}_S\, u_S(\mathbf{r}_S)\exp\left[\frac{-ik}{z}\,\mathbf{r}\cdot\mathbf{r}_S\right] \tag{5.5}
$$

Here $\mathbf{r}$ is a point in the plane z in the far field of S and $\mathbf{r}_S$ is a point on S. The far-field correlation function can now straightforwardly be calculated:

$$\langle u(\mathbf{r}_1)u^*(\mathbf{r}_2)\rangle = \int d\mathbf{r}_s \int d\mathbf{r}_s' \langle u_s(\mathbf{r}_s)u_s^*(\mathbf{r}_s')\rangle \exp\left[\frac{-ik}{z}(\mathbf{r}_1.\mathbf{r}_s-\mathbf{r}_2.\mathbf{r}_s')\right]$$

$$= \int d\mathbf{r}_s \int d\mathbf{r}_s' \gamma(\mathbf{r}_s-\mathbf{r}_s')\, I\left(\frac{\mathbf{r}_s+\mathbf{r}_s'}{2}\right)\exp\left[\frac{-ik}{z}(\mathbf{r}_1.\mathbf{r}_s-\mathbf{r}_2.\mathbf{r}_s')\right]$$

Introducing new coordinates $\mathbf{x} = \mathbf{r}_s-\mathbf{r}_s'$ and $\mathbf{X} = \frac{1}{2}(\mathbf{r}_s+\mathbf{r}_s')$ we find:

$$\langle u(\mathbf{r}_1)u^*(\mathbf{r}_2)\rangle = \int d\mathbf{x} \int d\mathbf{X}\, \gamma(\mathbf{x})\, I(\mathbf{X}) \exp\left[\frac{-ik}{z}\left\{\mathbf{X}.(\mathbf{r}_1-\mathbf{r}_2)+\frac{1}{2}\mathbf{x}.(\mathbf{r}_1+\mathbf{r}_2)\right\}\right] \tag{5.6}$$

The r.h.s of (5.6) is immediately recognised as the product of the Fourier transforms of γ and I:

$$\langle u(\mathbf{r}_1)u^*(\mathbf{r}_2)\rangle = \hat{I}(\mathbf{r}_1-\mathbf{r}_2)\, \hat{\gamma}\left(\frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)\right), \tag{5.7}$$

where $\hat{I}$ and $\hat{\gamma}$ are the Fourier transforms of I and γ, respectively. The result (5.7) is a generalisation of the Van Cittert–Zernike theorem as we will show in a moment. We see from eq. (5.7) that the far-field coherence function has the same form as the coherence function of the source as specified in eq. (5.4).

We therefore conclude that the far field coherence is described by the Fourier transform of the intensity distribution across the source, while the far-field intensity distribution is the Fourier transform of the source coherence function γ.

The Van Cittert–Zernike theorem deals with the special case of a completely incoherent source: $\gamma(\mathbf{r}_s-\mathbf{r}_s') = \delta(\mathbf{r}_s-\mathbf{r}_s')$. Eq.(5.7) then reduces to

$$\langle u(\mathbf{r}_1)u^*(\mathbf{r}_2)\rangle = \hat{I}(\mathbf{r}_1-\mathbf{r}_2) \tag{5.8}$$

The far-field correlation function is the Fourier transform of the source intensity distribution, which is the content of the Van Cittert–Zernike theorem.

Also without calculations we can understand why a completely incoherent source generates a partially coherent field. Let us consider two arbitrary source points S_1 and S_2. We study the field in two points in space P_1 and P_2. These points receive light *both* from S_1 and S_2 and therefore become correlated. This correlation is made more quantitative by the Van Cittert-Zernike theorem.

6. PROPAGATION OF COHERENCE.

We consider a general isoplanatic imaging system as depicted in Fig. 6.1. The transfer relation between input and output fields reads:

$$u_{out}(\mathbf{r}_o) = \int h(\mathbf{r}_o-\mathbf{r}_i)\, u_{in}(\mathbf{r}_i)\, d\mathbf{r}_i \tag{6.1}$$

For the relation between the correlation functions of the input and output distributions we find:

$$\langle u_{out}(\mathbf{r}_o)u^*_{out}(\mathbf{r}_o')\rangle = \int d\mathbf{r}_i \int d\mathbf{r}_i' \langle u_{in}(\mathbf{r}_i)u^*_{in}(\mathbf{r}_i')\rangle\, h(\mathbf{r}_o-\mathbf{r}_i)\, h^*(\mathbf{r}_o'-\mathbf{r}_i')$$

$$\Gamma_{out}(\mathbf{r}_o, \mathbf{r}_o') = \int d\mathbf{r}_i \int d\mathbf{r}_i'\, \Gamma_{in}(\mathbf{r}_i-\mathbf{r}_i')\, h(\mathbf{r}_o-\mathbf{r}_i)\, h^*(\mathbf{r}_o'-\mathbf{r}_i') \tag{6.2}$$

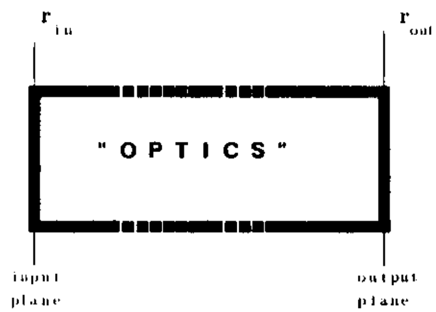

Fig. 6.1 A general imaging system

The transfer function of the correlation function Γ of the out-field is $h(\mathbf{r}_o-\mathbf{r}_i)\, h^*(\mathbf{r}_o'-\mathbf{r}_i')$.

The transfer functions describe the properties of the optical system; e.g. they also contain the aberrations, at least as long as the imaging is isoplanatic.

7. IMAGE FORMATION AS A SUCCESSION OF FOURIER TRANSFORMS.

We consider the system as sketched in Fig. 7.1. The distance between the object plane

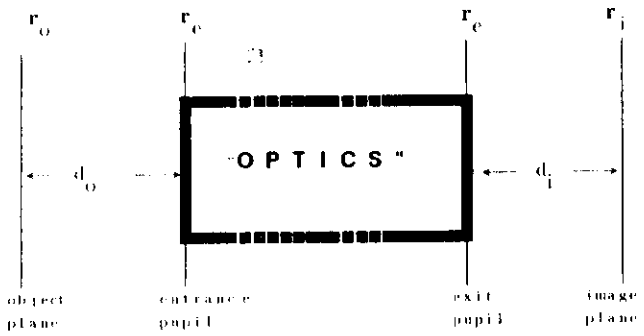

Fig. 7.1 Image formation

and the entrance pupil is d_o (the object distance), d_i is the distance between the exit pupil and the image plane (the image distance).

Wave propagation through this system proceeds as follows:

Assuming that the entrance pupil is in the far field of the object plane, the distribution in the entrance pupil is the Fourier transform of the complex amplitude in the object plane:

$$u_e(\mathbf{r}_e) = \frac{1}{i\lambda d_o}\int d\mathbf{r}_o\, u_o(\mathbf{r}_o)\exp\left[\frac{-ik}{d_o}\,\mathbf{r}_o\cdot\mathbf{r}_e\right] \qquad (7.1)$$

$\mathbf{r}_e$ is the coordinate in the entrance pupil. As the exit pupil is the geometric optic image of the entrance pupil we can use $\mathbf{r}_e$ also as the coordinate in the exit pupil. When choosing the units of length in the exit pupil appropriately, (7.1) also represents the complex amplitude in the exit pupil. The integration in (7.1) extends over the aperture in the object plane. If the image distance d_i is so large that the image plane is in the far field of the exit pupil, the complex amplitude in the image plane is given by:

$$u_i(\mathbf{r}_i) = \frac{1}{i\lambda d_i}\int_{-\infty}^{+\infty} d\mathbf{r}_e\, P(\mathbf{r}_e)\, u_e(\mathbf{r}_e)\exp\left[-\frac{ik}{d_i}\,\mathbf{r}_i\cdot\mathbf{r}_e\right]. \qquad (7.2)$$

where we now explicitly introduced the pupil function $P(\mathbf{r}_e)$ in order to emphasise the finite width of the aperture in the exit pupil. This allows us to take the limits of integration as $+\infty$ and $-\infty$: $P(\mathbf{r}_e) = 1$ inside the diaphragm and zero outside. Combining (7.1) and (7.2) we see that the transition from the object plane to the image plane

consists of two successive Fourier transforms. It is instructive to consider the case when there are no diaphragms in the system: $P(\mathbf{r}_e)=1$ everywhere. In such a case the imaging is perfect: the resolving power is ideal, because two successive Fourier transforms reproduce the object distribution up to a scale factor (which can be absorbed by choosing an appropriate scale for the coordinates). Aberrations can now easily be taken into account by incorporating them in the exit pupil function: replace $P(\mathbf{r}_e)$ by $P(\mathbf{r}_e)\exp[iW(\mathbf{r}_e)]$, where $W(\mathbf{r}_e)$ is the wave aberration function in the exit pupil.

8.RESOLVING POWER.

Let us first study a coherently and monochromatically illuminated object. If the system has no aberrations, the spatial frequency content of the complex image amplitude distribution is given by the Fourier transform of the complex image amplitude: $U_i(\mathbf{k})$, where $\mathbf{k}$ is the spatial frequency. According to eq. (7.2) $U_i(\mathbf{k})$ is proportional to the complex amplitude distribution in the exit pupil: $U_i(\mathbf{k}) = u_e(\mathbf{k})\,P(\mathbf{k})$, where $\mathbf{k} = \mathbf{r}_e/(\lambda d_i)$.

As the resolving power is defined by the highest transmitted spatial frequency, the resolving power is determined by the width of the exit pupil and the transmission function is unity for an aberration-free system. Aberrations degrade the performance of the system, leading to values of the transmission function smaller than unity. This means that the strength of these spatial frequencies is not fully transmitted. A qualitative example is sketched in Fig.8.1.

If, as another example, we consider the case of an incoherently illuminated object, we find from eq. (6.2) the following relation between the intensity distributions in the image and object plane, $I_i(\mathbf{r}_i)$ and $I_o(\mathbf{r}_o)$, respectively:

$$I_i(\mathbf{r}_i) = \int I_o(\mathbf{r}_o)\,|h(\mathbf{r}_i-\mathbf{r}_o)|^2\, d\mathbf{r}_o \qquad (8.1)$$

In this example, the object is characterised by its intensity distribution. Such a situation arises e.g. in fluorescence microscopy. The transmission function is now $|h(\mathbf{r}_i-\mathbf{r}_o)|^2$, the Fourier transform of which describes the transmission of the spatial frequencies. The corresponding transfer function is the Fourier transform of $|h|^2$, which is the auto correlation function of H:

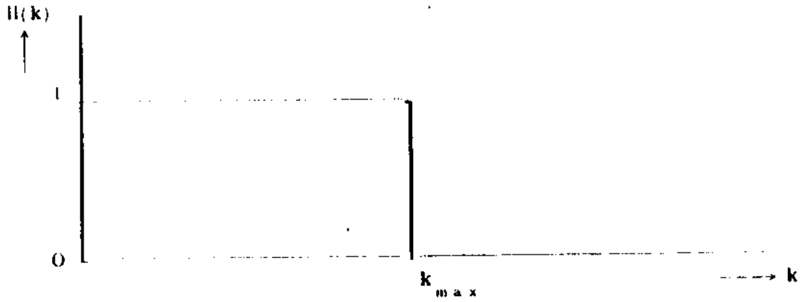

Fig. 8.1 Examples of transfer functions

$$AH(\mathbf{k}) = \int d\mathbf{k}' \, H(\mathbf{k}') \, H^*(\mathbf{k}+\mathbf{k}') \qquad (8.2)$$

For the case of ideal imaging ($H(\mathbf{k})=1$ for $|\mathbf{k}|<k_{max}$) $AH(\mathbf{k})$ is given by a triangular function sketched in Fig. 8.2. The cut-off is now $2k_{max}$, twice the value of the coherent case.

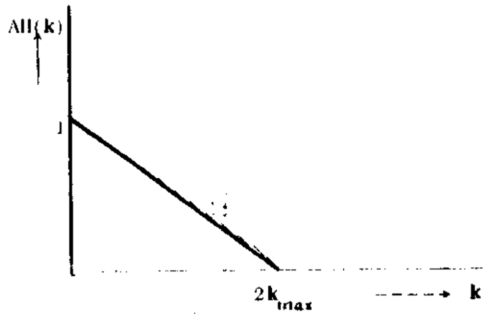

Fig. 8.2 Transfer function for an incoherently illuminated object

However, it is wrong to say that the resolving power obtained with incoherent illumination is twice the resolving power of the coherent case: in the coherent case we deal with the spatial frequencies in the complex *amplitude* distribution, while in the incoherent case we deal with the spatial frequencies in the *intensity* distribution.
The treatment of wave propagation for the case of partial coherence is a direct generalisation of the present treatment and will not be discussed here.

9. HOLOGRAPHY.

Holography is another way of recording an object. The methods of Fourier Optics can also be fruitfully applied here. A general arrangement for holographic recording has been sketched in Fig. 9.1. A fully coherent laser beam is split by a beam splitter into two beams, one of which is directed towards the object, the other being directed

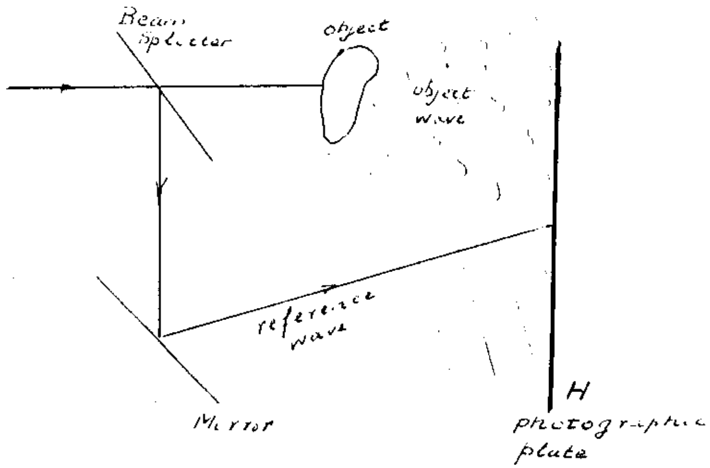

Fig. 9.1 A general arrangement for holographic recording

towards a mirror which becomes after reflection from the mirror the *reference beam*. The object scatters the incident beam and gives rise to an object wave. Reference beam and object wave have a common origin and consequently have a fixed phase relationship. If the beams are made to interfere on a photographic plate H an interference pattern is recorded. If the film is developed appropriately, the amplitude transmittance of the plate is proportional to the recorded interference pattern. The developed photographic plate is called the hologram and contains information about the object in a coded form. Let us look at the underlying formalism:
Let the object wave at the photographic plate be denoted by $o(\mathbf{r})$, where $\mathbf{r}$ denotes an

arbitrary point on H. The reference wave on H is $r(\mathbf{r})$. The total complex amplitude on H is

$$u(\mathbf{r}) = o(\mathbf{r}) + r(\mathbf{r}) \qquad (9.1)$$

The interference pattern on H is proportional to the absolute square of $u(\mathbf{r})$:

$$I(\mathbf{r}) = |\, o(\mathbf{r}) + o(\mathbf{r})\, |^2. \qquad (9.2)$$

where we omitted the proportionality constant. Evaluation of (9.2) gives:

$$I(\mathbf{r}) = |r(\mathbf{r})|^2 + |o(\mathbf{r})|^2 + o(\mathbf{r})r(\mathbf{r})^* + o^*(\mathbf{r})r(\mathbf{r}) \qquad (9.3)$$

The first term on the right hand side is a background term due to the reference wave. The remaining terms in (9.3) contain information on the object. This information is retrieved when we illuminate the hologram with a replica of the reference beam: the amplitude transmittance $t(\mathbf{r})$ is proportional to $I(\mathbf{r})$, where we will ignore the proportionality constant: the wave emerging from the hologram is now described by:

$$u(\mathbf{r}) = r(\mathbf{r})\, I(\mathbf{r})$$
$$= r(\mathbf{r})|r(\mathbf{r})|^2 + r(\mathbf{r})|o(\mathbf{r})|^2 + |r(\mathbf{r})|^2 o(\mathbf{r}) + r^2(\mathbf{r})o^*(\mathbf{r}) \qquad (9.4)$$

We will now interpret the different waves on the right hand side of eq. (9.4):

$r(\mathbf{r})|u(\mathbf{r})|^2$: if $r(\mathbf{r})$ is a plane wave, $r(\mathbf{r})=R_o \exp(i\mathbf{k}_r.\mathbf{r})$, or when $|r(\mathbf{r})|^2$ is constant, this terms is a replica of the reference wave, apart from a constant multiplicative factor.

$|r(\mathbf{r})|^2 o(\mathbf{r})$: if $r(\mathbf{r})$ is a plane wave or when $|r(\mathbf{r})|$ is constant, this term is a replica of the object wave. For this reason, holography is sometimes referred to as *object wave reconstruction*, a much better name. The observer receiving this wave will perceive the object as if he were standing in front of the object: he can move around and observe parallax. The observation is degraded by the other unwanted terms.

$r^2(\mathbf{r})o^*(\mathbf{r})$: this term gives rise to another type of reconstruction of the object, called the *twin image*. We will discuss this term in some more detail in a moment.

$r(\mathbf{r})|o(\mathbf{r})|^2$: has a non-linear relation with the object and does not have a directly appealing physical interpretation.

We will revisit the terms in more detail for the important particular case that the

reference wave is a plane wave $r(\mathbf{r}) = R_o \exp(i\mathbf{k}_r.\mathbf{r})$, where $\mathbf{k}_r$ is the wave vector.

It is important for the reconstruction of the object wave $o(\mathbf{r})$ from the total field $u(\mathbf{r})$ behind the hologram, that the different terms on the right hand side of (9.3) can be observed separately. We will show that this can be achieved by having the plane reference wave make a sufficiently large angle of incidence on the hologram. This has been fully appreciated for the first time by Leith and Upatnieks, who developed this technique which turns out to be the spatial counterpart of the carrier wave concept in the time domain well known from communication technique. This becomes clear when we study the Fourier transforms of the terms considered a moment ago:

$|r(\mathbf{r})|^2 o(\mathbf{r}) = R_o^2\, o(\mathbf{r})$ is a direct reconstruction of the object wave. Its Fourier transform is the spatial frequency spectrum of the object.

$r^2(\mathbf{r})o^*(\mathbf{r})$: taking the Fourier transform with respect to $\mathbf{r}$ we find its spatial frequency spectrum: $O^*(\mathbf{k}+2\mathbf{k}_r)$. Apart from complex conjugation this is the original spatial frequency spectrum shifted by an amount $2\mathbf{k}_r$. We will comment on the physical meaning of the complex conjugation in a moment.

$r(\mathbf{r})|o(\mathbf{r})|^2$: taking the Fourier transform leads to the *autocorrelation function* of the object spectrum shifted by an amount $\mathbf{k}_r$.

For a reliable object wave reconstruction we have to be able to observe the reconstructed object wave without cross-talk from the other terms. This can be done by choosing the offset angle of the reference beam appropriately: let the reference beam be a plane wave whose wave vector makes angles θ and ϕ with the normal on the plane of the hologram: $r(\mathbf{r}) = R_o \exp(i\mathbf{k}_r.\mathbf{r})$. The components of $\mathbf{k}_r$ are: ($k \sin\theta \cos\phi$, $k \sin\theta \sin\phi$, $k \cos\theta$), where $k=\omega n/c$. Let us continue the discussion with only one transversal coordinate, e.g. x. At the plane of the hologram we choose $z=0$ so that the reference wave is described there by the complex amplitude:

$$r(x) = R_o \exp(ik_r x) \qquad (9.5)$$

The spatial frequency spectrum of the object at H is determined from:

$$o(x) = (2\pi)^{-1/2} \int_{-W/2}^{W/2} O(k_x) \exp(ik_x x)\, dk_x \qquad (9.6)$$

where we assumed that the object has a finite bandwidth W.

The spatial frequency spectrum of $r^*(x)o(x)$ is now given by:

$$r^*(x)o(x) = (2\pi)^{-1/2}\int_{-W/2}^{W/2} O(k_x)\exp\left[i(k_x-k_r)x\right]dk_x$$

$$= (2\pi)^{-1/2}\int_{-W/2-k_r}^{W/2-k_r} O(\ell-k_r)\exp(i\ell x)\,d\ell \qquad (9.7)$$

The spectrum of this term is the same as the object spectrum shifted over an amount k_r.
We now calculate the spectrum of $o(x)o^*(x)$:

$$\mathcal{F}\{o(x)o^*(x)\} = (2\pi)^{-1/2}\int_{-\infty}^{+\infty} o(x)o^*(x)\exp(ikx)\,dx$$

$$= (2\pi)^{-3/2}\int_{-W/2}^{W/2} dk_x\int_{-W/2}^{W/2} dk'_x\; O(k_x)O^*(k'_x)\exp\left[i(k_x-k'_x+k)x\right]dx$$

$$= (2\pi)^{-1/2}\int_{-W/2}^{W/2} dk_x\int_{-W/2}^{W/2} dk'_x\; O(k_x)O^*(k'_x)\,\delta(k_x-k'_x+k)$$

$$= \int_{-W}^{+W} dk_x\, O(k_x)O^*(k_x+k)\,\theta(k_x+k), \qquad (9.8)$$

where $\theta(k)=1$ for $|k|<W/2$ and 0 otherwise.
The spatial frequency spectra of the terms discussed above have been sketched schematically in Fig. 9.2. We see that the spatial frequency spectra of the terms do

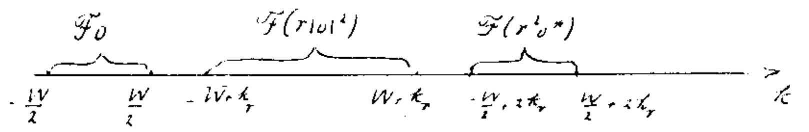

Fig 9.2 Spatial frequency spectra in the wave emerging from the hologram

not overlap if the following condition is satisfied

$$k_r > \frac{3}{2}W \qquad (9.9)$$

So the offset angle θ has to satisfy $\sin\theta > \frac{3c}{2\omega n}W$ (9.10)

We now want to explain how the different fields in the wave behind the hologram manifest themselves:
In particular, we want to investigate the physical meaning of the complex conjugate field $o^*(\mathbf{r})$. This can best be done for the special case of a point source. This is hardly a restriction, because a general object can be considered as a collection of point sources (see eq.(2.17)):
Let the object point source be located at the point $(\mathbf{r}_s, -z_s)$ (see Fig. 9.3). The object wave at a point $\mathbf{r}$ in the plane of the hologram is given by:
$o(\mathbf{r}, z=0) = a\exp(ikr)/kr$, where $r=[|\mathbf{r}-\mathbf{r}_s|^2+z_s^2]^{1/2}$. If we take into account the time dependence of the object wave, we have the object field: $\exp[i(kr-\omega t)]/kr$, which denotes a spherical wave diverging from S.
The field $o^*(\mathbf{r})$ is a field that propagates in the direction of the positive z-axis. At the hologram this corresponds to the following time-dependent field:

$$\bar{o}(\mathbf{r},t) = o^*(\mathbf{r})e^{-i\omega t} = a^*\exp\left[-i(kr+\omega t)\right]/kr.$$

This field corresponds to a converging spherical wave which converges to the point $(\mathbf{r}_s, z_s)$. Now we understand that the reconstructed object field corresponds to a virtual object, while the twin image corresponds to a real object as shown in Fig. 9.3.

Fig. 9.3 The wave fields behind the hologram

The two images are each other's mirror images with respect to the plane of the hologram.
The non linear term does not have a simple physical interpretation.

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WAVE PROPAGATION AND FOURIER OPTICS

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March 199[illegible]

1. TO BE DISCUSSED

- wave propagation through uniform space
- diffraction from apertures
- image formation by thin lenses transfer function
- image formation by a general optical system
- coherence
- basic principles of holography

2. WAVE PROPAGATION. DIFFRACTION

Wave equation:

$$\nabla^2 \vec{e} - \frac{1}{c^2}\frac{\partial^2 \vec{e}}{\partial t^2} = 0 \qquad \vec{e}\text{: electric field}$$

Scalar version:

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

Monochromatic light: freq. ω

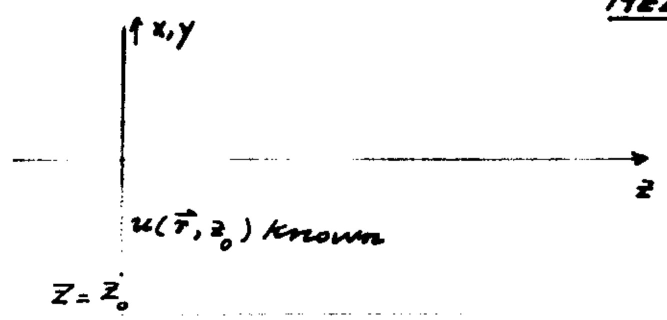
$$u(\vec{r},t) = u(\vec{r})e^{-i\omega t} + u^*(\vec{r})e^{i\omega t}$$

$u(\vec{r})$ to be solved from:

$$(\nabla^2 + k_0^2)\,u(\vec{r}) = 0 \quad ; \quad k_0 = \frac{\omega}{c}$$

Helmholtz eq.

A. PROPAGATION THROUGH HOMOGENEOUS MEDIUM.



To be calculated: $u(\vec{r}, z)$ from $u(\vec{r}, z_0)$

Fourier transf. w.r.t. $\vec{r}$:

$$u(\vec{r}, z_0) = \frac{1}{2\pi} \int d\vec{k}_t \; U(\vec{k}_t, z_0)\, e^{i\vec{k}_t \cdot \vec{r}}$$

$$\vec{r} = (x, y) \; ; \; \vec{k}_t = (k_x, k_y)$$

Plane-wave solutions of Helmh. eq.:

$$e^{i(\vec{k}_t \cdot \vec{r} + k_z z)}$$

$$k_z^2 = k_0^2 - k_t^2$$

H. eq. is a linear equation:

$$u(\vec{r}, z) = \frac{1}{2\pi} \int d\vec{k}_t \; U(\vec{k}_t, z_0)\, e^{i[\vec{k}_t \cdot \vec{r} + k_z (z - z_0)]}$$

$$k_z = \sqrt{k_0^2 - k_t^2}$$

$k_t \leq k_0$: homogeneous waves

$k_t > k_0$: evanescent waves
(decay exponentially)

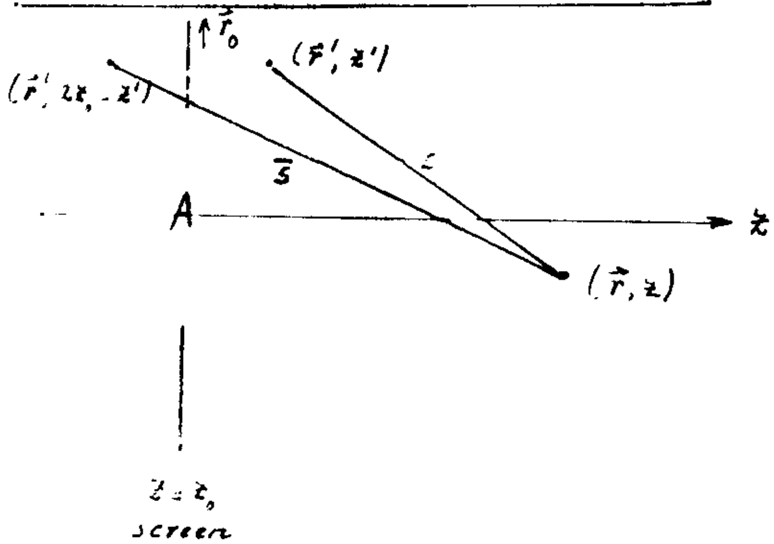
$$U(\vec{k}_t, z_0) = \frac{1}{2\pi} \int d\vec{r} \; u(\vec{r}, z_0)\, e^{-i\vec{k}_t \cdot \vec{r}}$$

$u(\vec{r}, z)$ can now be expressed in $u(\vec{r}, z_0)$:

$$u(\vec{r}, z) = \int \underbrace{h(\vec{r} - \vec{r}\,')}_{\text{transfer function}} u(\vec{r}\,', z_0)\, d\vec{r}\,' \qquad h(\vec{r}) \otimes u(\vec{r}, z_0)$$

$$h(\vec{r} - \vec{r}\,') = \frac{1}{(2\pi)^2} \int d\vec{k}_t \; \exp\left[i\vec{k}_t \cdot (\vec{r} - \vec{r}\,') + i k_z (z - z_0)\right]$$

B. DIFFRACTION FROM APERTURE



Kirchhoff approximation
for a perfectly conducting screen

$u = 0$ on opaque part of screen

$u = u(\vec{r}, z_0)$ in aperture A, the same as when screen were absent $(d \gg \lambda)$

$$u(\vec{r},z) = \int_A \frac{\partial \mathcal{G}(\vec{r},z,\vec{r}\,',z_0)}{\partial z_0}\, u(\vec{r}\,',z_0)\, d\vec{r}\,'$$

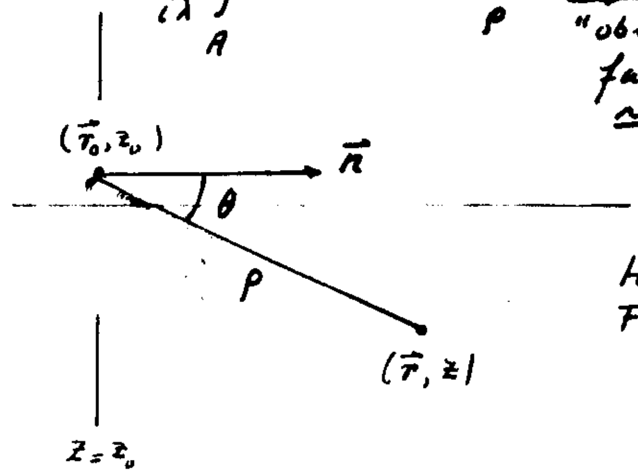
where

$$\mathcal{G}(\vec{r},z,\vec{r}\,',z') = \frac{1}{4\pi}\left[\frac{e^{iks}}{s} - \frac{e^{ik\bar{s}}}{\bar{s}}\right]$$

Rayleigh-Sommerfeld's Green's function

$$u(\vec{r},z) = \frac{1}{i\lambda}\int_A d\vec{r}_0\, u(\vec{r}_0,z_0)\,\frac{e^{ik\rho}}{\rho}\,\underbrace{\cos\theta}$$

"obliquity factor" $\simeq 1$



Huygens-Fresnel

$$\rho = \sqrt{|\vec{r}-\vec{r}_0|^2 + (z-z_0)^2}$$

Far-field $\quad \rho \simeq |z-z_0| + \frac{1}{2}\frac{|\vec{r}-\vec{r}_0|^2}{|z-z_0|}$

$$u(\vec{r},z) = \frac{e^{ik|z-z_0|}}{i\lambda|z-z_0|}\int_A d\vec{r}_0\, \exp\left[\frac{ik|\vec{r}-\vec{r}_0|^2}{2|z-z_0|}\right] \times u(\vec{r}_0,z_0)$$

Fresnel approximation

$$|\vec{r}-\vec{r}_0|^2 = r^2 - 2\vec{r}\cdot\vec{r}_0 + \cancel{r_0^2}$$

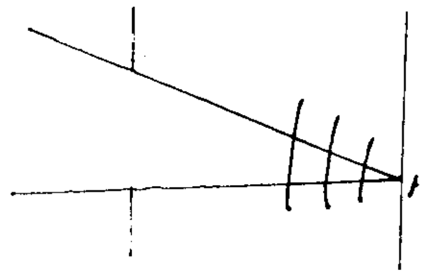
$$u(\vec{r},z) = \frac{e^{ik|z-z_0|}}{i\lambda|z-z_0|} \cdot \exp\left[\frac{ikr^2}{2|z-z_0|}\right] \cdot$$

const. phase/ampl. factor $\qquad$ phase curvature

$$\cdot \int_A d\vec{r}_0\, u(\vec{r}_0,z_0)\exp\left[\frac{-ik\vec{r}\cdot\vec{r}_0}{|z-z_0|}\right]$$

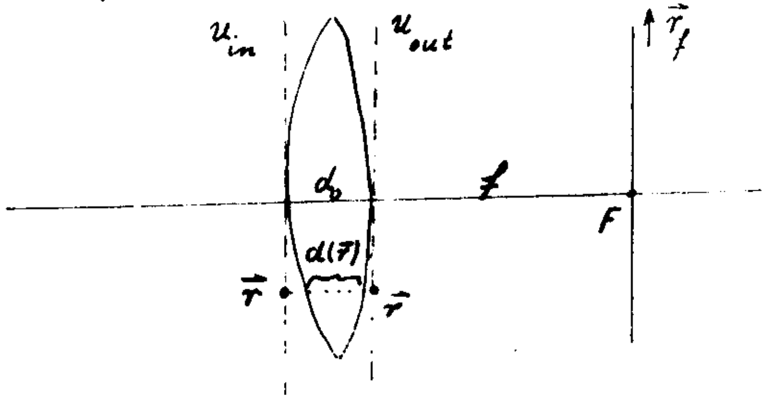
Fourier transform

Fraunhofer approximation



compensation of phase curvature

3. IMAGE FORMATION BY THIN LENSES



$$u_{out}(\vec{r}) = t(\vec{r})\, u_{in}(\vec{r})$$; $t(\vec{r})$ independent from u_{in}!

$t(\vec{r})$: independent of incident field

$$t(\vec{r}) = \exp\left[ik(n-1)d(\vec{r})\right]$$

$$d(\vec{r}) = d_0 - R_1 + \sqrt{R_1^2 - r^2} + R_2 - \sqrt{R_2^2 - r^2}$$

thickness function

$$\approx d_0 - \frac{1}{2} r^2 \left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$u_{out}(\vec{r}) = e^{ik(n-1)d_0} \exp\left[\frac{-ik}{2f} r^2\right] u_{in}(\vec{r})$$

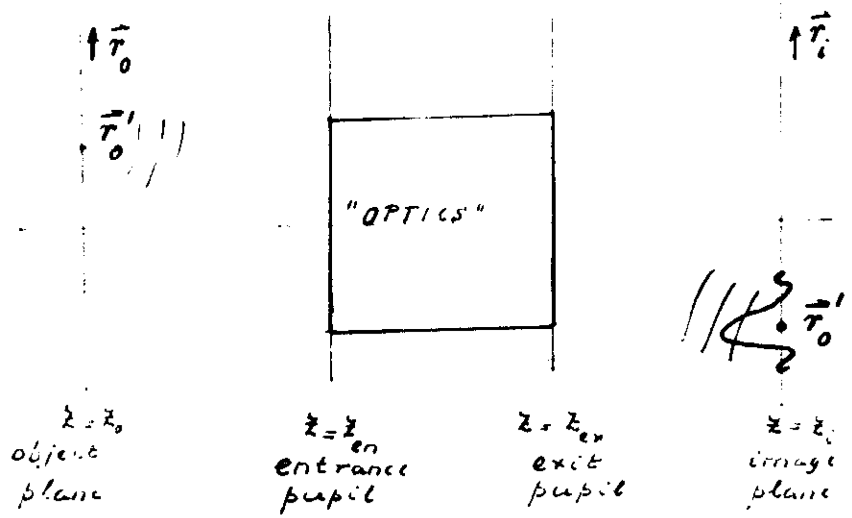
Wave field in back focal plane:

$$u_f(\vec{r}_f) = \frac{\exp\left[\frac{ik\, r_f^2}{2f}\right]}{i\lambda f} \int_A d\vec{r}\, u_{out}(\vec{r}) \exp\left[\frac{-ik}{f}\, \vec{r}\cdot\vec{r}_f\right]$$

$$u_f(\vec{r}_f) = \frac{e^{i(k)(n-1)d_0}}{i\lambda f} \int_A u_{in}(\vec{r}) \exp\left[\frac{-ik}{f}\, \vec{r}\cdot\vec{r}_f\right] d\vec{r}$$

$$u_f(\vec{r}_f) \;(:)\; \mathcal{F} u_{in}$$

4. IMAGE FORMATION IN GENERAL



exit pupil is geom. opt. image of entrance pupil

$$u_i(\vec{r}_i) = \int d\vec{r}_0\; h(\vec{r}_i - \vec{r}_0)\, u_0(\vec{r}_0)$$ [illegible]

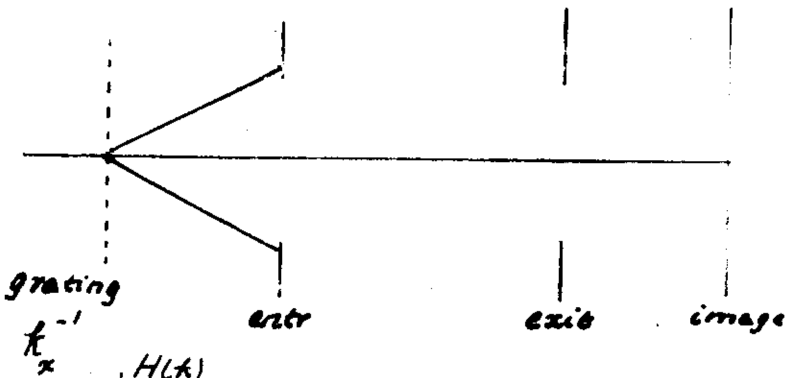
image amplitude — transfer function "point spread function" — object amplitude

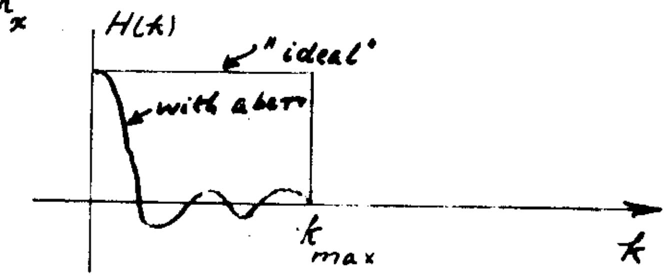
[illegible]

Fourier transform:

$$U_i(\vec{k}) = \underbrace{H(\vec{k})}_{\text{transfer function}} U_o(\vec{k})$$

Abbe:





k_{max} : resolving power

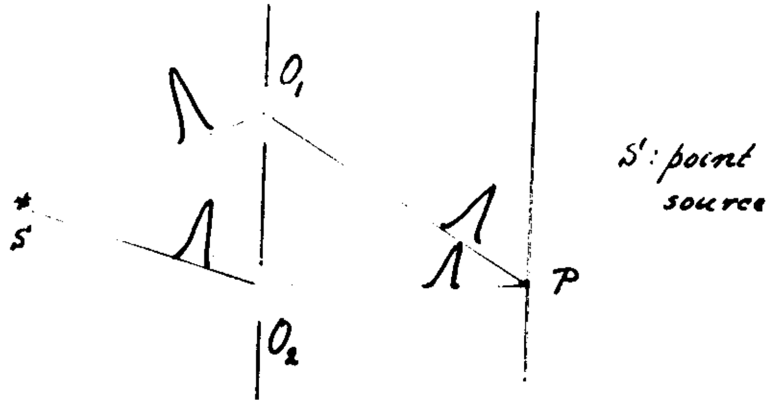
considered from entrance pupil: due to acceptance angle of entrance p.

Considered from exit pupil: highest transmitted spatial frequency is determined by the width of the exit pupil.

5. TEMPORAL AND SPATIAL COHERENCE

A. Temporal coherence

Atoms emit random light pulses → spectral width τ^{-1}
τ = duration light pulse



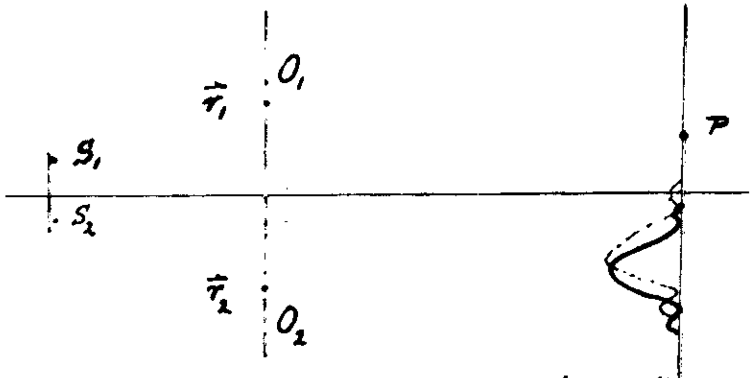
$\delta = SO_1P - SO_2P$

$\frac{\delta}{c} < \tau$: interference

$\frac{\delta}{c} > \tau$: no interference

Interference pattern gives information on spectrum of the light source.

B. SPATIAL COHERENCE



Assume a collection of monochromatic light sources S_j.
Radiating atoms may or may not be correlated.

$$S_j \rightarrow u_{1j}(\vec{r}_1) \longrightarrow u_{1j}(P)$$

$$u_{2j}(\vec{r}_2) \longrightarrow u_{2j}(P)$$

Interference pattern in P:

$$I(P) = \langle \Big| \sum_j \{ \underbrace{u_{1j}(P)}_{u_1} + \underbrace{u_{2j}(P)}_{u_2} \} \Big|^2 \rangle$$

$$= \langle |u_1|^2 \rangle + \langle |u_2|^2 \rangle + \langle u_1^* u_2 \rangle + \langle u_1 u_2^* \rangle$$

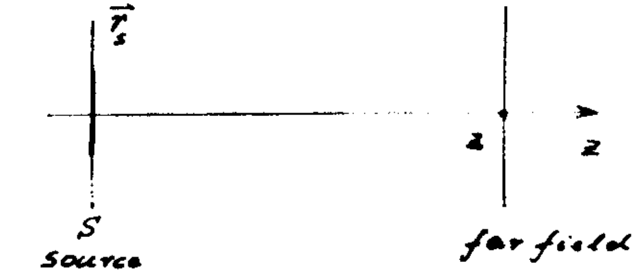
$\langle \ldots \rangle$: ensemble average

$$\gamma(\vec{r}_1, \vec{r}_2) \overset{df}{=} \frac{\langle u_0(\vec{r}_1) u_0^*(\vec{r}_2) \rangle}{\sqrt{I(\vec{r}_1) I(\vec{r}_2)}} \; ; \; \gamma(\vec{r}_1, \vec{r}_1) = 1$$

spatial degree of coherence

$\gamma(\vec{r}_1, \vec{r}_2)$ measures correlation between the values of the field at $\vec{r}_1$ and $\vec{r}_2$.

VAN CITTERT – ZERNIKE THEOREM

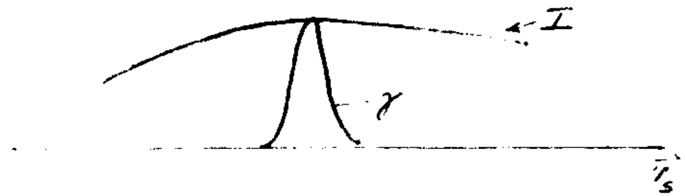


Source correlation function

$$\Gamma(\vec{r}_s, \vec{r}_s') = \langle u_s(\vec{r}_s) u_s^*(\vec{r}_s') \rangle$$

$$= \gamma(\vec{r}_s - \vec{r}_s') \; I\left(\tfrac{1}{2}(\vec{r}_s + \vec{r}_s')\right)$$

statistically homogeneous source



In far field:

$$u(\vec{r}) = \int_S d\vec{r}_s \; u_s(\vec{r}_s) \exp\left[-\frac{ik}{z} \vec{r} \cdot \vec{r}_s\right]$$

Far-field correlation function:

$$\Gamma(\vec r_1, \vec r_2) = \langle u(\vec r_1, z)\, u^*(\vec r_2, z)\rangle$$

$$= \int d\vec x \int d\vec X\, \gamma(\vec x) I(\vec X) \times \qquad \vec x = \vec r_s - \vec r_s{}' \quad \vec X = \tfrac{1}{2}(\vec r_s + \vec r_s{}')$$

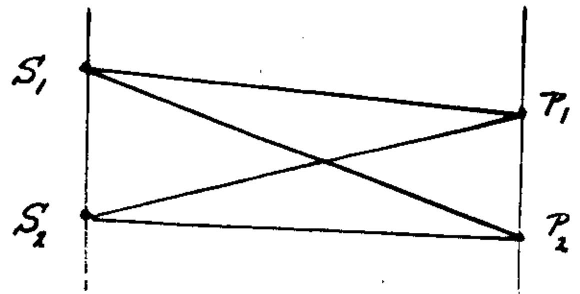
$$\times \exp\left[-\frac{ik}{z}\left\{\vec X \cdot (\vec r_1 - \vec r_2) + \tfrac{1}{2}\vec x \cdot (\vec r_1 + \vec r_2)\right\}\right]$$

$$= \hat I(\vec r_1 - \vec r_2)\, \hat\gamma\left(\tfrac{1}{2}(\vec r_1 + \vec r_2)\right)$$

FF correlation (:)	$\mathcal{F} I$
FF intensity (:)	$\mathcal{F} \gamma$

Van Cittert-Zernike deals with the special case of a fully incoherent source:

$$\gamma(\vec r_s - \vec r_s{}')\ (:)\ \delta(\vec r_s - \vec r_s{}')$$



The incoherent sources S_1 and S_2 produce correlated vibrations at P_1 and P_2.

6. PROPAGATION OF COHERENCE

$$u_{out}(\vec r_o) = \int h(\vec r_o - \vec r_i)\, u_{in}(\vec r_i)\, d\vec r_i$$

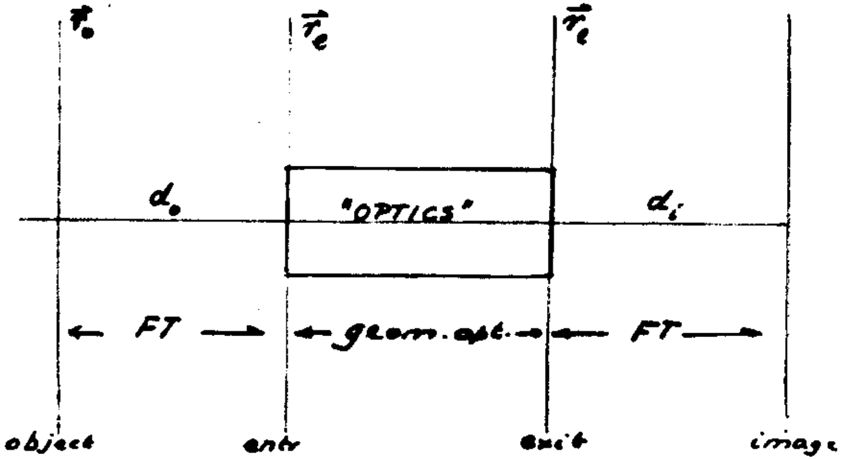
$$\Gamma_{out}(\vec r_o - \vec r_o{}') = \langle u_{out}(\vec r_o)\, u^*_{out}(\vec r_o{}')\rangle$$

$$= \int d\vec r_i \int d\vec r_i{}'\, h(\vec r_o - \vec r_i)\, h^*(\vec r_o{}' - \vec r_i{}')\, \Gamma_{in}(\vec r_i - \vec r_i{}')$$

In particular, for the intensity in the "out"-plane: $(\vec r_o = \vec r_o{}')$

$$I_{out}(\vec r_o) = \int d\vec r_i \int d\vec r_i{}'\, h(\vec r_o - \vec r_i)\, h^*(\vec r_o - \vec r_i{}')\, \Gamma_{in}(\vec r_i - \vec r_i{}')$$

7. IMAGE FORMATION AS A SEQUENCE OF FOURIER TRANSFORMS.



$$u_e(\vec r_e) = \frac{1}{i\lambda d_o} \int d\vec r_o \; u_o(\vec r_o) \exp\left[-\frac{ik}{d_o}\, \vec r_e \cdot \vec r_o\right]$$

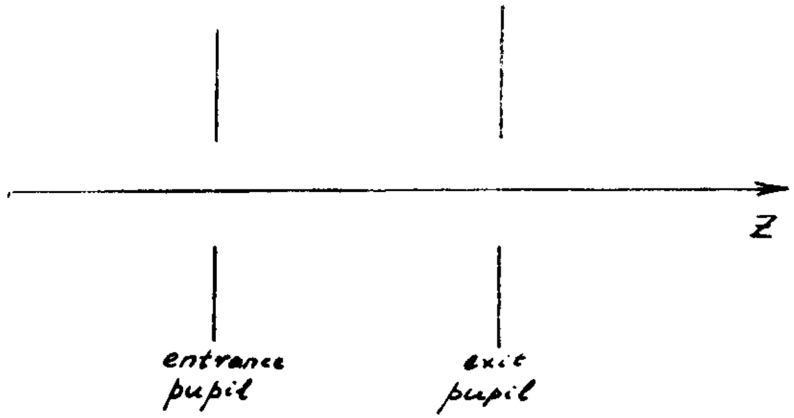
$$u_i(\vec r_i) = \frac{1}{i\lambda d_i} \int_{-\infty}^{\infty} P(\vec r_e)\, u_e(\vec r_e) \exp\left[-\frac{ik}{d_i}\, \vec r_e \cdot \vec r_i\right] d\vec r_e$$

$$\left.\begin{array}{ll} P(\vec r_e) = 1 & \text{for } \vec r_e \text{ in exit pupil} \\ \quad\;\; = 0 & \text{else} \end{array}\right\} \text{"pupil function"}$$

Aberrations : $P(\vec r_e) \to P(\vec r_e)\exp\left[i W(\vec r_e)\right]$

wave aberration

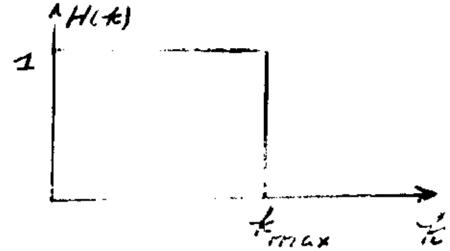
8. RESOLVING POWER



Resolving power :

- limited acceptance angle of entrance pupil
- finite width of exit pupil.

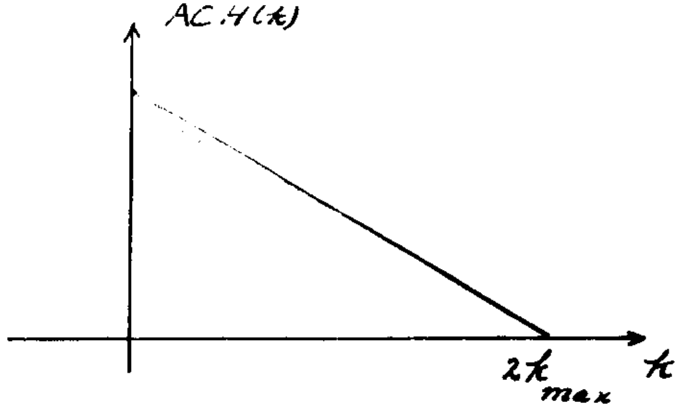
Transfer function for coherent illumination without aberrations :



Consider fully incoherently illuminated object (or fluorescing object). Then the image and object intensities are related by :

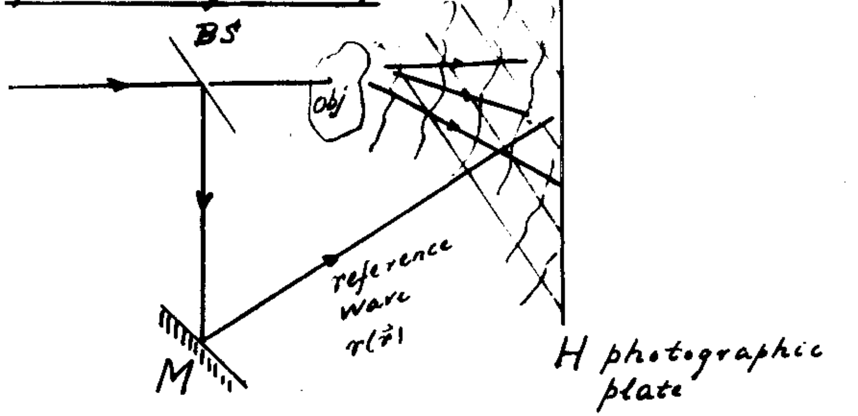
$$I_i(\vec{r}_i) = \int I_0(\vec{r}_0)\,|h(\vec{r}_i - \vec{r}_0)|^2\, d\vec{r}_0$$

Transfer function in spatial frequency space is the autocorrelation function of $H(k)$, which is a triangular function:



<u>Not</u> true: resolution for incoherent object is twice the resolution of a coherent object.

9. HOLOGRAPHY



On the holographic plate H we have the total complex amplitude

$$u(\vec{r}) = \underbrace{o(\vec{r})}_{\text{object wave}} + \underbrace{r(\vec{r})}_{\text{reference wave}} \; ; \; \vec{r} \in H$$

Interference pattern on H:

$$I(\vec{r}) = |o(\vec{r}) + r(\vec{r})|^2$$

$$= |r(\vec{r})|^2 + |o(\vec{r})|^2 + o(\vec{r})\,r^*(\vec{r}) + o^*(\vec{r})\,r(\vec{r})$$

Development of photographic plate $\longrightarrow$ hologram

Amplitude transmission function:

$$t(\vec{r}) = I(\vec{r}) = |r|^2 + |o|^2 + o^* r + o\, r^*$$

Illuminate hologram with replica of reference wave: wave field behind the hologram is (on the hologram)

$$u(\vec{r}) = t(\vec{r})\, r(\vec{r}) \qquad \vec{r} \in H$$

$$= |r|^2 \vec{r} + |o|^2 r + |r|^2 o + o^* r^2$$

We take a plane wave for the ref. wave:

$$r(\vec{r}) = R_0 \exp(i\vec{k}_r \cdot \vec{r}) \quad ; \vec{r} \in H$$

$$u(\vec{r}) = \underbrace{R_0^2 r(\vec{r})}_{I} + \underbrace{R_0 \exp(i\vec{k}_r \cdot \vec{r})\, |o(\vec{r})|^2}_{II} +$$

$$+ \underbrace{R_0^2\, o(\vec{r})}_{III} + \underbrace{R_0^2 \exp(2i\vec{k}_r \cdot \vec{r})\, o^*(\vec{r})}_{IV}$$

Contributions I – IV have to be observed separately $\rightarrow$ off-axis holography

III: <u>reconstruction of object wave $o(\vec{r})$</u>

holography = object wave reconstruction

I: reproduction of reference wave

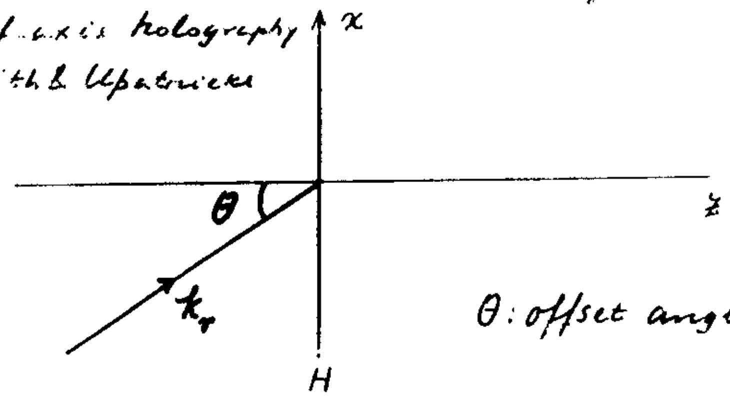
Contributions II + IV can best be studied by considering their spatial frequency spectrum.

Spatial frequency spectrum of object:

$$o(x) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{W}{2}}^{\frac{W}{2}} O(k_x)\, e^{ik_x x}\, dk_x$$

W = bandwidth of object

Off-axis holography
Leith & Upatnieks



θ: offset angle

For the calculation of the spatial frequency content of the contributions II and IV we need the following properties of Fourier transforms:

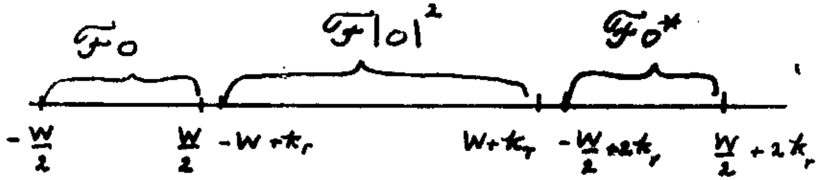
If $\mathcal{F} o(x) = O(k)$, then

$$\mathcal{F}[e^{i\alpha x} o(x)] = O(k-\alpha)$$

$$\mathcal{F} o^*(x) = O^*(-k^*)$$

The supports of the spatial frequency spectra can now easily be sketched:

Supports of spatial frequency spectra of contributions I – IV :



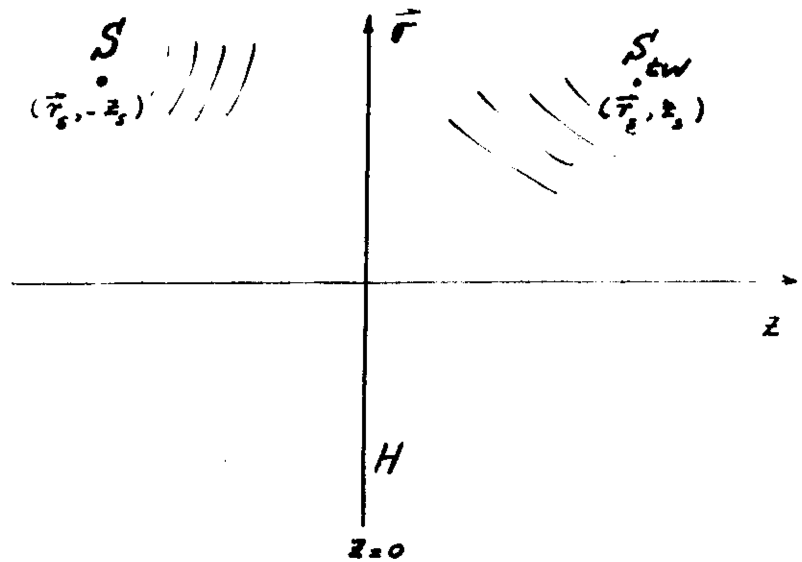
Contributions are separated if:

$$\frac{W}{2} < -W + k_r \longrightarrow k_r > \frac{3}{2} W$$

$$k_r = k \sin\theta = \frac{\omega n}{c} \sin\theta$$

$$\sin\theta > \frac{3c}{2\omega n} W$$

<u>Twin images</u> : O and O^*



Let original object be a point source:

$$\underline{O(\vec{r}, z=0)} = a \frac{e^{ik\rho}}{\rho} \quad ; \ \rho = \sqrt{|\vec{r} - \vec{r}_s|^2 + z_s^2}$$

Include time-dependence $\exp(-i\omega t)$:

yields outgoing spherical wave, coming from S:

$$a \frac{e^{i(kr - \omega t)}}{r}$$

$$r = \sqrt{|\vec{r} - \vec{r}_s|^2 + (z + z_s)^2}$$

The object is virtual.

The other term is

$$O^*(\vec{r}, z=0) = a\,\frac{e^{-ik\rho}}{\rho}$$

Inserting the time dependence $e^{-i\omega t}$

we find the wave

$$a\,\frac{e^{-i(kr+\omega t)}}{r}$$

which is a spherical wave converging to

the real object S_{tw}.