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THE QUANTUM HALL AND FRACTIONAL QUANTUM HALL EFFECT

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Abstract

A review of the current understanding of the quantum Hall effect and fractional quantum Hall effect is presented.

Contents

1 Introduction

The range of phenomena observed in electronic systems in magnetic fields is large and spectacular. Perhaps most spectacular of all are the quantum Hall effect, (QHE), and fractional quantum Hall Effect. These three lectures will be concerned with the theoretical description of these phenomena. Many of the aspects presented are treated in greater detail in the The Quantum Hall Effect [1987].

The plateaux observed at low temperatures in the Hall resistance of semiconductor heterostructures at multiples $1/i$ of h/e^2, with i integer, [von Klitzing et al 1980] and the related drastic reduction in the longitudinal resistance are known as the quantum Hall effect. The QHE is a macroscopic quantum phenomenon. The QHE can be understood in terms of the energy levels for a single

electron or quasiparticle in the presence of a magnetic field, host lattice and electric field and the interplay between them.

Plateaux are also observed in the Hall resistance of some semiconductor heterostructures at rational multiples m of h/e^2, [Tsui et al 1982]. These and the related anomalies in the longitudinal resistance are known as the fractional quantum Hall effect. Again this is a macroscopic quantum phenomenum, but unlike the QHE cannot be understood in terms of a modified one-electron picture.

The observation of the fractional quantum Hall effect implied an unsuspected role for many-body effects and at first appeared mysterious. The initial mystery has now been more or less resolved, although the theory of the effect is certainly not complete. The most widely accepted explanation invokes the existence of a correlated incompressible ground state, which is preferentially stabilized close to certain magnetic field strengths.

The quantum Hall Effect and the fractional quantum Hall effect are presented in the next two sections. Standard results for a single electron in a magnetic field are included as appendices.

2 The Quantum Hall Effect

The quantum Hall effect, QHE, discovered by von Klitzing et al [1980], is now reasonably well understood as a macroscopic phenomenon. This section reviews the basis for our understanding of the QHE.

2.1 The Measurement

Plateaux in the Hall resistance of suitably doped inversion layers and heterostructures are observed at low temperatures when plotted as a function either of applied magnetic field or of particle density, Fig.1. At the same time the longitudinal resistivity appears to vanish, dropping to values seven orders of magnitude or more smaller than at magnetic fields at which there are no plateaux.

The experiments are usually performed in a geometry similar to the one shown in Fig. 2. I will use the orientation of the x, y and z axes shown in Figure 2.

2.2 Interpretation of the Measurement

Not surprisingly the observation of plateaux in the Hall resistance of a system in a magnetic field says something fundamental about that system. Quite what, was first stated by Laughlin [1981].

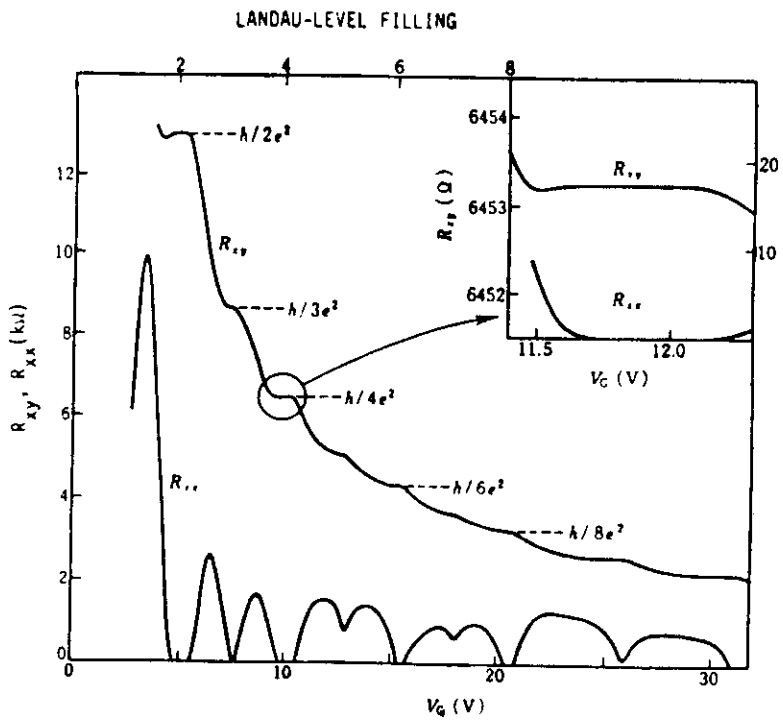

Figure 1: The Hall resistivity, ρ_{xy}, and longitudinal resistivity, ρ_{xx}, of a Si-MOSFET, as a function of applied gate voltage V_g. V_g controls the number density of electrons in the quasi two-dimensional electron gas in the channel region of the transistor, [after von Klitzing 1982].

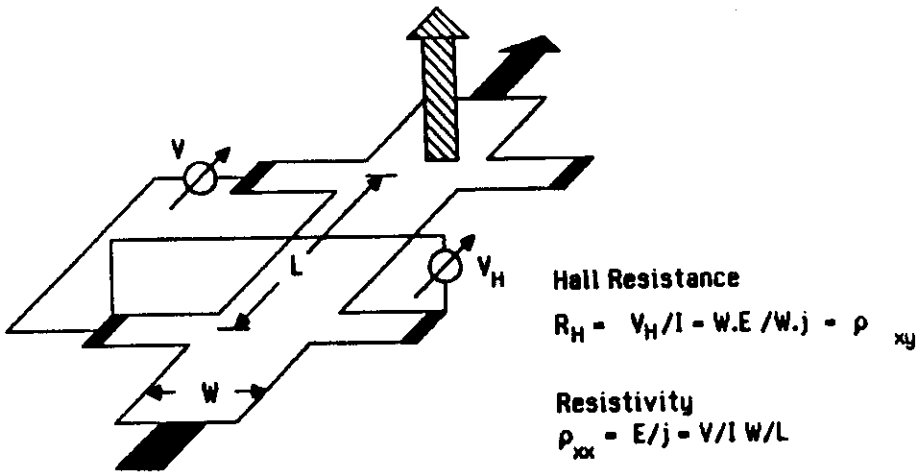

Figure 2: A standard sample configuration for the Hall measurement. [After Störmer and Tsui 1983].

Laughlin deduced from the implications of a Gedankenexperiment, outlined below, that the Hall response, σ_{xy} , of a system of charged particles must be quantized in units of e^2/h, if the Fermi energy lies in a mobility gap. Alternatively one may state that the observation of a quantized Hall resistance implies the existence of a mobility gap. By itself the original argument of Laughlin does not give the level quantization, i.e. it does not give the value of n where

$$\sigma_{xy} = ne^2/h. \tag{1}$$

This quite general result coupled with hindsight explains why some approximate treatments like that of Ando et al [1975] appeared to anticipate the discovery of the Quantum Hall Effect. Irrespective of the details of any calculation or approximation scheme the response to crossed electric and magnetic fields the Hall conductivity, σ_{xy}, can only take values consistent with equation 1 when the Fermi energy is in a mobility gap.

2.3 Laughlin's Gedankenexperiment

Suppose a system of electrons is confined on the surface of a cylinder with a constant magnetic field, B, perpendicular to the cylinder surface at all points, Figure 3. Assume also that the total flux threading the 'circuit' of the cylinder be Φ_x, but that none of the flux lines of Φ_x come close to the surface of the

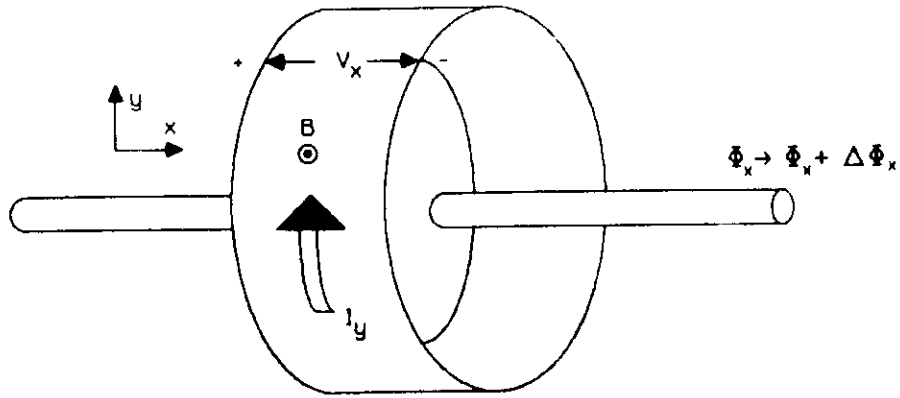

Figure 3: Laughlin's [1981] Gedankenexperiment. With the Fermi energy in a mobility gap, the Voltage V_x induces a current I_y to flow around the ring. By considering the gauge change $\Phi_x \rightarrow \Phi_x + \Delta\Phi_x$ one can deduce that V_x/I_y takes only quantized values.

cylinder. Let the application of the transverse voltage, V_x, induce a current to flow round the ring as shown.

Imagine causing the flux, Φ_x, to be changed adiabatically by an amount $\Delta\Phi_x = h/e$. The change in Φ_x implies a change in the vector potential for the electrons on the cylinder. The change in vector potential,

$$\mathbf{A} \rightarrow \mathbf{A} - \frac{h}{e}\frac{\hat{\mathbf{y}}}{L}(= \mathbf{A} + \frac{\hbar}{e}\nabla\theta), \tag{2}$$

corresponds to the gauge change (see Appendix 2, ($q = -e$))

$$\psi \rightarrow \psi exp(-i\theta(x,y)) = exp(-2\pi i\frac{y}{L})\psi. \tag{3}$$

But the gauge change accompanying the change, $\Phi_x \rightarrow \Phi_x + \Delta\Phi_x$, does not change the energy eigen values of H_0 and so the physical properties of the system must be the same after the gauge change as they were before. When the Fermi energy of the system lies within a band of localized states the occupied bands of delocalized states before the gauge change must be occupied afterwards. The occupation numbers of the localized states certainly do not change. The only possible change is in the occupation numbers of the edge states, i.e. particles can be transferred from one edge to the other.

Suppose n particles with charge q are transferred from one edge to the other,

then the potential energy of the system is increased by ΔU,

$$\Delta U = nqV_x, \tag{4}$$

as a result of the gauge change. As the change was effected adiabatically this energy must compensate exactly the work done by the system, which by Faraday's law is ΔW, with

$$\Delta W = -\int dt I_y \frac{d\Phi}{dt} = -I_y \frac{h}{e}. \tag{5}$$

So

$$0 = \Delta U + \Delta W = nqV_x - I_y \frac{h}{e},$$

or

$$I_y/V_x = -\sigma_{xy} = n\frac{qe}{h}. \tag{6}$$

Equation 6 is Laughlin's result. For electrons or holes, $\mid q \mid = e$, the Hall response $\mid \sigma_{xy} \mid = ne^2/h$, with n integer. This quantization of the Hall resistance followed directly from the assumption that the Fermi energy was within some mobility gap.

2.4 Aspects of a Microscopic Theory of the Quantum Hall Effect

The motion of electrons or holes in the inversion layers or at the interface in a semiconductor heterostructure is essentially two-dimensional because the motion perpendicular to the interface is quantized into discrete levels. The behaviour of the system is then determined by the motion in the plane of the interface. One may write an effective Hamiltonian, H_0, describing the motion of the particles in the plane of the interface with coordinates x and y, see Appendix 1.

The energy spectrum for these two-dimensional particles in a perpendicular magnetic field, B, but in the absence of impurities or an electric field is just that of the Landau levels:

$$\varepsilon_n = \hbar\omega_c(n + \frac{1}{2}). \tag{7}$$

In each Landau level there are ρ_0 states per unit area with, (see Appendix 1),

$$\rho_0 = B\frac{e}{h}. \tag{8}$$

If one were to plot the density of states in energy one would find a series of delta functions at the energies ε_n.

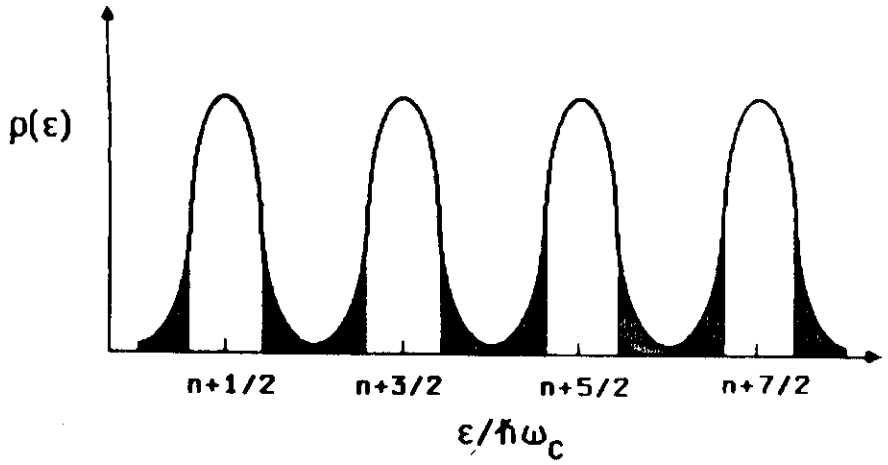

Figure 4: Possible density of electronic states in a magnetic field of a quasi two-dimensional electron gas containing impurities. The shaded regions indicate bands of localized states.

In the presence of an electric field, E, parallel say to the x-direction the energies form bands centred on the energies ε_n, (Appendix 4), with energies

$$\varepsilon_{nk} = \varepsilon_n - kv, \tag{9}$$

where $v = eEl_0/\hbar\omega_c$. The delta-functions in the density of states are replaced by these bands.

If there are also impurities in the system these may give rise to energy shifts for those cyclotron orbits which remain delocalized. They may also give rise to some localized states centred on the individual scattering centres. These localized states will have been split off from the continua of delocalized states. The density of states could then be expected to look something like as shown in Figure 4.

The localized states will not contribute to any net current in the system, so that between the lower end of one continuum and the next there will be a mobility gap. If the Fermi energy is in one of these mobility gaps then Laughlin's argument tells us that σ_{xy}, $(= I_y/V_x)$, is ne^2/h. This is independent of the number of localized states.

2.4.1 The Contribution of the Delocalized States

The contributions to the current from the individual delocalized states can be calculated as follows. The argument is adapted from Prange's original argument

[Prange 1981, Joynt & Prange 1984, Brenig 1983].

For a system of electrons write the Hamiltonian H_0 in the Landau gauge including an additional gauge parameter, a, and an impurity scattering potential, V_{sc}, as

$$H_0(a) = \frac{p_x^2 + [p_y + eB(x-a)]^2}{2m^*} + eEx + V_{sc}(x,y). \tag{10}$$

Equation 10 is as Equation 101 in Appendix 4 except for the parameter a. A change in the parameter, a , corresponds to a simple gauge transformation, (see Appendix 2 and remember q=-e),

$$\begin{aligned} a &\rightarrow a + \Delta a \\ \mathbf{A} &\rightarrow \mathbf{A} - B\Delta a\hat{\mathbf{y}} \\ \psi &\rightarrow exp(+i\frac{\Delta a}{l_0}y')\psi, \end{aligned} \tag{11}$$

with $y' = y/l_0$.

The eigen states of $H_0(a)$ in the absence of V_{sc} may be written as

$$\psi_{nk} \sim e^{i(k-v)y'} H_n(x' + k - \eta) e^{-\frac{(x'+k-\eta)^2}{2}}, \tag{12}$$

where $x' = x/l_o, y' = y/l_o, v = eEl_o/\hbar\omega_c, \eta = a/l_o$, and H_n are the Hermite polynomials. The energies are

$$\varepsilon_{nk} = \hbar\omega_c(n + \frac{1}{2}) - eEl_0k + eEa. \tag{13}$$

These energies form quasi-continuous bands centred on the cyclotron energies $\hbar\omega_c(n + \frac{1}{2})$ and bounded by

$$\varepsilon^l < \varepsilon_{nk} < \varepsilon^u, \tag{14}$$

where, if the system is assumed bounded at $x = \pm x_0$,

$$\begin{aligned} \varepsilon^l &= \hbar\omega_c(n + \frac{1}{2}) - eEx_0 \\ \varepsilon^u &= \hbar\omega_c(n + \frac{1}{2}) + eEx_0. \end{aligned} \tag{15}$$

Direct differentiation of equation 9 gives

$$\frac{\partial H_0}{\partial a} = -\frac{p_y + eB(x-a)}{m^*}eB = Bj_y, \tag{16}$$

an operator equation for the current j_y.

Now the Feynman-Hellmann theorem states that for some eigen state of $H_0(a)$ characterized by quantum numbers α and with energy ε_α,

$$< \psi_\alpha(a) \mid \frac{\partial H_0}{\partial a} \mid \psi_\alpha(a) >= \frac{\partial \varepsilon_\alpha}{\partial a}. \tag{17}$$

This gives

$$j_y^\alpha = \frac{1}{B}\frac{\partial \varepsilon_\alpha}{\partial a}, \tag{18}$$

for the current carried by the state characterized by α.

For the eigen states of the Hamiltonian H_0 with $V_{sc} = 0$, we have that the states $\psi_\alpha = \psi_{nk}$. Inserting equation 13 into equation 17 gives

$$j_y^\alpha = j_y = e\frac{E}{B}. \tag{19}$$

Each state makes the same contribution to the current.

The total current, j_y, is the sum of the contributions from all occupied states. In each Landau level there are $N = \frac{B}{h/e}$ states per unit area so that with exactly n Landau levels full

$$j_y = nNj_y^0 = n\frac{e^2}{h}E, \tag{20}$$

in agreement with equation 6.

For $V_{sc} \neq 0$ one can treat the problem as one in scattering theory. The effect of impurity scattering on the electronic states ψ_{nk} is severely restricted in the presence of the magnetic and electric fields as the states ψ_{nk} are non-degenerate. The usual degeneracies associated with time reversal and rotational symmetries have been lifted. Reflection of a particle in state ψ_{nk} is therefore impossible.

The only possibility for an electron in a delocalized state, with energy in one of the continua of equation 14 is to be scattered into itself. Its asymptotic form can therefore differ from its form in the absence of the scattering potential by at most a phase factor. At energies outside of these continua there may be localized states, cf. Fig. 4.

Suppose the system is bounded somewhere in the y-direction far from the region in which V_{sc} is non-zero, say at $y = \pm L$. We may then write the asymptotic form for an eigen state, ψ_α, including the effect of a possible gauge change $a \rightarrow a + \Delta a$, (see eq. 11), as

$$\begin{aligned} y &\rightarrow +L, \\ \psi_\alpha &\quad (\mathbf{r}, \varepsilon_\alpha) \sim \psi_{nk}(\mathbf{r}, \varepsilon_{nk}) \sim C_\alpha e^{i(k_\alpha - v + \frac{\Delta a}{l_0}y')} \\ y &\rightarrow -L, \\ \psi_\alpha &\quad (\mathbf{r}, \varepsilon_\alpha) \sim e^{-i\delta(\varepsilon_\alpha)}\psi_{nk}(\mathbf{r}, \varepsilon_{nk}) \sim C_\alpha e^{i(k_\alpha - v + \frac{\Delta a}{l_0}y' - \delta(\varepsilon_\alpha))} \end{aligned} \tag{21}$$

for an energy $\varepsilon_\alpha = \varepsilon_{nk}$ for some nk in the continua of equation 14. The phase shifts, $\delta(\varepsilon_\alpha)$, can be found by solving first for the Green's function of the system without impurities and then for $\psi_\alpha(\mathbf{r}, \varepsilon_\alpha)$ from Dyson's equation, as outlined in Appendix 4.

The phase shifts are related by Levinson's theorem [Levinson 1949] to the number of bound states, N_b, split off from the quasi-continuum by the impurity potential. They also determine the contribution to the current from the state α as shown below.

Imposing some boundary condition at $y = \pm L$, allows one to label the states α by the integer variable m_α. For example one might choose periodic boundary conditions:

$$\psi_\alpha(+L) = \psi_\alpha(-L), \tag{22}$$

in which case

$$(k_\alpha - v)\frac{2L}{l_0} + \delta(\varepsilon_\alpha) = -2\pi m_\alpha - \Delta a \frac{2L}{l_0^2}. \tag{23}$$

In the absence of impurities m_α takes N successive values, with N the number of states in the Landau level. In the presence of impurities m_α will only take $(N - N_b)$ successive values where N_b is the number of localized states split off from the continuum.

It is also evident from equation 23 that a gauge change,

$$\Delta a = \frac{l_0^2 2\pi}{2L}, \tag{24}$$

takes the state labelled by m_α to the state labelled by $m_\alpha + 1$.

Treating the differentials in equation 18 as finite differences we may write

$$\begin{aligned} j_y^\alpha &= \frac{\Delta\varepsilon}{B\Delta a} = \frac{\varepsilon_{\alpha+1} - \varepsilon_\alpha}{B\Delta a} \\ &= -eEl_0 \frac{k_{\alpha+1} - k_\alpha}{B\Delta a} \\ &= -eEl_0 \frac{l_0}{2L} \frac{-2\pi - \delta(\varepsilon_{\alpha+1}) + \delta(\varepsilon_\alpha)}{B\Delta a} \\ &= j_y^0 (1 + \frac{\delta(\varepsilon_{\alpha+1}) - \delta(\varepsilon_\alpha)}{2\pi}), \end{aligned} \tag{25}$$

showing that the contribution to the current from state α has an added contribution determined by the phase shifts, $\delta(\varepsilon_\alpha)$, when compared to its value in the absence of impurities, equation 19.

The total current if all the delocalized states in the Landau level are occupied is then given by:

$$j_y = (N - N_b) j_y^0 + \frac{\delta^u - \delta^l}{2\pi} j_y^0. \tag{26}$$

where $\delta^{u,l}$ are the values of δ at the upper and lower edges of the band. The first term is as in equation 20 but with the contributions from the N_b localized states subtracted. The second term is the extra contribution of the remaining delocalized states in the presence of the impurities.

Using equation 23, and writing k^u and k^l for the momenta at the upper and lower edges of the band, gives

$$(k^u - k^l)\frac{L2}{l_0} + \delta^u - \delta^l = -2\pi(N - N_b). \tag{27}$$

When compared to the equivalent equation for the system without impurities this gives

$$\delta^u - \delta^l = 2\pi N_b. \tag{28}$$

Equation 26 and equation 28 give as expected the result that the total current is independent of N_b. The loss of contributions to the current from the states which become localized in the presence of impurities is exactly compensated by the extra contributions from the delocalized states.

Since this result was first derived by Prange it has been extended and made more rigorous. In particular Thouless et al [1982] have shown how to relate the response σ_{xy} to results in differential geometry. By placing the system on a torus, (imposing periodic boundary conditions in both the x and y directions), they were able to eliminate the rather vague account of the boundaries, which was used here. On the torus the quantity corresponding to σ_{xy} is a topological invariant independent of the number of localized states in the Landau level. This connection has not surprisingly attracted interest from other branches of physics.

Perhaps more interesting from the viewpoint of quantum transport in general is to look more closely at the rôle played by boundary states. This question has been addressed by Halperin [1982] and is discussed in the next section.

2.5 Edge States

For the sake of simplicity assume the simple boundary conditions at the edges $x = \pm x_0$, [Halperin 1982]:

$$\psi(\pm x_0) = 0. \tag{29}$$

For x close to x_0 the eigen states of $H_0(a)$, equation 10, with no applied electric field and $V_{sc} = 0$ will now be of the form:

$$\psi_{nk} \sim e^{iky'} g_n((x' + k - \eta), x_k - x_0) e^{-\frac{(x' + k - \eta)^2}{2}}, \tag{30}$$

where

$$x_k = l_0(\eta - k) = a - kl_0 \tag{31}$$

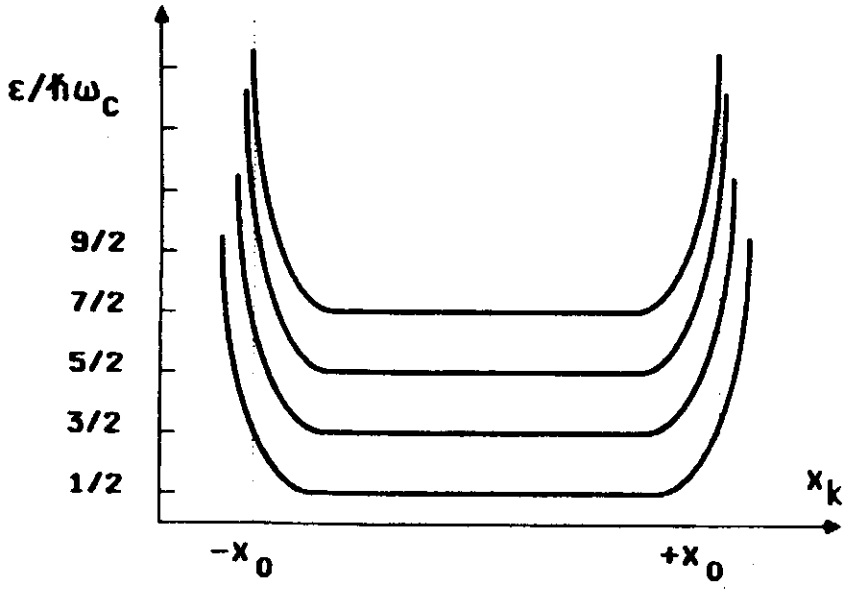

Figure 5: Energy levels of a two-dimensional electron gas confined to a ribbon, bounded at $\pm x_0$, as a function of the 'cyclotron orbit centre' coordinate, $x_k = a - kl_0$. [after Halperin 1982].

is the centre of the cyclotron orbit with 'wavevector' k. The other symbols are as in equation 12. The function g_n satisfies the same equation as the H_n in equation 12 but subject to the added boundary condition equation 29. The solutions for g_n depend on how far the centre of the cyclotron orbit, x_k, is from the boundary as well as on $x - x_k$.

For $x_k - x_0 \ll l_0$ g_n will be take its bulk form $H_n(x' - x'_k)$ with eigen value, $\varepsilon_{nk} = (n + \frac{1}{2})\hbar\omega_c$. As x_k approaches x_0 ε_{nk} will change as the boundary causes g to involve admixtures of the various $H_{n'}$. Precisely at $x_k = x_0$, only those solutions of $H_0(a)\psi = \varepsilon\psi$ are possible which have $g(0,0) = 0$. These are the $H_{n'}$ with $n' = 2n + 1$, so that

$$\varepsilon_n(x_k = x_0) = (2n + \frac{3}{2})\hbar\omega_c. \tag{32}$$

For $x_k \gg x_0$, ε_{nk} will vary as $(x_k - x_0)^2$. Figure 5 shows how ε_{nk} can be expected to vary as a function of x_k.

The variation of ε_{nk} with x_k and therefore with the gauge parameter, a, implies that these states carry non-zero current even in the absence of an applied electric field, (see equation 18):

$$j_y^k = \frac{1}{B}\frac{\partial\epsilon_{nk}}{\partial a} = -\frac{1}{B}\frac{\partial\epsilon_{nk}}{\partial x_k}. \tag{33}$$

Multiplying by the density of states per unit area $\frac{B}{h/e}$ and integrating equation 33 from the edges at $\pm x_0$ into the bulk gives that for each Landau level, m, there is a contribution to the current from the occupied edge states close to $-x_0$:

$$j_y = +\frac{e}{h}\delta E_n^-, \tag{34}$$

and a contribution $-\frac{e}{h}\delta E_n^+$ from the states close to x_0. $\delta E_n^\pm$ is the difference in energy between the highest occupied state in the n'th level and the bulk energy for the level at the two edges. If $\delta E_n^+ \neq \delta E_n^-$ there will be a net current.

When a voltage is applied between the edges one expects that $\delta E_n^+ \neq \delta E_n^-$ so that there is a contribution from the edge states to the total current. The voltage dropped across the edge states, ΔV, as a proportion of the total applied voltage, V, can be written approximately in terms of the effective capacitance per unit length of the edge states, C. With the Fermi energy in the bulk of the sample between the n'th and (n-1)'th Landau level, the net charge accumulated per unit length in at the edges, σ, will be given approximately by

$$\sigma \approx \frac{e}{l_0}\frac{e\Delta V}{\hbar\omega_c(n+1)}. \tag{35}$$

The second term in equation 35 measures the proportion of states of the n'th level whose occupation is changed by the voltage drop ΔV. Comparing this with $\sigma = CV$ gives

$$\frac{\Delta V}{V} \sim \frac{l_0}{e^2}\hbar\omega_c(n+1)C. \tag{36}$$

For a disordered sample, ($V_{sc} \neq 0$) in equation 10, the effect of any scattering on edge states will be much as for bulk states. If localized states are formed close to $x = \pm x_0$ then the remaining delocalized edge states compensate exactly for the lost contribution of the localized states, as shown explicitly for bulk states in subsection 2.4.1. This case has been argued by Halperin [1982].

3 The Fractional Quantum Hall Effect

At low temperatures plateaux can be observed in the Hall resistivity, ρ_{xy}, of some $GaAs - GaAl_xAs_{1-x}$ heterostructures [Tsui et al 1982]. These tend to be those suitably doped structures with very high electron mobilities $>\sim 10^5 cm^2/V sec$. The plateaux are observed to occur at rational multiples, m,

of h/e^2, with m^{-1} non-integer. This phenomenon and related effects are known as the fractional quantum Hall effect. Figure 6 shows some of the experimental results of Chang et al [1984].

If the quantum Hall effect can be thought of as the quantized response of filled bands of (delocalized) states of quasi two-dimensional electrons, the Landau levels, the fractional quantum Hall effect appears to be the quantization of the response of an apparently partially filled lowest Landau level. The plateaux are found for magnetic fields close to those at which one would expect a fraction, ν, of the states in the lowest Landau level to be occupied. For these magnetic fields the resitivity, ρ_{xy}, is found to be

$$\rho_{xy} = \frac{h}{\nu e^2}. \tag{37}$$

Plateaux have been observed for many rational ν, all with odd denominators. The most pronounced and readily observed is at $\nu = 1/3$, followed in approximate order by 2/3, 2/5, 2/7, 3/5 ...[Chang et al 1984]. Plateaux have also been observed at filling fractions 5/3, 4/3 and a few others with $1 < \nu < 2$, and 'forerunners' of plateaux at $2 < \nu < 3$ [Clark et al 1986].

At the high fields involved it was originally assumed that the electrons in these systems are completely spin polarized. On this basis one would expect that the results for a spin-up ν-partially filled zero'th Landau level, $0 < \nu < 1$, to be mirrored approximately by those for the corresponding spin-down $(2-\nu)$-partially filled zero'th Landau level, $(1 < \nu < 2)$. The results for $2 < \nu < 3$ would then correspond to a partially filled first Landau level.

Measurements of Clark et al [1989] have shown that there are states which are not spin polarized. Systems switch from being totally polarized at some filling fractions, for example $\nu = \frac{1}{3}$ to a spin singlet configuration at $\nu = \frac{4}{3}$ to partially polarized at $\nu = 11/7$.

NOTE : In the following all quantities will be assumed to be in magnetic units, see Appendix 1.

3.1 Interpretation of the Measurement: Many-Body Gap and Fractional Charge

Faced with the observed quantization one's first reaction might be to invoke Laughlin's Gedankenexperiment.The observation of a quantization of the Hall resistance would then imply the existence of a mobility gap in the excitation spectrum as in the case of the quantum Hall effect. This gap would have to be caused by the particle-particle interaction as the single-particle states, the cyclotron orbits, are degenerate. With the Fermi energy in this gap the

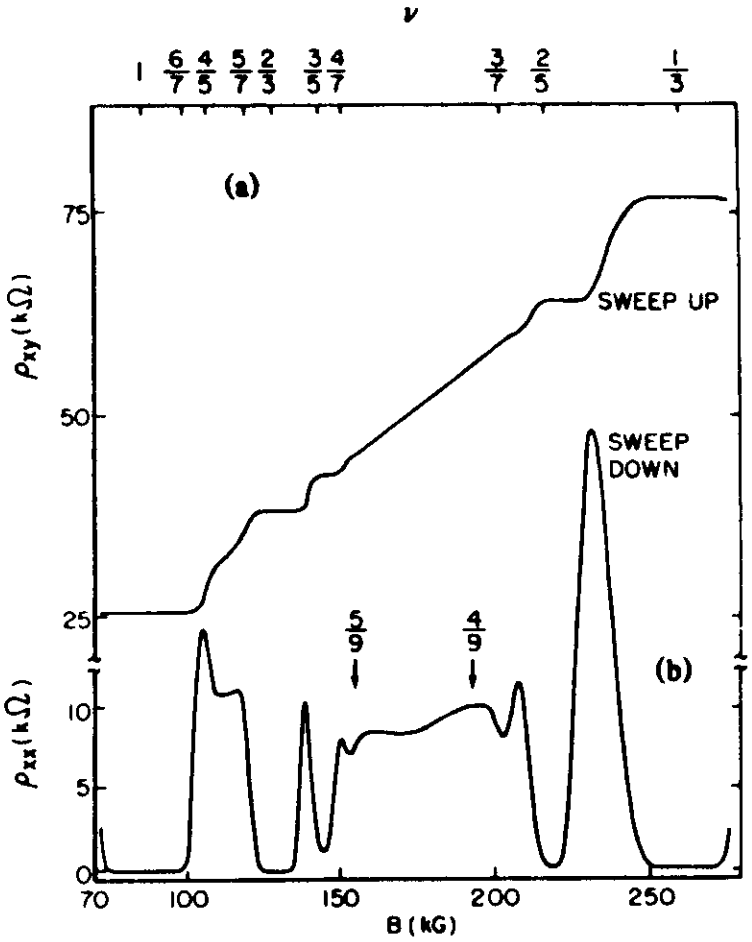

Figure 6: The Hall resistivity, ρ_{xy}, and longitudinal resistivity, ρ_{xx}, of a $GaAs - Al_{0.3}Ga_{0.7}As$ heterojunction as a function of applied magnetic field or filling fraction of the lowest Landau level, ν. Plateaux observed in ρ_{xy} and minima in ρ_{xx} indicate fractional quantized Hall states. [After Chang et al 1984.]

resistivity, ρ_{xy}, is given (neglecting signs) by equation 6

$$\rho_{xy} = \mid \frac{V_x}{I_y} \mid = \mid \frac{h}{nqe} \mid . \tag{38}$$

Comparing with the observed value at say ν=1/3 gives

$$n \mid \frac{q}{e} \mid = \frac{1}{3}. \tag{39}$$

n is the number of particles of charge q transferred in the Gedankenexperiment between edge states when the flux threading the ring is changed by h/e.

Simple application of this result at $\nu = 1/3$ implies the existence of fractionally charged particles with charge $\mid q \mid = \frac{e}{3}$. These particles might be expected to behave like the electrons in the case of the normal quantum Hall effect. Impurities would be expected to localize these particles giving rise to a mobility gap and so on. This appealing interpretation was developed originally by Laughlin [1983], who identified possible forms for the ground state and excitation wavefunctions, which do indeed describe fractionally charged objects.

The Gedankenexperiment presupposes that the bulk occupation numbers cannot change when the gauge is changed. However if there are more than one equivalent ground states it could be that the gauge change causes the electrons in the bulk to be mapped not into the same ground state but into one of the equivalent ground states. Indeed it turns out that at rational filling fraction, $\nu = p/t$. and with the application of certain types of boundary condition there is a t-fold centre of mass degeneracy of the ground state. Only changes of flux by multiples of $t\frac{h}{e}$ map these systems into themselves again with a corresponding change of one electron/hole in the occupation of the edge states. Relating the measurement to this centre of mass degeneracy was suggested later by Niu et al [1985] and Tao[1986].

The connection between ground state degeneracy and topological defects with fractional quantum numbers have been discussed in this and other contexts by Schrieffer [1985].

Either way both interpretations of the Gedankenexperiment imply the existence of an excitation gap. (In fact this result can also be deduced on quite general thermodynamic grounds, [Halperin 1983].) We can also expect to find locally well-defined excitations with apparent fractional charge.

3.2 Zeroes and Flux Quanta

All current theories associate the gaps in the excitation spectra implied by the experiments with the density of 'zeroes' permitted in the N-particle wavefunctions, $\psi(\mathbf{r}_1, ..., \mathbf{r}_N)$, as a function of filling function of the lowest Landau level, ν.

To identify the rôle of zeroes in the wavefunction suppose a system of electrons is confined in two dimensions to some area Ω, and a large magnetic field leads to complete spin polarization of the electrons. At filling fraction, ν, there are then ν electrons per cyclotron orbit. Equivalently one may say that there are ν electrons per flux quantum threading the area Ω, (see appendix A1).

Now imagine performing the following Gedankenexperiment: take one of the electrons, say the i'th one, anticlockwise along the closed path, $\sum_l \mathbf{a}_l$, which, assume, defines the perimeter of the system enclosing the area Ω. Then $\psi(\mathbf{r}_1, ..., \mathbf{r}_N)$ is transformed to $\psi'(\mathbf{r}_1, ..., \mathbf{r}_N)$):

$$\psi(\mathbf{r}_1, ..., \mathbf{r}_N) \rightarrow \psi'(\mathbf{r}_1, ..., \mathbf{r}_N) = (\prod_l T(\mathbf{a}_l))\psi(\mathbf{r}_1, ..., \mathbf{r}_N), \tag{40}$$

where the $T(\mathbf{a}_l)$ are the magnetic translation operators defined in Appendix 3. Successive application of the combination rule for the $T(\mathbf{a}_l)$, equation 87 in Appendix 3, yields with q=-e

$$\psi' = e^{2\pi i \frac{q}{e} \oint \frac{\mathbf{A}.d\mathbf{l}}{\phi_0}} \psi = e^{-2\pi i \frac{\Phi}{\phi_0}} \psi \tag{41}$$

where Φ, $(= B\Omega)$, is the total flux threading the area Ω and $\phi_0 = h/e$. But

$$\frac{\Phi}{\phi_0} = N\nu^{-1} \equiv Nm. \tag{42}$$

The phase of ψ therefore changes as a result of the Gedankenexperiment by $-2\pi m$ for each electron contained in the area Ω, $(m = \nu^{-1})$.

The wavefunction of an electron in the lowest Landau level can be written in the symmetric gauge, $\mathbf{A} = (y, -x, 0)/2$ (see Appendix 2), as

$$\psi(\mathbf{r}_i) \sim P(z_i) e^{-\frac{|\mathbf{r}_i^2|}{4}}, \tag{43}$$

with $P(z_i)$ a polynomial in the variable

$$z_i = x_i + i\frac{q}{e} y_i = x_i - iy_i. \tag{44}$$

$\psi(\mathbf{r}_i)$ changes phase by -2π around a point at which $P(z_i)$ has a simple zero, and by $-2\pi n$ around a point at which $P(z_i)$ has a zero of order n. From equation 42 we then see that for each electron within the area Ω, the N-particle wavefunction, $\psi(\mathbf{r}_1, ..., \mathbf{r}_N)$, must have Nm zeroes as a function of the position coordinates, z_i. This is equivalent to one zero per flux quantum, or m zeroes per particle.

The Fractional Quantum Hall Effect is thought to result from the 'binding' of the zeroes in the wavefunction as a function of the coordinate, z_i, to the

coordinates of each of the other electrons. This will only be possible at certain odd integer m, $(= \nu^{-1})$, if the wavefunction is to retain an analytic form and describe fermions. The binding of m zeroes coincidentally to particle positions will tend to minimize the interaction energy of two particles, as the probability of particle i approaching particle j vanishes as

$$| \psi(\mathbf{r}_i, \mathbf{r}_j) |^2 \sim | \mathbf{r}_i - \mathbf{r}_j |^{2m} \tag{45}$$

as $| \mathbf{r}_i - \mathbf{r}_j | \rightarrow 0$. If the zeroes were somehow bound close but not coincidentally the probability of two particles approaching would also be small.

The system would then be expected to have a gap in the excitation spectrum for ν close to 1/m, because it would not be possible to vary the number of electrons or flux quanta without adding or removing zeroes to the wavefunction. Excitations involving the addition or removal of flux quanta from the system would lead to excitations which would appear at least to carry fractional charge. For, whereas in the ground state wavefunction m zeroes are associated with the electron positions, an added or missing zero would be associated with $\pm 1/m$ electronic charges.

3.3 Laughlin's Wavefunction

In 1983 Laughlin constructed his trial wavefunction:

$$\psi_L(\mathbf{r}_1, ..., \mathbf{r}_N) = \prod_{i<j}^{N} (z_i - z_j)^m e^{-\sum_i \frac{|z_i|^2}{4}}. \tag{46}$$

in which all the zeroes as a function of coordinate z_i are clearly coincidental with the position coordinates of the other electrons. ψ_L is of the form of equation 43 for each z_j and so describes electrons in the lowest Landau level. ψ_L describes a fluid as can be seen by writing $| \psi_L |^2$ as a classical distribution function:

$$| \psi_L |^2 \sim e^{-\beta V}, \tag{47}$$

with $\beta = 2/m$ and

$$V = m^2 \left(\sum_{i<j}^{N} -ln | z_i - z_j | + \frac{1}{4m} \sum_{l}^{N} | z_l |^2 \right). \tag{48}$$

V is the potential energy of the two-dimensional one component plasma, 2DOCP. This has particles of charge m interacting via a logarithmic interaction in a neutralizing background with number density $1/(2\pi m)$. For $2m <\sim 140$ it is known that equation 47 describes a fluid [Caillol et al 1982]. This fluid is incompressible as the length scale, l_0, cannot change without introducing or removing flux quanta. As stated above this would have to involve excitations across a gap.

Laughlin [1983] also suggested explicit forms for excitations he called quasiparticles and quasiholes. Imagining removing and adding flux quanta from the system at some point z_0, $(= x_0 + i\frac{q}{e}y_0)$, he suggested the wavefunctions

$$\begin{aligned} \psi_L^{+z_0} &= \prod_i (z_i - z_0)\psi_L \\ \psi_L^{-z_0} &= \prod_i (2\frac{\partial}{\partial z_i} - z_0)\psi_L. \end{aligned} \tag{49}$$

$| \psi_L^{+z_0} |^2$ describes a classical system as before, but with an added particle with charge 1 at z_0. This will be screened by the charge m particles, so that in the electron system there will be an absence of 1/m'th of an electron in the neighborhood of z_0, and hence a local excitation with effective charge +e/m. Interpretation of $| \psi_L^{-z_0} |^2$ along these lines is also possible though not so straightforward.

$| \psi_L^{+z_0} |^2$ together with $| \psi_L |^2$ and $| \psi_L^{-z_0} |^2$ have been extensively studied by Monte Carlo simulation and hypernetted chain calculations for the corresponding classical systems. All these studies show that the Laughlin picture is at least consistent. There appears to be a well-defined thermodynamic limit for the quasiparticle, quasihole and gap excitation energies, ε^+, ε^- and ε_g. Perhaps the most extensive studies to date have been made by Morf and Halperin [1986], who find $\varepsilon^+ \approx .026\frac{e^2}{\epsilon l_0}$ and $\varepsilon^- \approx .073\frac{e^2}{\epsilon l_0}$.

Their Monte Carlo simulations also showed that a 'pairing' ansatz for ψ^- :

$$\psi^- \sim A\frac{1}{(z_1 - z_2)^2} \prod_{j=3}^{N} \frac{z_j - \frac{z_1 + z_2}{2}}{(z_1 - z_j)(z_2 - z_j)} \psi_L. \tag{50}$$

gives at least as low an estimate for the quasiparticle excitation as Laughlin's $\psi_L^{-z_0}$. Here A is the antisymmetrizing operator. Both wavefunctions describe an excitation with a local accumulation of charge -e/m distributed on a ring about z_0 for $\psi_L^{-z_0}$ or about the origin for ψ^-, again consistent with the Laughlin's original interpretation [Morf and Halperin 1986].

The low-lying collective excitations at constant field and particle number might be thought of as quasiexcitons. The quasiexciton dispersion, $\varepsilon(\mathbf{k})$, would then be the energy-wavevector dependence of bound pairs of fractionally charged quasiparticles and quasiholes.

In the spirit of Laughlin's theory Girvin, MacDonald and Platzmann (GMP) [1986] have suggested using a Feynman-Bijl Ansatz to describe the quasiexcitons. They obtain an $\varepsilon(\mathbf{k})$, which is in agreement with results from the direct diagonalization of the Hamiltonian for small systems.

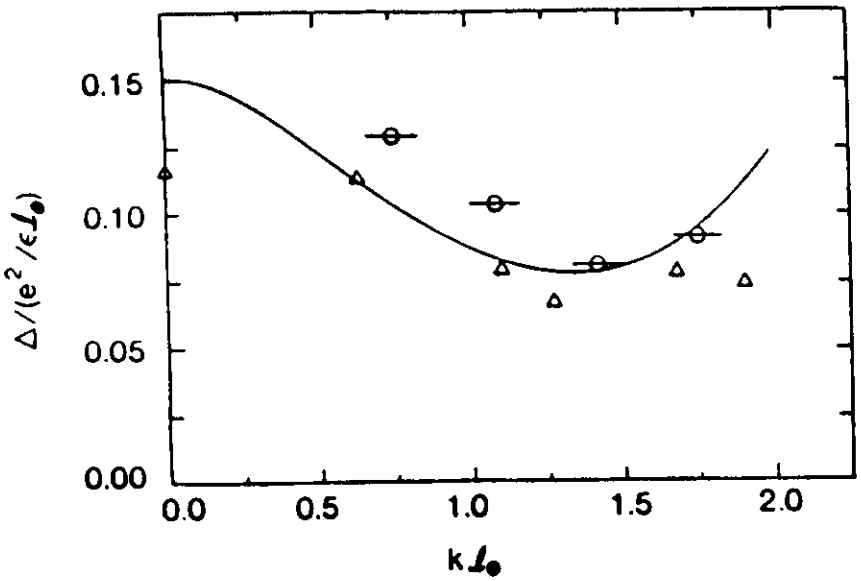

Figure 7: The dispersion relation, $\epsilon(k)$, for excitations, (quasiexcitons), of the two-dimensional electron gas at constant magnetic field and particle number for $\nu = 1/3$. The continuous line is the variational estimate of $\epsilon(k)$ based on a single-mode approximation. The points are results from exact calculations on a system of six particles. [After Girvin et al 1986 and Haldane and Rezayi 1985.]

The Feynman-Bijl Ansatz assumes that the density operator $\rho(\mathbf{k})$ only couples a single-mode to the homogeneous ground state, ψ_{gs}. So

$$\psi_{\mathbf{k}} = \rho_{\mathbf{k}}\psi_{gs}. \tag{51}$$

is an eigen state of the system with energy $\epsilon(\mathbf{k})$. The approximation is sometimes known as the Single-Mode Approximation or SMA. One may also think of $\psi_{\mathbf{k}}$ as a trial wavefunction for the lowest-lying energy state with wavevector $\mathbf{k}$.

GMP make the further approximations of treating ψ_L as the exact ground state and replacing $\rho_{\mathbf{k}}$ by its the projection onto the lowest Landau level, $\bar{\rho}_{\mathbf{k}}$. The variational estimate for $\epsilon(\mathbf{k})$, $(=< \psi_{\mathbf{k}} \mid H - \varepsilon_{gs} \mid \psi_{\mathbf{k}} > / < \psi_{\mathbf{k}} \mid \psi_{\mathbf{k}} >)$, is then a function of the static structure factor s(k). GMP took s(k) from results obtained for the 2DOCP. Their results together with exact results for small systems obtained by Haldane and Rezayi are shown in Figure 7.

3.4 Haldane's Argument

A slightly different justification of the Laughlin picture has been given by Haldane and Rezayi [1985] and others. They pointed out that ψ_L is in fact the exact ground-state wavefunction at $\nu = 1/m$ for a system of particles interacting through a kind of hard-core interaction.

One might then imagine a Gedankenexperiment in which the range of the interaction was adiabatically changed from this hard-core interaction to the interaction of the physical system. Provided no crossover occurred between states of different symmetry the physical system would be qualitatively the same as the system for which exact solutions are known. There would only be some renormalization of parameters such as excitation energies. This would be as in the Landau Fermi Liquid theory of interacting systems, where of course the system whose solutions are known exactly is the one made up of non-interacting particles.

Haldane and Rezayi simulated by direct numerical diagonalization of the Hamiltonian for small systems the adiabatic variation of the interaction. They defines the interaction:

$$V(\lambda) = V^{Coulomb} + \lambda(V^{h.c.} - V^{Coulomb}), \tag{52}$$

and adiabatically varies λ. $V^{Coulomb}$ describes a Coulomb interaction and $V^{h.c.}$ the hard-core interaction defined in Appendix 5. The ground state and excited states of six particles on the surface of a sphere, labelled by total angular momentum, are plotted as a function of λ in Figure 8.

Figure 8 shows that at least for a system of only six particles there is no crossover between states with different quantum numbers for the range of interactions likely in physical systems. It is also suggestive of an excitation gap between the Laughlin-like ground state, (L=0), and the first excited states.

Other numerical studies of small systems at $\nu = 1/m$ have been reported by many authors for various boundary conditions. All lend support to Laughlin's interpretation at $\nu = 1/m$: all show large overlaps between the exact ground state with a Coulomb interaction and the Laughlin ground state and almost equally large overlaps between the exact excitations and those suggested by Laughlin. Table 1 shows as an example the results obtained by Fano et al [1986] in a spherical geometry first introduced by Haldane.

3.5 Other Filling Fractions - the Hierarchy

'Binding' ν^{-1} zeroes directly to the particle positions, as in the Laughlin wavefunction, is only possible for ν^{-1} an odd integer. The theory needs generalizing if it is to explain the observation of plateaux at $\nu = 2/5$, $\nu = 2/7$ and so on.

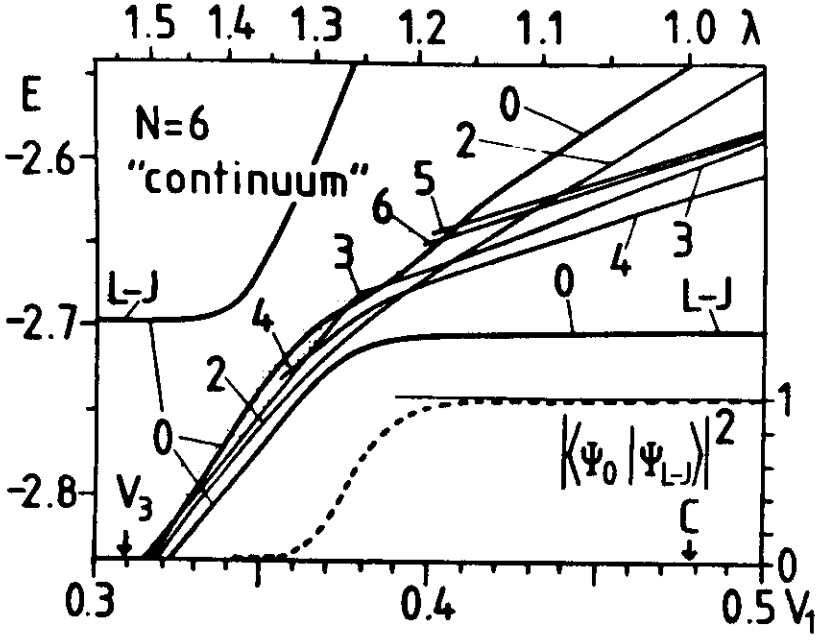

Figure 8: The energies of the low-lying states of six electrons confined to the surface of a sphere are plotted for the case $\nu = 1/3$ labelled according to their total angular momentum. The inter-particle interaction is varied from hard-core, ($\lambda = 0$), through Coulomb, ($\lambda = 1$), to longer-range interactions, ($\lambda > 1$). The strength of the hard-core component V_1 is also shown, see Appendix A.5 equation 116. Also shown is the overlap between the ground-state and the Laughlin wavefunction ψ_{LJ}. For the range of likely physical interactions the ground state remains qualitatively the same as the Laughlin state. [After Haldane and Rezayi 1985.]

N	3	4	5	6	7	8	9
$\nu = 1/3$	1	.99804	.99906	.99644	.99636	.99540	.99406
$\nu = 1/5$	1	.98405	.99743	.94865	.97681		

Table 1: Overlap between ψ_L and exact ground state with a Coulomb interaction, ψ_C, at $\nu = 1/3$ and $\nu = 1/5$. [After Fano et al 1986.]

A suggestion of Halperin [1983, 1984] and Haldane[1983] was to consider the properties of the excitations from ψ_L. If these were point charges, with coordinates η_a^1, they would constitute a system similar to the original one of electrons/holes, but with effective charge $q_1 = \alpha_1 e/m$ and magnetic length $l_1^2 = l_0^2 m$. $\alpha_1 = \pm 1$ according to whether the excitations are quasiholes or quasiparticles. One could therefore imagine a pseudohamiltonian for these excitations including some kind of repulsive interaction between them, and a pseudowavefunction, $\bar{\psi}_1$, written in the coordinates η_a^1. The η_a^1 would be the z_0 of the Laughlin excitations or the 'centre of mass' coordinates of the pair excitations equation 50.

Also if one imagined taking one of these particles around the perimeter of the system enclosing flux Φ, the pseudowavefuntion would change to $\bar{\psi}_1'$, (see equation 41 with $q = \alpha_1 e/m$):

$$\bar{\psi}_1' = \bar{\psi}_1 e^{\alpha_1 2\pi i \frac{\Phi}{\Phi_0}}. \tag{53}$$

If no other excitation is enclosed $\Phi = mN_e$ and $\bar{\psi}_1' = \bar{\psi}_1$. If one other excitation is enclosed $\bar{\psi}_1' = \bar{\psi}_1 e^{i\alpha_1 \frac{2\pi}{m}}$. Arovas et al [1984] have argued by analogy with electrons: When one electron encircles another the phase change is -2π corresponding to a phase change of $-\pi$ for interchange of two particles. The statistic for interchange of two fractionally charged excitations would then be:

$$\bar{\psi}_1(\eta_a^1, \eta_b^1) = \bar{\psi}_1(\eta_b^1, \eta_a^1) e^{i\alpha_1 \frac{\pi}{m}}. \tag{54}$$

This result can be incorporated into a trial ground state pseudowavefunction for N_1 particles:

$$\bar{\psi}_1(\eta_1^1, ..., \eta_{N_1}^1) = \prod_{a<b}^{N_1} (\eta_a^1 - \eta_b^1)^{p_1 + \frac{\alpha_1}{m}} e^{-\sum_c \frac{|\eta_c|^2}{4m}}. \tag{55}$$

Only even p_1 are consistent with equation 54.

The excitations from this pseudo-wavefunction, $\psi_1^{\pm\eta_0}$, could be expected to be described by operators similar to those in equation 49 operating on $\bar{\psi}_1$. These excitations would form the particles of the second level of a hierarchy of excitations, carrying charge

$$q_2 = \pm \frac{q_1}{p_1 + \frac{\alpha_1}{m}} = \frac{e}{mp_1 \pm 1}. \tag{56}$$

This follows by analogy with the Laughlin fluid at $\nu = 1/m$. The ratio of the charge of the particles of 'parent' fluid to that of its excitations is equal to the number of zeroes in the the wavefunction bound to the coordinates of the other particles. This is just the exponent in equation 55.

This process leads one to the hypothesis of a hierarchy of Laughlin-like fluid ground states at different filling fractions for the lowest Landau level. For a hierarchy of n levels this leads to a filling fraction, $\nu(m;p_1,p_2,...,p_n)$, derived in appendix 6:

$$\nu(m;p_1,p_2,...,p_n) = \cfrac{1}{m+\cfrac{\alpha_1}{p_1+\cfrac{\alpha_2}{p_2+\cfrac{\alpha_3}{p_3+\cfrac{\ldots+\alpha_n}{p_n}}}}} \qquad (57)$$

This interpretation implies a natural order in which the ground state fluids would become apparent as the temperature is reduced. First one would expect to see a 'parent' fluid at $\nu = 1/m$, or its hole counterpart at $\nu = 1-1/m$. These would be followed by first level fluids at $\nu = 2/5$ $(m = 3, \alpha_1 = -1, p_1 = 2)$, $\nu = 2/7$ $(m = 3, \alpha_1 = +1, p_1 = 2)$, or $\nu = 3/5$ and $\nu = 5/7$. Then second level fluids at $\nu = 3/7$ $(m = 3, \alpha_1 = -1, p_1 = 2, \alpha_2 = -1, p_2 = 2)$, and so on. This natural order seems to agree quite well with the apparent 'strength' of the experimentally observed plateaux [Chang et al 1984].

Béran and Morf [1990] have shown that the hierarchical model can be used to predict ground state energies and energy gaps for systems of electrons in a magnetic field. This requires careful treatment of the effective interaction between quasiparticles. They have also looked at hierarchical states involving spin reversed quasiparticles, see section 3.7.

3.6 Microscopic Trial Wavefunctions for the Hierarchy

The hierarchical picture is not on the same microscpic basis as the trial wavefunction, (MTW), ψ_L, for $\nu = 1/m$.

In an attempt to improve on the phenomenological nature of the hierarchical fluid Morf et al [1986] suggested a family of MTWs, which work explicitly with the electron coordinates, but which appear to incorporate the idea of the hierarchy. Taking the 'paired' excitation, equation 50, as the basis for the quasiparticle they suggested writing, with A the antisymmetrizing operator:

$$\begin{aligned} \psi_T &= A\phi_D \\ &= A\Big(\psi_L \prod_n^{\frac{N}{2}} (z_{2n} - z_{2n-1})^{-t} \prod_{n,n'}^{\frac{N}{2}} (z_{2n}z_{2n-1} + z_{2n'}z_{2n'-1} - 2Z_nZ_{n'})^s\Big) \end{aligned} \qquad (58)$$

Here Z_n, ($= (z_{2n} + z_{2n-1})/2$), denotes the centre of mass coordinate of the pair formed by particles 2n and 2n-1. To ensure that the wavefunction is antisymmetric with respect to interchange of particles 2n and 2n-1 in the 'pair' n, m-t must be odd, (m is the exponent in ψ_L). Also $m - t \geq 1$ and $s \geq 0$. By evaluating the maximum power of z_i for any i and dividing by N ψ_T can be seen to be a trial wavefunction for a filling factor $\nu = 1/(m + \frac{s}{2})$.

For $t > 0$ the partially antisymmetric part ϕ_D can be thought of as a Laughlin fluid with N/2 quasiparticles nucleated at the coordinates Z_n. These excitations are essentially prevented from approaching each other by the term in the bracket raised to the s'th power. For $t < 0$ the partially antisymmetric part ϕ_D might be thought of as ψ_L with N/2 quasiholes at the coordinates Z_n, although the interpretation is not so clear in this case.

Morf et al [1986] have made extensive studies both by Monte Carlo and CI type calculations of these wavefunctions. Fano et al [1986] have also made similar studies. The results for the overlaps between the trial wavefunctions and the exact wavefunction with the Coulomb interaction are presented in Table 2 for $\nu = 2/5$, $(m = 2, s = t = 1)$, and $\nu = 2/7$, $(m = 3, s = 1, t = -2)$. Again the overlaps between the trial wavefunctions and the exact ground-state wavefunction with a Coulomb interaction add further support to the picture of a series of Laughlin-type fluids.

		$\nu = 2/5$			$\nu = 2/7$	
N	$E(\psi_T)/N$	$E(\psi_C)/N$	$< \psi_C \mid \psi_T >$	$E(\psi_T)/N$	$E(\psi_C)/N$	$< \psi_C \mid \psi_T$
4	-.426104	-.426104	1	-.386004	-.386012	.999 939
6	-.427641	-.428517	.98840	-.384527	-.384626	.997163
8	-.428 283	-.429 543	.97712	-.383 671	-.383 811	.996293
10	-.428 939	-.430 258	.97154			
12	-.429 327					
∞	-.4310	-.4330				

Table 2: Coulomb energy and overlap for the trial wavefunction, ψ_T, (equation 58), and overlap with exact ground state with a Coulomb interaction, ψ_C, at $\nu = 2/5$ and $\nu = 2/7$. N is the particle number. Energies are measured in units of $e^2/\varepsilon l_0$. [After Morf et al 1986.]

3.7 Spin Polarization

The measurements of Clark et al [1989] showed that not all fractional quantum Hall states observed in GaAs-GaAlAs heterostructures are spin polarized.

Spin unpolarized and partially polarized states have also been found to minimize the ground state energy of systems of electrons in a magnetic field corresponding to some of the filling fractions at which such states are observed experimentally. Such states are clearly Laughlin liquids in the sense that they are incompressible.

Béran and Morf [1990] have recently shown that the gaps and ground state energies can be accurately predicted within the hierarchical scheme. This involves taking as the parent fluid say the $\nu = 1/3$ Laughlin state and introducing spin reversed quasiparticles. The interaction between excitations has to take account of the charge distribution of a quasiparticle for large separations and quantum corrections for small separations. The satisfying result is that one needs only to account for two-quasiparticle interactions.

The plateau observed at filling fractions $\nu = 5/2$ [Clark et al 1986, Willett et al 1987, Eisenstein et al 1988] is also thought to correspond to a spin-singlet ground state. The understanding of this state is still incomplete.

3.8 Higher Landau Levels

It is no problem to write down the equivalent wavefunctions to ψ_L for higher Landau levels. In the lowest Landau level the operators z_i and $\frac{\partial}{\partial z_i}$ have the same matrix elements as the raising and lowering operators, Z_i^+ and Z_i^- defined in Appendix A3. The obvious generalization of ψ_L to the n'th Landau level is [Haldane 1987]

$$\psi_L^n = \prod_{i<j}(Z_i^+ - Z_j^+)^m \mid n,0 >, \tag{59}$$

where $\mid n,0 >$ is the symmetric product of single particle states in the n'th Landau level with $L_{z_i} = 0$.

With n=1 when any two particles i and j approach each other the norm of the wavefunction $\mid \psi_L^1 \mid^2$ rises only as [d'Ambrumenil and Reynolds 1987]

$$\mid \psi_L^1 \mid^2 \sim r_{ij}^{2(m-2)} \left(L_2^{m-2}\left(\frac{r_{ij}^2}{2}\right)\right)^2. \tag{60}$$

For *both* m=1 and m=3 $\mid \psi_L^1 \mid^2$ increases as r_{ij}^2.

For m=5 $\mid \psi_L^1 \mid^2$ increases as r_{ij}^6. ψ_L is therefore the exact non-degenerate ground state for a hard-core interaction at $\nu = 1/m$ only for $m = 5, 7, \ldots$, but not for $m = 3$, see MacDonald and Girvin [1986]. One might therefore expect to find an approximate analogy between states at filling fractions in the first Landau level, $1/5 < \nu < 1/3$, and states in the lowest Landau level at $1/3 < \nu < 1$. This is supported by the numerical calculations of d'Ambrumenil and Reynolds [1988] on small systems. As an example Figure 9 shows the equivalent curve to Figure 7 for the first Landau level at filling fraction of the first Landau level $\nu = 1/5$. As in the lowest Landau level there is a well-defined excitation dispersion and gap.

Results of calculations for small systems have lead Haldane [1987] and d'Ambrumenil and Reynolds [1987] to conclude that at a filling fraction of the n=1 Landau level $\nu = 1/3$ the system is close to the borderline between having a incompressible Laughlin-like fluid ground state and compressibility.

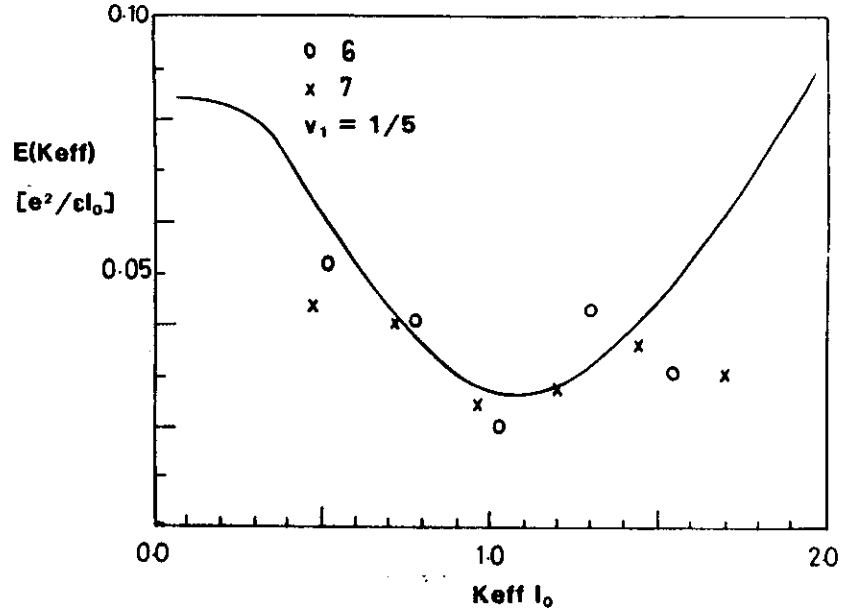

Figure 9: The excitation energies at $\nu_1^{-1} = 5$ for six and seven particles. The energies are defined as the total energy difference between the lowest energy state for the various L and the exact ground state. The solid line represents the spectrum predicted by MacDonald and Girvin [1986] on the basis of the Feynman-Bijl Ansatz. After d'Ambrumenil and Reynolds [1988]

d'Ambrumenil and Reynolds [1988] also found evidence that there should be a fractional quantized Hall state at a filling fraction of the n=1 Landau level $\nu = 2/7$. This is again compatible with the premise that there should be at least an approximate analogy between the states in the n=1 Landau level at filling fractions $1/5 < \nu < 1/3$ and those in the lowest, (n=0), Landau level at filling fractions $1/3 < \nu < 1$.

The observations of Clark et al [1986] are however still only on the way to being understood.

3.9 Ring Exchange

An alternative approach to understanding the FQHE has been suggested by Kivelson et al [1986], (KKAS), which does not seek to describe the systems

with explicit wavefunctions.

KKAS found from an approximate expansion scheme for the free energy that at certain rational filling fractions there were large contributions to the Free energy from coherent exchanges of particles round large rings. These ring exchanges (CRE) add coherently when the ratio of particles to flux quanta is rational with odd denominator, and preferentially stabilize the system at the corresponding filling-fraction.

Initially their calculations seemed to imply that the ground state would have to have the kind of long-range order generally associated with a solid. However Baskaran [1986] later argued that CRE would stabilize any incompressible state, including of course ψ_L.

3.10 Summary

It is generally accepted that the the gap implied by the experiments for rational filling factors is tied up with 'binding' of zeroes in the wavefunction to particle positions, (sections 3.2 and 3.7). The evidence supports Laughlin's picture of an incompressible fluid ground state at $\nu = /3$, (sections 3.3 and 3.4). The hierarchy of states proposed to account for other filling-fractions is not on quite such firm ground (section 3.5), but is at the very least plausible, especially in view of the microscopic trial wavefunctions (section 3.6). The recent studies of Béran and Morf show that the hierarchical scheme can be adapted to account for spin unpolarized states observed in experiments.

A Dictionary of Standard Results (more or less)

A.1 Hamiltonian and Energy Spectrum

The Hamiltonian for a particle of charge q, with coordinates (x, y, z) and effective mass, m^*, is given by

$$H_0' = \frac{(\mathbf{p} - q\mathbf{A})^2}{2m^*} = \frac{\boldsymbol{\pi}^2}{2m^*}, \tag{61}$$

where $\mathbf{A}$ is the vector potential and $\mathbf{p}$ the momentum. The Hamiltonian will also include tems describing the particle-impurity interaction, H_i, the particle-lattice interaction, H_{p-l}, the lattice dynamics important in the magnetophonon effect, H_L, and the particle-particle interaction, H_{p-p}.

It is usual to separate the Hamiltonian H_0' into a part describing motion parallel to the magnetic field and a part describing motion perpendicular to the magnetic field. Taking the magnetic field to be parallel to the z-direction H_0' separates and the eigen functions can be written as

$$\Psi = \psi(x, y)\phi(z), \tag{62}$$

with

$$H_0'\Psi = (E_z + H_0)\psi, \tag{63}$$

where H_0 operates on the x and y coordinates only. In a three-dimensional jellium model $\phi(z)$ would be just a plane wave with $E_z = \frac{\hbar^2 k_z^2}{2m^*}$. In a semiconductor heterostructure $\phi(z)$ would be the envelope wavefunction of the subband.

It is often helpful to write H_0 in so-called magnetic units. Introduce the cyclotron frequency $\omega_c = eB/m^*$ and the magnetic length l_0, $l_0^2 = \hbar/eB$, where e is the magnitude of the electron charge, and write

$$\begin{aligned} x' &= \frac{x}{l_0}, \\ y' &= \frac{y}{l_0}, \\ \boldsymbol{\nabla}' &= l_0\boldsymbol{\nabla}, \\ \mathbf{A}' &= \frac{\mathbf{A}}{l_0 B}. \end{aligned} \tag{64}$$

Primed operators have been used to denote operators and variables in magnetic units.

Then

$$H_0 = \frac{\hbar\omega_c}{2}\,\boldsymbol{\pi}'^2, \tag{65}$$

with

$$\boldsymbol{\pi}' = (\frac{1}{i}\boldsymbol{\nabla}' - \frac{q}{e}\mathbf{A}') = \frac{l_0}{\hbar}\boldsymbol{\pi}. \tag{66}$$

The components, π_α, of the conjugate momentum $\boldsymbol{\pi}$, do not commute with one another in the presence of a magnetic field:

$$[\pi_\alpha, \pi_\beta] = iq\hbar\varepsilon^{\alpha\beta\gamma}B^\gamma. \tag{67}$$

With B parallel to the z-direction this reduces to the simple relation

$$[\pi_x, \pi_y] = iq\hbar B. \tag{68}$$

The eigen functions of H_0 which are known as the cyclotron orbits have the well-known energy spectrum

$$E(n) = (n + \frac{1}{2})\hbar\omega_c, \tag{69}$$

where n is known as the Landau level index. One may also show that the density of states per unit area in each Landau level, ρ_0, is given by

$$\rho_0 = \frac{B}{h/e} = \frac{1}{2\pi l_0^2}. \tag{70}$$

N_A, the total number of states in each Landau level, is therefore

$$N_A = \rho_0 A = \frac{\Phi}{h/e}, \tag{71}$$

where Φ is the total flux threading the system of area A. Noting that h/e is just the magnetic flux quantum, we may say that each flux quantum threading a system gives rise to one cyclotron orbit per Landau level.

A.2 Gauge Choice

The form of the eigen functions of H_0 are dependent on the choice of gauge for **A**. The two most popular choices are the Landau gauge and the symmetric gauge. In the Landau gauge an orthonormal basis can be easily constructed which makes translational invariance parallel to some direction, the y-direction say, explicit. In the symmetric gauge the construction of an orthonormal basis of eigen-states of angular momentum is simple.

In the Landau gauge,

$$\mathbf{A} = (0, Bx, 0), \tag{72}$$

one may write the cyclotron orbits, ψ, as

$$\psi = \mid nk > \sim e^{iky'} H_n(x' - k\frac{q}{e}) e^{-(x'-k\frac{q}{e})^2}, \tag{73}$$

where the H_n are the Hermite polynomials [Gradshteyn and Ryzhik, 1980] and again primes denote the use of magnetic units.

In the symmetric gauge,

$$\mathbf{A} = (-B\frac{y}{2}, B\frac{x}{2}, 0), \tag{74}$$

$$\psi = \mid nm > = A_{n,m}(x' + i\frac{q}{e}y')^m e^{-\frac{r'^2}{4}} L_n^m(\frac{r'^2}{2}), \tag{75}$$

$$A_{n,m} = (2^m \frac{(m+n)!}{n!})^{-\frac{1}{2}} \tag{76}$$

where $r'^2 = x'^2 + y'^2$, and the L_m^n are the Laguerre polynomials [Gradshteyn and Ryzhik, 1980]. The $\mid nm >$ are eigen states of the angular momentum L_z defined in Appendix A3, equation 91.

However there is no unique choice for the cyclotron orbits in any particular gauge, nor is there any unique choice of gauge. All results for physical quantities must always be derivable independent of gauge. However it is often possible to exploit the freedom in the choice of gauge and basis to derive certain results more easily and so I note here the form of a gauge transformation:

Some observable, O, with matrix elements $< \psi O \psi >$, will transform to O' under the gauge transformation

$$\psi' = e^{i\frac{q}{e}\theta(x,y)}\psi, \tag{77}$$

with

$$O' = e^{i\frac{q}{e}\theta(x,y)} O e^{-i\frac{q}{e}\theta(x,y)}, \tag{78}$$

which has the same matrix elements in the transformed basis, ψ', as O in the original basis.

In particular the operator $\boldsymbol{\pi}$ becomes

$$\boldsymbol{\pi}_1 = (\frac{\hbar}{i}\nabla - q\mathbf{A}_1), \tag{79}$$

where

$$\mathbf{A}_1 = \mathbf{A} + \frac{\hbar}{e}\nabla\theta. \tag{80}$$

As an example the transformation from the symmetric gauge to the Landau gauge is as follows:

$$\frac{h}{e}\nabla\theta = \mathbf{A}_{Landau} - \mathbf{A}_{Symmetric} = (B\frac{y}{2}, B\frac{x}{2}, 0), \tag{81}$$

$$\theta = \frac{B}{e/\hbar}\frac{xy}{2} = \frac{x'y'}{2} \tag{82}$$

So an eigen function of angular momentum, $\mid nm >$, in the symmetric gauge, becomes after the gauge transformation

$$\mid \widetilde{nm} > = A_{n,m}(x' + i\frac{q}{e}y')^m e^{-\frac{r'^2}{4}} L_m^n(\frac{r'^2}{2}) e^{i\frac{q}{e}\frac{x'y'}{2}}, \tag{83}$$

which is an eigen state of H_0 in the Landau gauge as may easily be checked by acting on the state $\mid \widetilde{nm} >$ with H_0 explicitly.

A.3 Conserved Momenta, Magnetic Translations and Rotations

In the presence of a magnetic field the quantities $\boldsymbol{\pi}$ and $\mathbf{J} = \mathbf{r} \times \boldsymbol{\pi}$ are not conserved. However the conserved quantities corresponding to momentum and angular momentum are easily constructed.

The time-dependence of $\boldsymbol{\pi}$ in a magnetic field is

$$\frac{d\boldsymbol{\pi}}{dt} = q(\frac{d\mathbf{r}}{dt} \times \mathbf{B}). \tag{84}$$

In a constant magnetic field we may define the conserved quantity

$$\boldsymbol{\pi}^B = \boldsymbol{\pi} - q\mathbf{r} \times \mathbf{B}. \tag{85}$$

The $\boldsymbol{\pi}^B$ can then be used to define magnetic translation operators, $T(\mathbf{a})$, with, [Brown 1968],

$$T(\mathbf{a}) = e^{-i\frac{\boldsymbol{\pi}^B.\mathbf{a}}{\hbar}} \tag{86}$$

which commute with the Hamiltonian. In the absence of a magnetic field $T(\mathbf{a})$ reduces to the standard translation operator.

However for two non-parallel vectors **a**, **b** magnetic translation operators $T(\mathbf{a})$ and $T(\mathbf{b})$ only commute up to a phase factor:

$$\begin{aligned} T(\mathbf{a})T(\mathbf{b}) &= exp(-i\frac{\boldsymbol{\pi}^B.\mathbf{a}}{\hbar})exp(-i\frac{\boldsymbol{\pi}^B.\mathbf{b}}{\hbar}) \\ &= exp(-i\frac{\boldsymbol{\pi}^B.(\mathbf{a}+\mathbf{b})}{\hbar})exp(i\frac{q\mathbf{B}.(\mathbf{a}\times\mathbf{b})}{2\hbar}) \\ &= T(\mathbf{a}+\mathbf{b})exp(i2\pi\frac{q}{e}\frac{\mathbf{B}.(\mathbf{a}\times\mathbf{b})}{2\phi_0}) \\ &= T(\mathbf{a}+\mathbf{b})exp(i2\pi\frac{q}{e}\frac{\Phi}{\phi_0}). \end{aligned} \tag{87}$$

The phase factor, Φ/ϕ_0, is just the flux in units of $\phi_0 = h/e$ threading the triangle formed by the vectors **a**, **b**, and **-(a+b)**.

The angular momentum **J**,

$$\mathbf{J} = \mathbf{r} \times \boldsymbol{\pi} = \hbar\mathbf{r}' \times \boldsymbol{\pi}', \tag{88}$$

is also time-dependent. One finds

$$\begin{aligned} \frac{d\mathbf{J}}{dt} &= \mathbf{r} \times \frac{d\boldsymbol{\pi}}{dt} = \mathbf{r} \times q(\frac{d\mathbf{r}}{dt} \times \mathbf{B}) \\ &= q((\mathbf{r}.\mathbf{B})\frac{d\mathbf{r}}{dt} - \mathbf{B}(\mathbf{r}.\frac{d\mathbf{r}}{dt})). \end{aligned} \tag{89}$$

For particles confined to a plane $\mathbf{r}.\mathbf{z} = c$, c constant, and with **B** parallel to the z-direction, we may take c=0, and define a conserved quantity L_z, given by

$$L_z = \frac{q}{e}(\mathbf{r} \times \boldsymbol{\pi})_z + eB\frac{r^2}{2} + n\hbar, \tag{90}$$

which in magnetic units is

$$L_z = \hbar(\frac{q}{e}(\mathbf{r}' \times \boldsymbol{\pi}')_z + \frac{r'^2}{2} + n). \tag{91}$$

Characterization of eigen states of H_0 by angular momentum is useful when discussing the fractional quantum Hall effect. n is the Landau level index. Its inclusion in the definition of L_z is to ensure that L_z is positive. Operating with L_z on equation 76 shows that the $\mid nm >$ are indeed eigen states of L_z.

One may also define the so-called guiding centre coordinate, **R** with components (X,Y),

$$\mathbf{R} = \mathbf{r} - \frac{q}{e}\frac{l_0^2}{\hbar}\mathbf{z} \times \boldsymbol{\pi} \tag{92}$$

which classically is just the centre of the cyclotron motion in the plane perpendicular to the magnetic field. In the absence of any electric fields the quantity **R** is a constant of the motion. In a constant magnetic field the components of **R**. R_α, have the following commutation relations:

$$[R_\alpha, R_\beta] = -il_0^2\frac{q}{e}\epsilon^{\alpha\beta\gamma}\Omega^\gamma, \tag{93}$$

or with the magnetic field parallel to the z-direction

$$[X, Y] = -i\frac{q}{e}l_0^2 = -i\frac{\hbar}{qB}. \tag{94}$$

As the magnetic field is made stronger it becomes an increasingly accurate approximation to treat the quantities **R** as classical variables, for the commutator, [X,Y], tends to zero as B is increased. This has been the basis for an intuitively appealing description of the rôle of localized states in the quantum Hall effect [see Halperin 1986].

In magnetic units, $X' = X/l_0, Y' = Y/l_0$. equation 94 becomes

$$[X', Y'] = -i\frac{q}{e}. \tag{95}$$

For the case of motion in the plane $\mathbf{z}.\mathbf{r} = 0$ one may also write

$$L_z = (\frac{eBr^2}{2} + \frac{q}{e}(\mathbf{r} \times \boldsymbol{\pi})_z + n\hbar) = \frac{eB}{2}\mathbf{R}^2 - \frac{\hbar}{2} \tag{96}$$

which in magnetic units can be written as

$$\frac{L_z}{\hbar} = \frac{\mathbf{R}'^2 - 1}{2}. \tag{97}$$

One may use the components X' and Y' to form operators $Z^{\pm}$,

$$Z^{\pm} = \frac{X' \pm i\frac{q}{e}Y'}{\sqrt{2}}, \tag{98}$$

and then

$$[L_z, Z^{\pm}] = \pm\hbar L_z. \tag{99}$$

The operators $Z^{\pm}$ are raising and lowering operators for the eigen states of the angular momentum L_z defined in equation 91 and equation 97.

A.4 The Single-Particle Green's Function

In the presence of an electric field the single-particle Green's function for electrons confined to two dimensions, $G_0(\mathbf{r}_1, \mathbf{r}_2, \varepsilon)$, has simple asymptotic forms for $\mid \mathbf{r}_1 - \mathbf{r}_2 \mid \gg l_0$, [Joynt and Prange 1984].

Assume that the electric field is applied parallel to the x-direction, then in the Landau gauge H_0 is given for electrons (q = -e) by

$$H_0 = \frac{p_x^2 + (p_y + eBx)^2}{2m^*} + eEx, \tag{100}$$

which in magnetic units has the simple form:

$$H_0 = \frac{\hbar\omega_c}{2}\left(-\frac{\partial^2}{\partial x'} + \left(\frac{1}{i}\frac{\partial}{\partial y'} + x'\right)^2 + 2vx'\right) \tag{101}$$

with $v = eEl_0/\hbar\omega_c$.

H_0 has eigen states, ψ_{nk}, with

$$\begin{aligned} \psi_{nk} &= A_{nk} e^{i(k-v)y'} H_n(x' + k) e^{-\frac{(x'+k)^2}{2}}, \\ A_{nk} &= (2^n n!\sqrt{\pi})^{-\frac{1}{2}}, \end{aligned} \tag{102}$$

where H_n is the n'th Hermite polynomial [Gradshteyn and Ryzhik 1980]. The ψ_{nk} can be normalized for a system of length L', $(= L/l_0)$, in the y-direction. The eigen energies, ε_{nk}, measured from $\frac{\hbar\omega_c}{2}(1 + v^2)$ are

$$\varepsilon_{nk} = (n - kv)\hbar\omega_c \tag{103}$$

k is a continuous real variable and n takes the values 0,1,.... Imagining that the system is bounded at $x = \pm x_0$, the energies ε_{nk}, form bands bounded by ε^u and ε^l with

$$\begin{aligned} \varepsilon^u &= n + vx_0 \\ \varepsilon^l &= n - vx_0 \end{aligned} \tag{104}$$

The Green's function, $G_0(\mathbf{r}_2', \mathbf{r}_1', \varepsilon)$, with primes denoting magnetic units, is given by:

$$G_0(\mathbf{r}_2', \mathbf{r}_1', \varepsilon) = \sum_{n=0}^{\infty} \int \frac{dk}{2\pi} \frac{\psi_{nk}(\mathbf{r}_2')\psi_{nk}^*(\mathbf{r}_1')}{\varepsilon - \varepsilon_{nk} + i\delta}, \tag{105}$$

which can be written as

$$G_0(\mathbf{r}_2', \mathbf{r}_1', \varepsilon) = \frac{X}{2\pi^{\frac{3}{2}}} \sum_{n=0}^{\infty} 2^{-n}(n!)^{-1} \int dk \frac{H_{nk}(x_2' + k)H_{nk}(x_1' + k)e^{-(k+\frac{\alpha}{2})^2}}{\varepsilon - \varepsilon_{nk} + i\delta}, \tag{106}$$

with

$$X = exp(-i(v + \frac{x_1' + x_2'}{2})(y_1' - y_2') - \frac{(r_2' - r_1')^2}{4}) \tag{107}$$

and

$$\alpha = x_1' + x_2' - i(y_2' - y_1'). \tag{108}$$

If ε lies in one of the bands m - kv this integral can be evaluated by integrating round the contour with the real axis as one edge and as the other the line $Imk = (y_2' - y_1')/2$. The contributions from off the real axis and all contributions with $n \neq m$ contribute only exponentially small terms in the limit $\mid y_1' - y_2' \mid \rightarrow \infty$, $\mid x_1' - x_2' \mid \rightarrow \infty$. The asymptotic form for G is determined by the pole at $k = k_0$, where $\varepsilon = m - kv_0$. With the electric field parallel to the positive x-direction as in equation 101, this only contributes for $y_2 < y_1$ so that using θ to denote the Heaviside function:

$$\begin{aligned} lim \mid \mathbf{r}_2' - \mathbf{r}_1' \mid &\rightarrow \infty \\ G_0(\mathbf{r}_2', \mathbf{r}_1', \varepsilon) &= \frac{-i}{v\sqrt{\pi}2^m m!}\theta(y_1' - y_2')H_m(x_2' + k_0)H_m(x_1' + k_0) \\ & exp\left(-\frac{(x_2' + k_0)^2}{2} - \frac{(x_1' + k_0)^2}{2} + i(k_0 - v)(y_2' - y_1')\right) \end{aligned} \tag{109}$$

Comparing with equation 102 shows that G_0 is of the form

$$G_0(\mathbf{r}_2', \mathbf{r}_1', \varepsilon) \sim i\theta(y_1' - y_2')\psi_{nk_0}(\mathbf{r}_2')\psi_{nk_0}^*(\mathbf{r}_1'). \tag{110}$$

The factoring of G_0 in equation 110 implies that only 'forward scattering' of a particle in state, ψ_{nk_0}, is possible for electrons with energies ε within one of the energy continua (equation 104). If the system included impurities close say to the origin one could insert G_0 into Dyson's equation and solve for the asymptotic form for ψ, the eigen state of the perturbed system. Again with an electric field parallel to the positive x direction one would find:

$$\begin{aligned} y' \gg 0, \quad & \psi(\mathbf{r}', \varepsilon = m - vk_0) \sim \psi_{mk_0}(\mathbf{r}'), \\ y' \ll 0, \quad & \psi(\mathbf{r}', \varepsilon = m - vk_0) \sim \psi_{mk_0}(\mathbf{r}')e^{-i\delta(\varepsilon)}, \end{aligned} \tag{111}$$

The impurity potential simply gives rise to the additional phase shift, $\delta(\varepsilon)$.

A.5 Exactness of Laughlin's Wavefunction

The sense in which the Laughlin's wavefuntion, ψ_L (equation 46), is exact is as follows [Haldane 1983]. Any inter-particle interaction of the form $v_{ij} = v(|\mathbf{r}_i - \mathbf{r}_j|)$ conserves the relative angular momentum of the particles i and j, L_z^{ij}, (see appendix A.3). The interaction in the lowest Landau level may therefore be written as

$$v_{ij} = \sum_{m=1,3,\ldots} V_m P_m(i,j). \tag{112}$$

$P_m(i,j)$ projects out from some wavefunction, $\psi(\mathbf{r}_1,\ldots,\mathbf{r}_N)$, those components with relative angular momentum m for particles i and j.

For any pair i and j the two particle wavefunction $\psi(\mathbf{r}_i,\mathbf{r}_j)$ can be written as products of eigen states of relative and centre of mass angular momenta, with quantum numbers M and m, as:

$$\psi(\mathbf{r}_i,\mathbf{r}_j) \sim \psi_M(\frac{\mathbf{r}_i+\mathbf{r}_j}{\sqrt{2}})\psi_m(\frac{\mathbf{r}_i-\mathbf{r}_j}{\sqrt{2}}) \tag{113}$$

In the lowest Landau level and in the symmetric gauge ψ_m takes the simple form, (as shown in appendix A.2, equation 75):

$$\psi_m(\frac{\mathbf{r}_i-\mathbf{r}_j}{\sqrt{2}}) \sim (z_i-z_j)^m e^{-\frac{|z_i|^2+|z_j|^2}{4}}, \tag{114}$$

where $z_i = x_i + i\frac{q}{e}y_i$. Clearly Laughlin's wavefunction, ψ_L, can be written for any particles i and j as:

$$\psi_L \sim (z_i-z_j)^m e^{-\frac{|z_i|^2+|z_j|^2}{4}} P(z_i,z_j), \tag{115}$$

with $P(z_i,z_j)$ some polynomial.

At $\nu = 1/m$ ψ_L is the only wavefunction that contains no relative angular momentum component less than m. If one were to define a hard-core interaction $V^{h.c.}$:

$$\begin{aligned} V_m^{h.c.} &= 1, m < n, \\ V_m^{h.c.} &= 0, m \geq n, \end{aligned} \tag{116}$$

then, at $\nu = 1/n$, ψ_L would be the exact non-degenerate ground state.

A.6 The Hierarchy

The following is based on the works of Haldane [1983] and Halperin [1984].

Suppose a hierarchical fluid with n levels describes the ground state at some filling factor, $\nu(m;p_1,p_2,\ldots,p_n)$, of the lowest Landau level. The N_i quasiparticles/holes at the i'th level with charge $\alpha_i q_i$, where $\alpha = \pm 1$ according to whether the excitations are hole-like or particle-like, and coordinates η_a^i, would have a Laughlin-like ground state:

$$\bar{\psi}_i \sim \prod_{a<b}^{N_i} (\eta_a^i - \eta_b^i)^{p_i+\alpha_i\frac{q_i}{q_{i-1}}} e^{-\sum_c \frac{|\eta_c^i|^2}{4q_i}}. \tag{117}$$

The exponent $p_i+\alpha_i\frac{q_i}{q_{i-1}}$ with p_i an even integer, ensures a consistent statistic for interchange of two excitations: Applying the same argument as the one leading up to equation 54 shows that the change in phase when one quasiexcitation is taken round another, is $\Delta\phi$, with

$$\Delta\phi = 2\pi\alpha_i q_i \frac{\Delta\Phi}{h/e} = 2\pi\alpha_i \frac{q_i}{q_{i-1}}. \tag{118}$$

$\Delta\Phi$, $(= h/q_{i-1})$, is the flux change to nucleate an excitation at level i. This is only consistent with a statistic for interchange of two quasiexcitations with coordinates, η_a^i and η_b^i:

$$\bar{\psi}_i(\eta_b^i,\eta_a^i) = \bar{\psi}_i(\eta_a^i,\eta_b^i)e^{i\pi\alpha_i\frac{q_i}{q_{i-1}}} \tag{119}$$

which is clearly satisfied by the particles described by ψ_i in equation 117.

The excitations from $\bar{\psi}_i$ form the particles of the (i+1)'th level. These carry charge

$$\alpha_{i+1}q_{i+1} = \alpha_{i+1}\frac{q_i}{p_i+\alpha_i\frac{q_i}{q_{i-1}}}, \tag{120}$$

and have a magnetic length, l_{i+1}^2, given by

$$l_{i+1}^2 = \frac{l_0^2}{q_{i+1}} = l_i^2\frac{q_i}{q_{i+1}}. \tag{121}$$

These ratios are determined by the exponent of the term $(\eta_a^i - \eta_b^i)$ in the Laughlin-like ground state proposed in equation 117.

Writing that

$$N_i = \nu(p_i,p_{i+1},\ldots,p_n)N_{i-1}, \tag{122}$$

we may use the condition that the fluid wavefunction describes particles uniformly distributed over the area of the system, A, to derive a recursion relation between the $\nu(p_i,p_{i+1},\ldots,p_n)$. The area, A_i, of a system described by ψ_i is related to the number of zeroes in ψ_i, via

$$A_i = l_i^2(p_i + \alpha_i\frac{q_i}{q_{i-1}} + \alpha_{i+1}\nu(p_{i+1},\ldots,p_n))N_i \tag{123}$$

and the area A_{i-1} is related to the number of zeroes in $\bar{\psi}_{i-1}$, via

$$A_{i-1} = l_{i-1}^2(p_{i-1} + \alpha_{i-1}\frac{q_{i-1}}{q_{i-2}} + \alpha_i\nu(p_i,\ldots,p_n))N_{i-1}. \qquad (124)$$

Requiring that $A_i = A_{i-1}$ and substituting for N_i/N_{i-1}, $(= \nu(p_i,\ldots,p_n)$), l_i^2/l_{i-1}^2, ($= q_{i-1}/q_i$), and $p_{i-1} + \alpha_{i-1}\frac{q_{i-1}}{q_{i-2}}$, ($= \frac{q_{i-1}}{q_i}$), yields the required relation:

$$\nu(p_i,\ldots,p_n) = \frac{1}{p_i + \alpha_{i+1}\nu(p_{i+1},\ldots,p_n)}. \qquad (125)$$

It is straightforward to show that equation 125 has the solution for $\nu(m;p_1,p_2,\ldots,p_n)$ given by equation 57.

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