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WORKSHOP ON ATMOSPHERIC LIMITED AREA MODELLING
15 October - 3 November 1990

"The Primitive Equations"

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1. INTRODUCTION

The state of the atmosphere is adequately characterised by the ambient pressure, temperature, humidity, wind direction and speed at various heights above a large number of fixed observing stations all over the world. These 'field variables' are assumed to have unique values at each point. Furthermore, their derivatives are also assumed to be continuous functions of space and time. The fundamental laws of fluid mechanics and thermodynamics which govern atmospheric motions can then be expressed in terms of the partial differential equations involving these field variables.

The fundamental laws are embodied in the physical principles of conservation of momentum, mass and (heat) energy. The form of the equations is such that there exists no general solutions to them. Nonetheless it is possible to obtain solutions to them for all time by the values of the variables at any instant and at every point in the atmosphere. In principle therefore, this provides the basis for 'mathematical' weather prediction, for the system of differential equations only need to be integrated in time, starting with known initial conditions.

However, because the set of equations governing atmospheric motions is very complex, it is necessary to develop models which are based on systematic simplifications of the fundamental equations. Thus, once a closed set of prediction equations relating the field variables are known, the two other types of information required to predict the future state of the atmospheric circulation are:

(a) the initial state of the field variables

(b) a method of integrating the equations in time to obtain the future distributions of the field variables.

In the following series of lectures, we shall therefore be concerned with the development of the various complete sets of equations containing the scalar field variables. The complete system of scalar equations contain the six unknowns p, T, u, v, α and w and represent the so-called Primitive Equations. An introductory preview of the requirement for an initial state of the variables will also be given, as a background to the Objective Analysis lectures.

2. THE PRIMITIVE EQUATIONS

For dry air, the primitive equation set in the (x, y, z, t) coordinate system may be written as:

$$\frac{\partial u}{\partial t} = -\underline{V}\cdot\nabla u - w\frac{\partial u}{\partial z} - \alpha\frac{\partial p}{\partial x} + fv + F_x \quad \ldots\ldots (1)$$

$$\frac{\partial v}{\partial t} = -\underline{V}\cdot\nabla v - w\frac{\partial v}{\partial z} - \alpha\frac{\partial p}{\partial y} - fu + F_y \quad \ldots\ldots (2)$$

$$\frac{\partial w}{\partial t} = -\underline{V}\cdot\nabla w - w\frac{\partial w}{\partial z} - \alpha\frac{\partial p}{\partial z} - g \quad \ldots\ldots (3)$$

$$\frac{\partial \rho}{\partial t} = -\nabla\cdot(\underline{V}\rho) \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$\frac{\partial T}{\partial t} = -\underline{V}\cdot\nabla T + \frac{Q}{c_p} + \frac{\alpha}{c_p}\frac{dp}{dt} \quad \ldots\ldots (5)$$

$$p\alpha = RT \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

$\underline{V}$ is the total wind vector, with components (u,v,w) in the east-west, north-south and vertical directions respectively, $\alpha\left(=\frac{1}{\rho}\right)$ is the specific volume, Q the rate of heat addition (which depends on the physical processes of absorption, radiation, condensation and eddy heat conduction), Cp the specific heat of air at constant pressure, f the coriolis parameter ($= 2\Omega \sin\theta$, θ is the latitude) and F is the frictional force. Equations (1) - (6) comprise six independent equations in six scalar unknowns u, v, T, p, α and w.

In the full primitive equation set, some inertial terms also appear on the right hand sides of (1) to (3). They arise as a result of the curvature of the earth but are negligibly small. The full derivation of these equations are given in Holton (1979).

As expressed, the primitive equations represent a synthesis of

(a) Newtons second Law for horizontal motion

(b) the ideal gas equation

(c) the First Law of Thermodynamics, and

(d) the Law of Conservation of Mass

The diabatic heating Q and friction F may either be specific as (known) external parameters or expressed in terms of the dependent (field) variables. If latent heat of condensation is important as a heat source, then modifications are necessary in the equations of state and the First Law of Thermodynamics. Also, an additional equation for the conservation of moisture would need to be incorporated into the equation system.

3. THE EQUATIONS IN OTHER COORDINATE SYSTEMS

The primitive equation set (1) - (6), expressed in geometrically fixed (x, y, z) coordinates make physical interpretations easy. Nevertheless it is mathematically more convenient to express them in other coordinate systems for reasons which will be enumerated in due course. The assumption, which is generally true of course, is that large-scale motions are in hydrostatic equilibrium, that is, vertical accelerations and the vertical component of the coriolis force are negligibly small.

Consider then a generalised vertical coordinate π, which is a single-valued monotonic function of pressure p or height z (Kasahara, 1974). In the geometrically fixed coordinates, $\pi = \pi(x, y, z, t)$ but in terms of π as the vertical coordinate the height z is the dependent variable. That is $z = z\ (x, y, \pi, t)$. Therefore any other dependent variable A (either a scalar or vector) is given by A (x, y, z, t) or A (x, y, π , t). Thus, we may write

$$A\ (x, y, \pi, t) = A\ (x, y, z\ (x, y, \pi, t), t)$$

Differentiating partially with respect to an arbitrary $S = S(x, y \text{ or } t)$ gives

$$\left(\frac{\partial A}{\partial s}\right)_\pi = \left(\frac{\partial A}{\partial s}\right)_z + \frac{\partial A}{\partial z}\left(\frac{\partial z}{\partial s}\right)_\pi \qquad \ldots\ldots (7)$$

and z or π denotes a particular vertical coordinate.

Also
$$\frac{\partial A}{\partial \pi} = \frac{\partial A}{\partial z}\frac{\partial z}{\partial \pi} \qquad \ldots\ldots (8)$$

Using (8) in (7) gives the transformation relation

$$\left(\frac{\partial A}{\partial s}\right)_\pi = \left(\frac{\partial A}{\partial s}\right)_z + \frac{\partial A}{\partial \pi}\frac{\partial \pi}{\partial z}\left(\frac{\partial z}{\partial s}\right)_\pi \qquad \ldots\ldots (9)$$

A simple physical relationship between S, π and z is given in Fig 1.

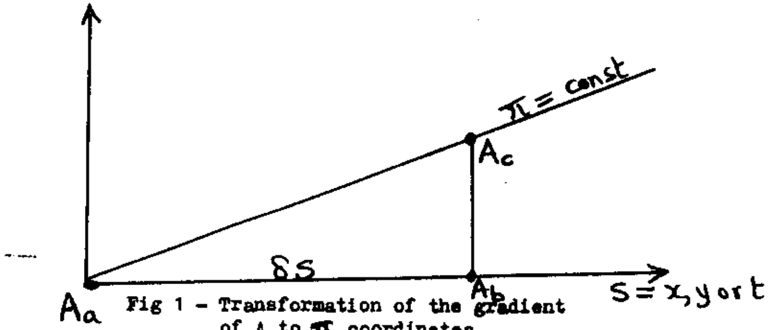

Fig 1 - Transformation of the gradient of A to π coordinates

Since π is the new independent vertical coordinate, the total derivative now is

$$\frac{d}{dt} \equiv \left(\frac{\partial}{\partial t}\right)_\pi + \left(\frac{\partial}{\partial x}\right)_\pi \frac{dx}{dt} + \left(\frac{\partial}{\partial y}\right)_\pi \frac{dy}{dt} + \frac{\partial}{\partial \pi}\frac{d\pi}{dt}$$

$$\Rightarrow$$

$$\frac{dA}{dt} \equiv \left(\frac{\partial A}{\partial t}\right)_\pi + u\left(\frac{\partial A}{\partial x}\right)_\pi + v\left(\frac{\partial A}{\partial y}\right)_\pi + \dot{\pi}\frac{\partial A}{\partial \pi} \qquad \ldots\ldots (10a)$$

or
$$\frac{dA}{dt} = \left(\frac{\partial A}{\partial t}\right)_\pi + \underline{V}\cdot\nabla_\pi A + \dot{\pi}\frac{\partial A}{\partial \pi} \qquad \ldots\ldots (10b)$$

If now S = x, s = y and s = t alternatively, then from (9)

$$\left(\frac{\partial A}{\partial x}\right)_\pi = \left(\frac{\partial A}{\partial x}\right)_z + \frac{\partial A}{\partial \pi}\frac{\partial \pi}{\partial z}\left(\frac{\partial z}{\partial x}\right)_\pi \qquad \ldots\ldots (11)$$

$$\left(\frac{\partial A}{\partial y}\right)_\pi = \left(\frac{\partial A}{\partial y}\right)_z + \frac{\partial A}{\partial \pi}\frac{\partial \pi}{\partial z}\left(\frac{\partial z}{\partial y}\right)_\pi \qquad \ldots\ldots (12)$$

$$\left(\frac{\partial A}{\partial t}\right)_\pi = \left(\frac{\partial A}{\partial t}\right)_z + \frac{\partial A}{\partial \pi}\frac{\partial \pi}{\partial z}\left(\frac{\partial z}{\partial t}\right)_\pi \qquad \ldots\ldots (13)$$

From (11) and (12), we have that

$$\nabla_\pi A = \nabla_z A + \frac{\partial A}{\partial \pi}\frac{\partial \pi}{\partial z}\nabla_\pi z \qquad \ldots\ldots (14)$$

The transformation relations (10) to (14) will be used to obtain the primitive equation set (1) - (6) in the two other coordinate systems in which they are commonly used for numerical weather prediction purposes.

3.1 THE ISOBARIC COORDINATE (case $\pi = p$)

The direct application of the unsimplified primitive equations results in the introduction of unwanted noise by large-amplitude sound and gravity waves which may result from errors in the initial data. The waves then amplify spuriously leading to the loss of the meteorologically important motions. The easiest way to remove these undesired sound waves is by the application of the hydrostatic balance which, as stated before, is an assumption of quasi-horizontal flow. In such a case, the equations can then be written in pressure coordinates.

When the expressions (10) - (12) are applied to equations (1) and (2) with $\pi = p$ and A = u or v, the horizontal momentum equations become

$$\frac{\partial u}{\partial t} = -\underline{V}\cdot\nabla_p u - \omega\frac{\partial u}{\partial p} - \frac{\partial \phi}{\partial x} + fv + F_x \qquad \ldots\ldots (15)$$

$$\frac{\partial v}{\partial t} = -\underline{V}\cdot\nabla_p v - \omega\frac{\partial v}{\partial p} - \frac{\partial \phi}{\partial y} - fu + F_y \qquad \ldots\ldots (16)$$

With hydrostatic balance, using (8) and putting $\phi = gz$, the geopotential, equation (3) reduces to

$$0 = \alpha + \frac{\partial \phi}{\partial p} \qquad \text{...... (17)}$$

The continuity equation becomes

$$\frac{\partial \omega}{\partial p} + \nabla_p \cdot \underline{V} = 0 \qquad \text{....... (18)}$$

The reader is referred to Haltiner & Williams (1980) for detail derivation of (18). The remaining equations for the complete set are

$$C_p \frac{d}{dt}(\ln\theta) = Q \;; \quad \theta = T\left(\frac{P_0}{P}\right)^k \qquad \text{...... (19)}$$

$$p\alpha = RT \qquad \text{...... (20)}$$

where θ is the potential temperature and

$$\dot{\pi} = \frac{dP}{dt} \equiv \omega = \frac{\partial P}{\partial t} + \underline{V} \cdot \nabla_z P + w\frac{\partial P}{\partial z}$$

is the pseudo-vertical velocity in pressure coordinates

The system (15) - (20), though free of undesired vertically propagating sound waves, can still support horizontal so-called Lamb waves which have their maximum amplitude at the lower boundary (ground) and which decay upwards, with hydrostatic balance obtaining everywhere. The isobaric formulation applies quite well in removing these waves with the use of the condition $\omega = 0$ at the lower boundary. This condition, together with the specification of hydrostatic balance, completely filters out sound waves. However, the system of equations still allows gravity wave propagation. We will deal with this problem next.

In spite of the above, some of the advantages of the isobaric coordinate system have now become obvious in equations (15) to (18). These are that:

(a) the continuity equation takes a simpler form

(b) density no longer appear explicitly in the momentum (time-dependent) equations

(c) the equations can be applied to the levels where meteorological data are normally referred and, more importantly,

(d) unwanted sound waves, which can amplify spuriously when equations (1) - (6) are integrated in time, have now been completely filtered out.

3.1.1 Filtering of Gravity Waves

One of the important conditions for gravity wave propagation is a changing and divergent horizontal velocity field. However, it is not immediately obvious that the primitive equation set (1) - (6) or (15) - (20) contain the time variation of divergence. This can be made to appear by obtaining the divergence and vorticity equations from the horizontal momentum equations.

Equations (15) and (16) are first combined into the vector form, neglecting friction F

$$\frac{\partial \underline{V}}{\partial t} = -(\underline{V}\cdot\nabla)\underline{V} - \omega\frac{\partial \underline{V}}{\partial p} - \nabla\phi - \underline{k}\wedge f\underline{V} \quad \ldots\ldots (21)$$

The advection term can be rewritten as

$$(\underline{V}\cdot\nabla)\underline{V} = \nabla\left(\tfrac{1}{2}\underline{V}\cdot\underline{V}\right) + \underline{k}\wedge\underline{V}\xi$$

where $\xi = \underline{k}\cdot\nabla\wedge\underline{V}$ is the vertical component of vorticity, or simply relative vorticity. In the atmosphere, large-scale flows are quasi-horizontal as the vertical velocity is much smaller than the horizontal components, making the vertical component of vorticity of prime importance. Hence only the horizontal wind vector and the vorticity component ξ are mostly needed. Equation (21) then becomes

$$\frac{\partial \underline{V}}{\partial t} = -\nabla\left(\phi + \tfrac{1}{2}\underline{V}\cdot\underline{V}\right) - \underline{k}\wedge\underline{V}(\xi+f) - \omega\frac{\partial \underline{V}}{\partial p} \quad \ldots\ldots (22)$$

Operating on (22) with $\underline{k}\cdot\nabla\wedge$ and $\nabla\cdot$ alternately give the vorticity and divergence equations, respectively.

$$\frac{\partial \xi}{\partial t} = -\underline{V}\cdot\nabla(\xi+f) - \omega\frac{\partial\xi}{\partial p} - (\xi+f)\nabla\cdot\underline{V} + \underline{k}\cdot\left(\frac{\partial \underline{V}}{\partial p}\wedge\nabla\omega\right) \quad \ldots\ldots (23)$$

$$\frac{\partial D}{\partial t} = -\nabla^2\left(\phi + \tfrac{1}{2}\underline{V}\cdot\underline{V}\right) - \nabla\cdot\left[\underline{k}\wedge\underline{V}(\xi+f)\right] - \omega\frac{\partial D}{\partial p} - \frac{\partial \underline{V}}{\partial p}\cdot\nabla \quad \ldots\ldots (24)$$

where $D = \nabla\cdot\underline{V}$

Equations (23) and 24) are independent scalar equations that can be used to replace the two scalar momentum equations (15) and (16) in the prediction set.

In order to filter out the time-dependent gravity waves, it is sufficient to neglect the local rate of change of divergence in (24). However, it is more advantageous to use scaling arguments to neglect all terms whose magnitudes are smaller than $\nabla^2\phi$ in order to remove the gravity waves. To do this, let the wind $\underline{V}$ be partitioned into its nondivergent $\underline{V}_\psi$ and irrotational $\underline{V}_r$ parts such that

$$\underline{V} = \underline{V}_\psi + \underline{V}_r \quad \ldots\ldots (25)$$

where $\nabla\wedge\underline{V}_r = 0 \;;\; \nabla\cdot\underline{V}_\psi = 0$

Then, for two-dimensional flow (since for synoptic-scale motions, vertical accelerations are negligible), a stream function ψ may be defined such that

$$\underline{V}_\psi = \underline{k}\wedge\nabla\psi \quad \ldots\ldots (26a)$$

giving $u_\psi = -\dfrac{\partial\psi}{\partial y} \;;\; v_\psi = \dfrac{\partial\psi}{\partial x}$

Hence

$$\xi = \underline{k}\cdot\nabla\wedge\underline{V} = \nabla^2\psi \quad \ldots\ldots (26b)$$

Also as stated above, the velocity field $\underline{V}$ is quasi-nondivergent for mid-latitude motions $\left(\nabla\cdot\underline{V} \lesssim 10^{-6}\ s^{-1}\right)$ so that

$$|\underline{V}_\psi| \gg \underline{V}_r$$

With all these, scaling analysis leads to the equation

$$\nabla^2\left(\phi + \tfrac{1}{2}\underline{V}_\psi\cdot\underline{V}_\psi\right) = \nabla\cdot\left\{k\cdot\underline{V}(\zeta+f)\right\}$$

Using (26a) and (26b) gives the non-linear balance equation.

$$\nabla^2\left\{\phi + \frac{1}{2}(\nabla\psi)^2\right\} = \nabla\cdot\left\{(f+\nabla^2\psi)\nabla\psi\right\} \quad \text{...... (27)}$$

The balance equation (27) is not usually used in numerical weather prediction in the middle latitude because of its complexity. A reasonable approximation to (27), again for synoptic scale flow, is the so-called linear balance equation

$$\nabla^2\phi = \nabla\cdot(f\nabla\psi) \quad \text{..... (28)}$$

By similar argument, the vorticity equation (23) may be shown to scale down to

$$\frac{\partial\zeta}{\partial t} = -\underline{V}_\psi\cdot\nabla(\zeta+f) - f\,\nabla\cdot\underline{V}_r \quad \text{..... (29)}$$

Note that $\zeta \ll f$ for the mid-latitude case but $\zeta \sim f$ in the tropics. However, outside precipitating (deep convective) systems $\nabla\cdot\underline{V} = 10^{-6}\ s^{-1}$ so that $(\zeta+f)\nabla\cdot\underline{V} \sim 10^{-4}s^{-1}$

Hence (29) further simplifies to the well-known barotropic vorticity equation

$$\frac{\partial\zeta}{\partial t} = -\underline{V}_p\cdot\nabla\zeta - v\beta \quad \text{...... (30)}$$

This equation has been used, with some success, in studying the evolution of tropical synoptic scale disturbances. We shall be dealing more with this later.

Though rather complex, the non-linear balance equation (27) is the more appropriate in the tropics where the stream function ψ cannot be evaluated from the geostrophic vorticity, as implied by (29), for a constant mid-latitude coriolis parameter. Solving the non-linear equation (27) at every time step is not only difficult but also involves considerable computer time. Because of all these and the fact that all the assumptions (quasi-geostrophy etc) limit the accuracy of the prediction even in higher latitudes makes it desirable to have predictive models with less restrictive assumptions.

3.2 THE SIGMA COORDINATES (case $\Pi = \sigma$)

Although the advantages of the isobaric system have been listed, a rather important disadvantage of this sytem is the assumption that $\omega(p_s) \simeq -\rho_s g w(z_s)$ as the lower boundary condition. This assumes that the height of the ground z_s coincides with the pressure P_s (usually taken as 1000 hPa). But this is not strictly true even if the ground is level. As the height of the ground generally varies, it is not correct to assume that the lower boundary condition applies at a constant P_s. Moreover, surface pressure also changes with time. Thus it should strictly be $P_s = P_s\,(x, y, t)$. However, it is not convenient to have a boundary condition to be applied at a horizontally variable surface. This is the basis for the so-called <u>sigma coordinate system</u> which is now commonly used in numerical modelling. In fact, most primitive equation models currently use sigma as the vertical coordinate. This was introduced by Phillips (1957).

In the sigma system, the vertical coordinate is a normalised pressure, viz,

$$\sigma = P/P_s$$

where P_s is the surface pressure. Thus σ, a nondimensional vertical coordinate, takes the value $\sigma = 1$ at the ground where $P = P_s$, and zero at the top of the atmosphere. The lower boundary condition will therefore always apply exactly at $\sigma = 1$ and the vertical velocity $\dot\sigma = d\sigma/dt$ will always be zero at the ground even when the terrain is not level.

Applying transformation (11) with $\pi = \sigma$ and $z = p$, we have that

$$\left(\frac{\partial A}{\partial x}\right)_\sigma = \left(\frac{\partial A}{\partial x}\right)_p + \frac{\sigma}{P_s}\frac{\partial P_s}{\partial x}\frac{\partial A}{\partial \sigma} \qquad \ldots\ldots (31)$$

$$\left(\frac{\partial A}{\partial y}\right)_\sigma = \left(\frac{\partial A}{\partial y}\right)_p + \frac{\sigma}{P_s}\frac{\partial P_s}{\partial y}\frac{\partial A}{\partial \sigma} \qquad \ldots\ldots (32)$$

or, in vector form

$$\nabla_p A = \nabla_\sigma A - \frac{\sigma}{P_s}\nabla P_s \frac{\partial A}{\partial \sigma} \qquad (33)$$

If (32) is now applied to (15) and (16), then combined after neglecting the friction term, we obtain

$$\frac{\partial \underline{V}}{\partial t} = -(\underline{V}\cdot\nabla_\sigma)\underline{V} - \dot\sigma\frac{\partial \underline{V}}{\partial \sigma} - \nabla_\sigma \phi + \frac{\sigma}{P_s}\nabla P_s \frac{\partial \phi}{\partial \sigma} \quad \ldots (34)$$

The hydrostatic equation is obtained by applying (8) to (17). The result is

$$\frac{\partial \phi}{\partial \sigma} + \sigma P_s = 0 \qquad \ldots\ldots (35)$$

The continuity equation is obtained as follows. Firstly apply (8) to (18) to get

$$P_s(\nabla\cdot\underline{V})_p + \frac{\partial \omega}{\partial \sigma} = 0 \qquad \ldots\ldots (36)$$

Now, in the σ-coordinates, the vertical velocity $\dot\sigma$ is given by

$$\begin{aligned}\dot\sigma = \frac{d\sigma}{dt} &= \frac{\partial \sigma}{\partial t} + \underline{V}\cdot\nabla\sigma + \omega\frac{\partial \sigma}{\partial p} \\ &= -\frac{\sigma}{P_s}\left(\frac{\partial P_s}{\partial t} + \underline{V}\cdot\nabla P_s\right) + \frac{\omega}{P_s}\end{aligned} \qquad \ldots\ldots (37)$$

Next, differentiate this w.r.t σ and rearrange to get

$$\frac{\partial \omega}{\partial \sigma} = P_s\frac{\partial \dot\sigma}{\partial \sigma} + \left(\frac{\partial P_s}{\partial t} + \underline{V}\cdot\nabla P_s\right) + \sigma\frac{\partial \underline{V}}{\partial \sigma}\cdot\nabla P_s$$

Using this last expression and (22) in (36) finally gives the transformed continuity equation:

$$\frac{\partial P_s}{\partial t} = -\nabla\cdot(P_s\underline{V}) - P_s\frac{\partial \dot\sigma}{\partial \sigma} \qquad \ldots\ldots (38)$$

Note that $\partial P_s/\partial\sigma = 0$, since $P_s \neq P_s(\sigma)$. The boundary conditions on the vertical velocity in this σ-system are:

at the surface $\dot\sigma = 0$ as $p = p_s$, giving $\sigma = 1$

at the top of the atmosphere $\dot\sigma = 0$ as $p = 0$ hence $\sigma = 0$

Equation (38) can be integrated over the entire atmosphere, using the above conditions, to obtain the surface pressure tendency equation:

$$\frac{\partial P_s}{\partial t} = -\int_0^1 \nabla\cdot(P_s \underline{V})\, d\sigma \qquad \text{...... (39)}$$

Integrating (38) from $\sigma = 1$ to an arbitrary level σ gives the vertical velocity at the level σ. Thus

$$P_s\dot{\sigma} = -\sigma\frac{\partial P_s}{\partial t} - \int_0^\sigma \nabla\cdot(P_s\underline{V})\, d\sigma \qquad \text{...... (40)}$$

Finally, the thermodynamic equation becomes

$$\frac{\partial\theta}{\partial t} = -(\underline{V}\cdot\nabla)\theta - \dot{\sigma}\frac{\partial\theta}{\partial\sigma} + Q\frac{\theta}{C_p} \qquad \text{...... (41)}$$

Equations (34), (35), (39), (40) and (41) contain the six independent scalar variables u, v, $\dot{\sigma}$, θ, ϕ and P_s and they represent the complete set of primitive equations in sigma coordinates which can then be used to predict the future state of the atmosphere.

Alternatively, equations (34) and (41) may be written in their <u>flux forms</u>, which are commonly used in the numerical integration. Multiply (41) by Ps and (38) by θ and add the results to obtain

$$\frac{\partial}{\partial t}(P_s\theta) = -\nabla\cdot(P_s\underline{V}\theta) - \frac{\partial}{\partial\sigma}(P_s\dot{\sigma}\theta) + \frac{P_s\theta}{C_p}Q \qquad \text{...... (42)}$$

Next, expand (34) into its components, multiply each in turn by Ps, next multiply (38) by u or v and then add together to get the flux forms of the momentum equations:

$$\frac{\partial}{\partial t}(P_s u) = -\nabla\cdot(P_s u\underline{V}) - \frac{\partial}{\partial\sigma}(P_s u\dot{\sigma}) + fP_s v - P_s\frac{\partial\phi}{\partial x} + \frac{\sigma}{P_s}\frac{\partial\phi}{\partial\sigma}\frac{\partial P_s}{\partial x} \qquad \text{..... (43)}$$

$$\frac{\partial}{\partial t}(P_s v) = -\nabla\cdot(P_s v\underline{V}) - \frac{\partial}{\partial\sigma}(P_s v\dot{\sigma}) - fP_s u - P_s\frac{\partial\phi}{\partial y} + \frac{\sigma}{P_s}\frac{\partial\phi}{\partial\sigma}\frac{\partial P_s}{\partial y} \qquad \text{...... (44)}$$

The alternate primitive equation set in flux forms now comprises equations (35), (39), (40), (42), (43) and (44). Note that an immediate result of the condition that $\dot{\sigma} = 0$ at $\sigma = 1$ is the concise form (equation (39)) for computing the surface pressure changes. Once $\partial P_s/\partial t$ is obtained, it can be used to evaluate the vertical velocity $\dot{\sigma}$ from (40) and hence the terms $\frac{\partial}{\partial\sigma}(P_s\dot{\sigma}\theta)$ in equation (42) and similar terms in (43) and (44).

4. <u>PRIMITIVE EQUATION MODELS</u>

Having assembled all the equations needed in the various coordinate systems, we will next briefly discuss the several variants of numerical prediction models which utilise the primitive equations and which are in current use. In sections 4.1 and 4.2, models specially suitable to the tropical regions are discussed while section 4.3 deals with models commonly used in the extra-tropics.

4.1 <u>ONE-LEVEL MODEL</u>

This model which allows divergent motions and is based on the shallow water equations is also known as the divergent barotropic model. It is the next in hierarchy to the so-called barotropic model whose equations are different only for the mass continuity. It is a special case of the shallow water model presented by Pedlosky (1979). In the present case, there is no bottom topography, that is, the terrain height h is zero.

The model contains three unknowns u, v and z (or ϕ).

The equations are

$$\frac{\partial u}{\partial t} = -\underline{V}\cdot\nabla u + fv - g\frac{\partial z}{\partial x} \qquad \text{...... (45)}$$

$$\frac{\partial v}{\partial t} = -\underline{V}\cdot\nabla v - fu - g\frac{\partial z}{\partial y} \qquad \text{...... (46)}$$

and the continuity equation

$$\frac{\partial z}{\partial t} = -\underline{V}\cdot\nabla_h z - z\nabla\cdot\underline{V} \qquad \text{...... (47)}$$

where z is the height of the free surface and ∇_h is the horizontal del operator. Except for the mass continuity equation (47), this set (45) - (47) is the same as for the non-divergent barotropic system. However, the latter model employs the more convenient and differentiated form of (45) and (46) as given by (29). As already discussed, the system allows for the propagation of gravity waves which, as part of the system, can grow spuriously as a result of imbalances in the initial data. The method of solving this problem will be discussed under Objective Analysis of Meteorological Fields (Tibaldi, 1990).

The initial values of the free height z (or the geopotential ϕ) is obtained by solving the non-linear balance equation (27) which we can also be written as:

$$\nabla^2\phi = \nabla\cdot f\nabla\psi + 2J\left(\frac{\partial\psi}{\partial x},\frac{\partial\psi}{\partial y}\right) \qquad \text{...... (48)}$$

where J is the Jacobian operator. The initial fields of u and v are obtained from streamline and isobach analysis. Equation (48) is called the reverse balance equation because the pressure (geopotential) field is determined from the motion field, as against the mid-latitude situation where the pressure field is used to determine the wind field.

Because the phase velocity of the gravity waves relative to the mean flow is given by

$$c = u \pm \phi^{1/2} \qquad \text{...... (49)}$$

the choice of ϕ (or z) is very crucial. From the work of Williamson (1976), the mean height z was fixed at 2000m, which is assumed to represent the height of the 700 hPa. The solution of the reverse balance equation is then followed by dynamic initialisation to bring the initial mass and wind fields into a state of balance. This is usually accomplished by repeated forward and backward integration.

Full details of this model (and the version with bottom topography in section 4.2 below) can be found in Krishnamurti and Pearce (1977), Adejokun and Krishnamurti (1983) and Krishnamurti (1986).

4.1.1 DOMAIN INVARIANTS OF THE MODEL

In developing a numerical model, it is very important to ensure that it conserves the integral properties of the flow when the model prediction equations are put into their finite difference analogue. It is usual to place emphasis on the flow properties in a closed domain into and out of which there can be no net mass flux, since it is within such a domain that the evolution of the systems are best seen.

Consider then the vorticity equation (29)

$$\frac{\partial \zeta_a}{\partial t} = -\underline{V}_h \cdot \nabla \zeta_a - \zeta_a \nabla \cdot \underline{V} \qquad \text{...... (50)}$$

where $\zeta_a = \zeta + f$ and $\underline{V}_h$ is the horizontal wind. The divergent term can be removed by the use of (47), with the result

$$\frac{\partial}{\partial t}\left(\zeta_a/z\right) + \underline{V}_h \cdot \nabla\left(\zeta_a/z\right) = 0$$

or

$$\frac{d}{dt}\left(\frac{\zeta_a}{z}\right) = 0 \qquad \text{...... (51)}$$

Equation (51) states that the potential vorticity $\zeta_p (= \zeta_a/z)$ is a domain invariant. By multiplying (51) by $\frac{1}{n}\zeta_a^{n-1}/z$, it can be shown that ζ_a^n is also an invariant.

A third domain invariant is the height of the mean free surface z. We first note that the continuity equation (47) can be expressed in the form

$$\frac{\partial z}{\partial t} + \nabla \cdot (z \underline{V}_h) = 0$$

Integration over a closed domain yields

$$\frac{\partial}{\partial t}\iint z \, dx\, dy = 0 \qquad \text{...... (52a)}$$

Yet another domain invariant is the total energy E. This can be shown as follows. Multiply equation (45) by u and (46) by v and then add.

The result is

$$\frac{\partial k}{\partial t} + \underline{V} \cdot \nabla k = -\underline{V} \cdot \nabla (gz) \qquad \text{...... (53)}$$

Now, multiply (47) by (k + gz) and (53) by z and add to obtain:

$$\frac{\partial}{\partial t}\left[z\left(k + \frac{gz}{2}\right)\right] + \underline{V} \cdot \nabla (kz) - z(k + gz)\nabla \cdot \underline{V} + z\underline{V} \cdot \nabla gz = 0$$

This can be rearranged to give the flux form

$$\frac{\partial}{\partial t}\left[z\left(k + \frac{gz}{2}\right)\right] + \nabla \cdot (kz\underline{V}) + \nabla \cdot \left(\underline{V} g \frac{z^2}{2}\right) = 0$$

Integrating over a closed domain leads to

$$\frac{\partial}{\partial t}\iint \left[z\left(k + \frac{gz}{2}\right)\right] dx\, dy = 0 \qquad \text{...... (54)}$$

showing that the total energy $E = z(k + g^z/2)$ is an invariant.

4.2 ONE-LEVEL-MODEL WITH TOPOGRAPHY

This is an extension of the above model with the height of the bottom terrain h incorporated into the equations. These are

$$\frac{\partial u}{\partial t} = -\underline{V}_h \cdot \nabla u + fv - g\frac{\partial}{\partial x}(z + h) \qquad \text{...... (55)}$$

$$\frac{\partial v}{\partial t} = -\underline{V}_h \cdot \nabla v - fu - g\frac{\partial}{\partial y}(z + h) \qquad \text{...... (56)}$$

$$\frac{\partial z}{\partial t} = -\underline{V}_h \cdot \nabla z - z\nabla \cdot \underline{V}_h \qquad \text{...... (57)}$$

h is the smoothed mountain height.

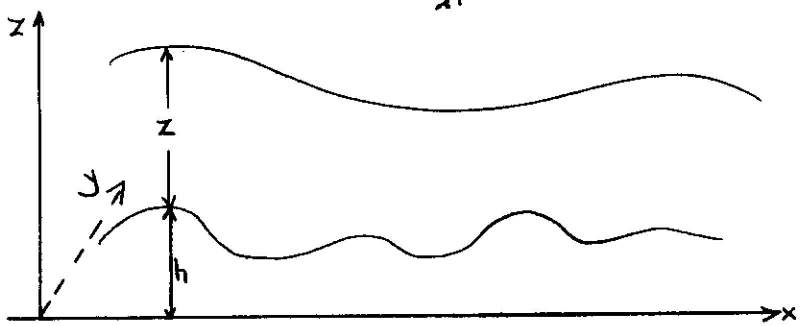

Fig 2 One-level with terrain height h and mean free surface height z.

4.2.1 INVARIANTS OF THE MODEL

As already presented for the special case of no bottom topography above (i.e h=o), the parcel and domain invariants are the potential verticity ζ_p its powers (e.g ζ_p^2) and the total energy E'. In this case however $E' = z\,(k + g^2/2 + gh)$. Proof of these are similar to the case h=o.

The two models described so far are widely used in the tropics. However, their performances vary from one region of the tropics to another. Discussion on their performance can be found in Krishnamurti et al (1987) and Yap (1987).

Attention will now be focussed on the primitive equation models that have become popular in the extra-tropics.

4.3 THE SIGMA-COORDINATE MODELS

Most current primitive equation models use the prediction equations formulated in the sigma (σ) coordinates. The equation set are (34), (35), (39) - (41) or their flux forms (35), (39), (40) and (42) - (44).

One of the important advantages of such models is that the variation of static stability parameter with time (an important atmospheric property) can be evaluated by computing $\partial\theta/\partial\sigma$ explicitly at each grid point and each time step. This parameter is usually specified as a constant in quasi-geostrophic models. There are currently two versions of the primitive equation models in sigma coordinates: the two- and six-layer models. A brief summary of these are presented here.

4.3.1 TWO-LAYER MODEL

In this model, ~~the~~ atmosphere is divided into two layers with the level surfaces labelled as 0 (ground), 2 and 4. Each layer is then subdivided into two with their mid-levels labelled 1 and 3. The momentum and thermodynamic equations ((43), (44) and (42) respectively), which are prognostic, are applied at levels 1 and 3. Initial data for the model (u and v) are obtained from the non-linear balance equation (27) since, for the mid-latitude case, the geopotential ϕ distributions are usually known.

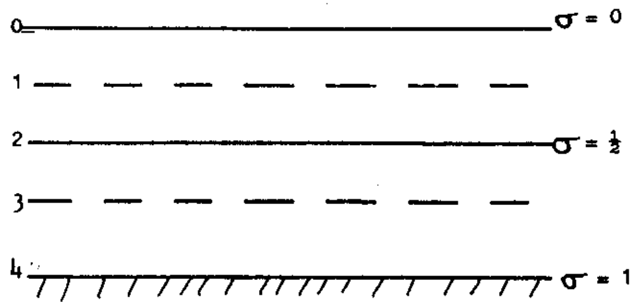

Fig 3 Vertical differencing Scheme for the two-layer model

To make a forecast with this model

(a) the finite difference analogues of (42) - (44) are first written. (This will be shown in the present workshop by Pearce (1990).

(b) step (a) is repeated for the surface pressure tendency equation (39).

(c) new set of values for u_1, v_1, θ_1, u_3, v_3, θ_3 and P_s are computed using suitable time differencing. The time step must be very small and should satisfy the condition

$$\Delta t = \frac{\Delta x}{c\sqrt{2}} \qquad \ldots\ldots (58)$$

c is the speed of the sound waves. This criterion takes care of the gravity and horizontal sound waves which are present in the primitive equations.

(d) the new values of u_1, v_1, θ_1 etc are then used to evaluate $\dot{\sigma}$, ϕ_1 and ϕ_3 diagnostically.

4.3.2 THE SIX-LAYER MODEL

This is a more complex model used for operational numerical prediction by the United States National Meteorological Centre. As its name suggests, it has six prediction levels in the vertical with a modified σ - system which allows better resolution in the stratosphere. In the model, the vertical coordinate σ is given by (Arakawa and Lamb (1977))

$$\sigma = \frac{P - P_m}{\pi} \qquad \ldots\ldots (59)$$

where

$$\pi = \begin{cases} \pi_u = P_m - P_T, & P_T \leq P \leq P_m \\ \pi_L = P_s - P_m, & P_m \leq P \leq P_s \end{cases}$$

P_m and P_T are constants.

Thus

$$\sigma \begin{cases} = 1, & P = P_s \\ = -1, & P = P_T \end{cases}$$

and behaves like the previously described σ below P_m but as p-coordinate above P_m. When $P_m = P_T = 0$, we recover the original σ - coordinate.

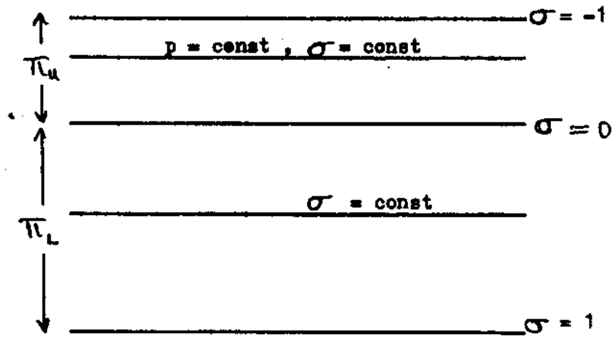

Fig 4 Layer definition in the model

The initial data for the model is also obtained from the balance equation (27).

The six-layer primitive equation model is suitable for both limited area prediction and general circulation studies.

4.4 SPECTRAL MODELS

Having described the common current primitive equation models, we conclude this lecture series with a short summary of spectral application to the primitive equations in numerical forecasting.

In the spectral model of the primitive equations, the momentum equations are replaced by the vorticity and divergence equations (29) and (24) but here written, respectively, in the forms

$$\frac{\partial \xi}{\partial t} = -\nabla \cdot \left[(\xi + f)\underline{V}\right] \qquad \text{...... (60)}$$

$$\frac{\partial D}{\partial t} = -\nabla^2\left(\phi + \tfrac{1}{2}\underline{V}\cdot\underline{V}\right) + \underline{k}\cdot\nabla_\wedge\left[(\xi+f)\underline{V}\right] \qquad \text{...... (61)}$$

Note that the divergence equation has been scaled so that the last two terms of equation (24) are neglected. Also,

$$\underline{V} = \nabla\chi + \underline{k}_\wedge\nabla\psi \qquad \text{...... (62a)}$$

and

$$\xi = \nabla^2\psi \; ; \; D = \nabla^2\chi \quad (\nabla\cdot\underline{V}_\psi = 0) \qquad \text{...... (62b)}$$

Equations (60) and (61) are more convenient and simpler when spherical harmonics (of orthogonal functions) are the basic functions. Also, the resulting (ordinary differential) equations then contain only scalar variables.

Further details of the use of spectral technique in the primitive equations numerical prediction can be found in Bourke (1972).

REFERENCES

Adejokun, J.A. and T.N. Krishnamurti 1983: Further Numerical experiments on tropical waves. TELLUS, 35A, 398-416.

Arakawa, A. and V.R. Lamb 1977: Computational design of the basic dynamical processes of the UCLA general circulation model. Methods in Computational Physics, 17, Academic Press, 174-265, 337 pp.

Bourke, W. 1972: An efficient, one-level, primitive-equation spectral model. Mon. Wea. Rev., 100, 683-689.

Haltiner, G.J. and R.T. Williams 1980: Numerical Prediction and Dynamic Meteorology (2nd Edition). John Wiley & Sons.

Holton, J.R. 1979: An Introduction to Dynamic Meteorology (2nd Edition). Academic Press, 391 pp.

Kasahara, A. 1974: Various vertical coordinate systems used for numerical weather prediction. Mon. Wa. Rev., 102, 504-522.

Krishnamurti, T.N. and R.P. Pearce 1977: The fundamentals of numerical weather prediction and the filtered barotropic model. Proceedings of WMO Seminar, Dakar, November 1976. WMO Report No. 492, Geneva, Switzerland.

Krishnamurti, T.N., A. Kumar and Xiang Li. 1987: Results of extensive integrations with simple NWP models over the tropics during FGGE. TELLUS.

Krishnamurti, T.N. 1986: Workbook on Numerical Weather Prediction for the Tropics for the Training of Class I and II Meteorological Personnel. WMO Tech. Pub. No. 669. Geneva, Switzerland.

Pearce, R.P. 1990: Numerical solutions of the Primitive Equations. ICTP/WMC Workshop on Limited Area Modelling. 15 October to 03 November 1990. Trieste, Italy.

Phillips, N.A. 1956: The general circulation of the atmosphere: A numerical experiment. Quart. J. Roy. Met. Soc., 82, 123-164.

Pedlosky, J. 1979: Geophysical Fluid Dynamics. Springer-Verlag, New York, 624 pp.

Tibaldi, S. 1990: Objective Analysis of Meteorological Fields.
ICTP/WMO Workshop on Limited Area Modelling.
15 October to 03 November 1990. Trieste, Italy.

Williamson, D.L. 1976: Normal mode initialisation procedure applied to forecasts with global shallow water equations. Mon. Wea. Rev., 104, 195-206.

Yap, Kok-Seng. 1987: Documentation on two simple tropical models. WMO Tropical Met. Programme Report Series No. 26.