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"The One-Sex Mixing Problem: A Choice of Solutions?"

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"THE ONE-SEX MIXING PROBLEM: A CHOICE OF SOLUTIONS?"

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Abstract

The transmission dynamics of HIV/AIDS, and of sexually transmitted diseases (STDs) in general, is highly dependent on the population social/sexual mixing structure, i.e., how many partners and who they are. STD epidemic models require mathematical descriptions of mixing which must satisfy various natural constraints. A few such solutions to the one-sex (homosexual) mixing problem have been found, and there has been some confusion as to which is the "correct" one to use. Only one of the solutions is completely general, and in addition reduces the problem of choosing a mixing function to one of choosing parameters.

The development of predictive models of the transmission dynamics of HIV, gonnorhea, syphilis, etc. has been severly hampered by the lack of a mathematical framework allowing a realistic description of social/sexual mixing. This is true both of one-sex (homosexual) and two-sex (heterosexual) epidemics. The problem has been discussed in detail in the literature (Barbour, 1978; Nold, 1980; Anderson et.al., 1989a,b,c; Blythe and Castillo-Chavez, 1989; Hyman and Stanley, 1989; Castillo-Chavez and Blythe, 1989; Jacquez et.al., 1989; Sattenspiel, 1989; Busenberg and Castillo-Chavez, 1989) and arises because descriptions of mixing (who everyone's partners are) are specified only by a set of constraints (eg. Castillo-Chavez and Blythe, 1989). In a one-sex population with N distinct groups, if $p_{ij}(t)$ is the mixing function (fraction of partners taken by people in group i among those in group j at time t), then the constraints amount to making the $\{p_{ij}(t)\}$ a set of probabilities, conserving the number of partnerships between each pair of groups, and ensuring that no one can take partners from an empty group. As N discrete groups are usually more convenient for practical modeling purposes than a continuum description, we shall use the former throughout, noting only that all results can easily be extended to the continuum case.

We shall also consider here only those mixing problems where all the possible links between groups exist (i.e. individuals in any group may form partnerships with individuals from any group, including their own). Some models with incomplete connectedness have been formulated (e.g. Jacquez et.al., 1989) and the general case and its analysis will be considered elsewhere (Blythe in prep.; Blythe et.al. in prep).

To date four solutions $p_{ij}(t)$ to the one-sex mixing problem have been found (Table I). The earliest, best known and most widely used is ((A) in Table I) proportionate or random mixing (Nold, 1980; Hethcote and Yorke, 1984), where mixing among groups occurs in proportion to the total number of partnerships formed by all the people in each group. Although easy to apply, (A) suffers from the disadvantage that human mixing behavior is not believed to be random. For this reason other solutions were sought.

In "preferred mixing" ((B) in Table I), the $p_{ij}(t)$ are formed by having the members of each group reserve a constant fraction of their partners among themselves, the rest being spread randomly over all groups (Jacques et.al., 1988). Hyman and Stanley's (1989) mixing function ((C) in Table I) is a much more general form of $p_{ij}(t)$, involving an arbitrary function of i and j (in our notation; it was originally formulated in the continuum case). Koopman et.al. (1989) also present a mixing model ((D) in Table I). In fact it may readily be shown by manipulation that (D) and (C) are the same solution written in slightly different ways, and interpreted quite differently by their originators. A recent approach due to Morris (pers. comm.) is also a particular case of this type of solution.

The fifth solution, ((E) in Table I) is generalized mixing (Blythe and Castillo-Chavez, 1989; Busenberg and Castillo-Chavez, 1989; Castillo-Chavez and Blythe, 1989). For a single-sex population consisting of N discrete groups, within each of which all individuals have the same level of sexual activity at any given time, and where they mix independently, (E) may be shown to be the general solution (Busenberg and Castillo-Chavez, in press). All other solutions may be written in this form, i.e., as multiplicative perturbations of random mixing. The key to (E) is the set of parameters $\{\phi_{ij}\}$. These must be chosen such that none of the functions $\{R_i(t)\}$ (see Table I) are negative. If the $\{\phi_{ij}\}$ are constants, then because they prescribe the multiplicative perturbation, they prescribe, in some sense, the deviation from ran dom mixing (in some cases a rough measure (Blythe and Castillo-Chavez, MS) of this deviation is the simple range $\phi_{max} - \phi_{min}$). We discuss the biological and behavioral interpretation of the $\{\phi_{ij}\}$ elsewhere (Palmer et.al., in prep.); roughly speaking, each ϕ_{ij} specifies the degree of non-randomness in the interaction link between groups i and j.

Demonstrating that the other solutions are special cases of the General Solution (D) involves finding a set of $\{\phi_{ij}\}$ which recover them. The third column in Table I lists appropriate $\{\phi_{ij}\}$ for the various solutions. Note that as the relationship is non-unique (Blythe et.al., MS), there exist many distinct sets of $\{\phi_{ij}\}$, which will recover any particular solution, if information on actual mixing proportions $\{p_{ij}\}$ is available only at one point in time. This is particularly clear for proportionate mixing. Note also that the equivalent $\{\phi_{ij}\}$ for "preferred mixing" (B) are time-dependent. Hence the distance from random mixing in this case varies with time (not surprising as the reserved fractions do not depend on group population sizes).

Solutions (C) and (D) in Table I, and Morris' (pers. comm.) solution which has $c_i = c$ (all i) and f_{ij} = a set of constants (which may be interpreted as saying that all individuals have exactly the same number of <u>social</u> contacts per unit time c, and that given they meet, an individual from group i and one from group j will have sex with fixed probability f_{ij}), are all clearly members of the same family. Note that if the general solution (E) is written in the form $p_{ij}(t) = \theta_{ij}(t)\,\bar{p}_j(t)$, then the (C/D) family is

$$\frac{f_{ij}}{\sum_{k=1}^{N} f_{ik}\bar{p}_k} = \theta_{ij}(t) \qquad (1)$$

It may readily be shown that if $f_{ij} = f_{ji} > 0$ and

$$\sum_{k=1}^{N} f_{ik}\bar{p}_k < 2 \qquad (2)$$

then (1) is a representation of the general solution. It should be clear, however, that constant $\{f_{ij}\}$ implies time dependent $\{\phi_{ij}\}$ (unless they are all equal, i.e. random mixing), so that the degree of non-randomness in each inter-group link changes with time. This representation thus has the same problem as does "preferred mixing": by holding an arbitrary set of parameters constant, the "distance" from random mixing, as prescribed by the $\{\phi_{ij}\}$, varies with time. Note that simply making the $\{f_{ij}\}$ time-dependent in an arbitrary manner (e.g. Koopman et.al., 1989) does not remove this problem; only choices of time-dependent $\{f_{ij}\}$ which are consonant with constant $\{\phi_{ij}\}$ maintain a constant "distance" from random mixing.

The source of the problem may be made abundantly clear as follows. Suppose (as do Koopman et.al., 1989, and Morris, pers. comm.) that sexual interaction is comprised of two separate processes. First, individuals mix socially, so that we have a "pre-cursor" social mixing matrix, $\{p_{ij}^{(1)}\}$ say. Then, individuals who have met decide whether or not to have sex, according to some set of rules which may be summed up in some matrix. Combining these two processes produces a sexual mixing matrix, say $\{p_{ij}^{(2)}\}$. This has two effects: first, the number of sexual contacts per unit time for each individual becomes, in general, a function of time (less than or equal to the number of social contacts in the "pre-cursor" mixing); this presents no problems. However, arbitrary choice of sexual acceptance parameters has a second effect, namely that because such a description does not take account of group-size changes with time (or does so at best in an ad hoc manner), the overall mixing process $\{p_{ij}^{(2)}\}$ in a sense decouples individual behavior from population dynamics. For example, the implication is that if a group i and a group j person meet, their overall probability of becoming sexual partners is unrelated to the abundance or scarcity of individuals in those, or any other, groups (or related in an ad hoc manner, c.f. Koopmen et.al., 1989). This produces time varying $\{\phi_{ij}\}$.

We feel that selection of sexual partners from social contacts will depend on the relative abundance of perceived groups; for example, people becoming more selective if a desirable group is abundant, and vice versa. Under the constant - $\{\phi_{ij}\}$ hypothesis, there is an underlying, invariant structure of mutual absolute preferences amongst all the groups in the population. Changes in activity (the $\{c_i(t)\}$ and/or group sizes (the $\{T_i(t)\}$) have no effect on this absolute preference structure. Where the $\{f_{ij}\}$ are constant, even if the $\{c_i\}$ do not change, the underlying preference structure does, and may be seen to do so as the $\{\phi_{ij}\}$ change with time. Arbitrarily changing $\{f_{ij}\}$ have exactly the same effect.

We note in passing that the empirical approach of Gupta et al. (17), where the $\{p_{ij}(t)\}$ are held constant at their initial (t=0) values by judiciously altering the group activity levels ($\{c_i(t)\}$ in Table I), does maintain a constant distance from proportionate mixing, but only at the cost of altering the proportionate mixing functions ($\{\bar{p}_i(t)\}$ in Table I) themselves through their assumption of adaptive sexual behavior change.

In conclusion we offer the following. If, in the absence of social/medical factors which cause a change in absolute preference, individuals in the population have some ranking of other individuals a priori as regards desirability for forming sexual partnerships, then a constant $\{\phi_{ij}\}$ description should be (at least approximately) correct. The implication is that the general solution (E) should be used as it stands for both modelling (Blythe and Castillo-Chavez, in prep.; Palmer et.al. in prep.) and estimation of mixing structures (Blythe et.al., MS). The alternative representation (using $\{f_{ij}\}$) is fine if the $\{f_{ij}(t)\}$ are known for all time, and are equivalent to constant $\{\phi_{ij}\}$ -- but in that case no advantage is to be gained from not using (E) directly. In the apparently beguiling case where the $\{f_{ij}\}$ are constructed from behaviors in a succession of mixing steps (á la Koopman et.al., 1989; Morris pers. comm.), the process of decoupling the steps (e.g. meet, talk, have sex) removes any invariance in the underlying structure of preference for sexual partners per se. We would suggest that while there is a great deal of choice for solutions of the mixing axioms, for descriptive modelling or parameter estimation the general representation solution (E) is the only one that makes sense under all contingencies.

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Table I

Relationship Between Mixing Functions

Mixing function	$p_{ij}(t)$	$\phi_{ij}(t)$
(A) Proportionate mixing[a]	$\bar{p}_j(t) \equiv \frac{c_j(t)T_j(t)}{\sum_{k=1}^{n} c_k(t)T_k(t)}$	α, all i, j $0<\alpha<1$
(B) Preferred mixing[b]	$\delta_{ij}f_i + (1-f_i)\frac{(1-f_j)\bar{p}_j(t)}{\sum_{k=1}^{n}(1-f_k)\bar{p}_k(t)}$	$\delta_{ij}f_i/\bar{p}_i(t)$
(C) Stanley's function[c]	$\left[1-\sum_{k=1}^{i} p_{ik}(t)\right]\frac{F_{ij}\bar{p}_j(t)}{\sum_{k=1}^{i} F_{ik}\bar{p}_k(t)}$, j>i $p_{ji}(t) = p_{ij}(t)\bar{p}_i(t)/\bar{p}_j(t)$, j<i	Any solution of $F_{ij}=R_i(t)R_j(t)+\phi_{ij}(t)\sum_{k=1}^{n} R_k(t)\bar{p}_k(t)$ where $R_i(t) = 1-\sum_{k=1}^{n}\bar{p}_k(t)\phi_{ik}(t)$
(D) Koopman et.al.	$\frac{f_{ij}\bar{p}_j(t)}{\sum_{k=1}^{n} f_{ik}\bar{p}_k(t)}$	$f_{ij} \equiv F_{ij}$ in (C)
(E) General solution	$\bar{p}_j(t)\left[\frac{R_i(t)R_j(t)}{\sum_{k=1}^{n}\bar{p}_k(t)R_k(t)} + \phi_{ij}(t)\right]$	Any $\phi_{ij}(t)$ such that $R_i(t)\geq 0$, all i and t, at least one $R_j(t) > 0$, $\phi_{ij}(t) = \phi_{ji}(t)$, all i, j, t.

Particular solutions (A-C) and the general solution (D) to the N-group one-sex mixing problem. The $\{p_{ij}(t)\}$ is the mixing matrix or function itself (explanation in text), and $\phi_{ij}(t)$ are parameters used in the General Solution to recover the particular solutions.

[a] $c_i(t)$ and $T_i(t)$ are respectively the number of new partners taken by an individual per unit time and the total population of group i.

[b] $\delta_{ij} = 1$ if i = j, zero elsewhere

[c] originally formulated in continuous case

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