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"Models for the Spread of Universally Fatal Diseases II"

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Models for the Spread of Universally Fatal Diseases II

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Abstract

We consider a simple model for a universally fatal disease with an infective period long enough to allow natural deaths during the infective period. The analysis of this model is considerably more complicated than the analysis of a model with an infective period short enough that the population dynamics are confined to the susceptible class. However, the basic result that in some circumstances the stability of an endemic equilibrium may depend on the distribution of infective periods is shared by both models.

1 Introduction

There are simple classical models for the spread of infectious diseases due to Soper (1929) and Wilson-Burke (1942) which can be interpreted as models for universally fatal diseases in a population which would grow exponentially in the absence of disease. For a disease with recovery it is possible to incorporate births and deaths into the model but to keep the total population size constant. This is not possible for a universally fatal disease; in order to incorporate births and deaths into such a model and keep the total population size constant it is necessary to assume nonlinear population dynamics. Recently, a start has been made on the study of disease models in populations of varying sizes see for example Pugliese (1990) and Busenberg and Van den Driessche (1990). The classical models for infectious diseases assume either an exponential distribution of infective periods as in the Soper model or an infective period of fixed length as in the Wilson-Burke model. The first study including an arbitrary distribution of infective periods is the model of Cooke & Yorke (1973), which also describes age-structured population dynamics and a variety of other applications. In the model of Cooke & Yorke, for a disease with recovery with no immunity against re-infection, the behavior of the model depends on the mean infective period but not on the distribution of infective periods. A model for a universally fatal disease with an arbitary distribution of infective periods has been studied by Brauer

(1990a), and it has been shown that in some circumstances the stability of an endemic equilibrium may depend on the distribution of infective periods. This analysis was carried out under the assumption that only the susceptible members of the populations contribute to the population dynamics, except for deaths fo infectives from the disease. Such an assumption is appropriate for rapidly debilitation animal diseases such as rabies but not for diseases with long infective periods such as AIDS. In this paper we formulate a model for a universally fatal disease with birth and death rates depending on total population size and with deaths other than from the disease distributed proportionally between the susceptible and infective classes. This model is of interest in its own right and is also a step towards a model for diseases which are fatal to some victims but from which others recover. There are diseases such as measles which are rarely fatal in developed countries but from which there is substantial mortality in less-developed countries. Another possible direction of extension would be towards a model for a disease with infectivity depending on age of infection but without a full age structure. Such a model may be of use in describing AIDS in a simple manner.

2 Basic Model

The model studied in Brauer (1990a) is

$$\begin{aligned} S'(t) &= gS(t) - \hat{C}\{S(t) + I(t)\}S(t)I(t) \\ I(t) &= \int_o^t \hat{C}\{S(x) + I(x)\}S(x)I(x)\}P(t-x)dx, \end{aligned} \tag{1}$$

for values of t large enough that members who were infective at $t = O$ have been removed. The hypotheses which led to this model are $(H1)$ The rate of change of population size in the absence of infection is a function g of population size. All births are in the susceptible class, all deaths other than from disease are in the susceptible class, and infective members do not contribute to the birth rate. The population has a carrying capacity K, with

$$g(K) = 0,\ g'(K) < 0,\ g(N) \le 0 \text{ for } N \ge K \tag{2}$$

(H2) The number of contacts per infective in unit time is a function $C(N)$ of total population size $N = S + I$, with

$$C(N) > 0,\ C'(N) \ge 0,\ [C(N)/N]' \le 0. \tag{3}$$

The rate of new infections is then $\hat{C}(N)SI$. It is convenient to define

$$\hat{C}(N) = C(N)/N$$

so that

$$\hat{C}'(N) \le 0)$$

and the rate of new infections is $\hat{C}(N)SI$. It is also convenient to define

$$\beta = \hat{C}(K). \tag{5}$$

(H3) The fraction of infectives remaining infective a time s after becoming infective is a function $P(s)$ with

$$P(0) = 1,\ P(s) \geq 0,\ \int_o^\infty P(s)ds = \tau < \infty,\quad P \text{ non} - \text{increasing}. \tag{6}$$

It has been shown [Brauer (1990a)] that for the model (1) there is a contact number

$$K\hat{C}(K)\int_o^\infty P(s)ds = \beta\tau K.$$

If the contact number is less than 1 the system (1) has a single asymptotic equilibrium, namely the disease-free equilibrium $S = K, /I = O$, and this equilibrium is asymptotically stable. If the contact number exceeds 1 the disease-free equilibrium is unstable, but there is also an endemic asymptotic equilibrium (S, I) with $S < K, I > O$. To analyze the stability of this equilibrium we form the character equation, which has the form

$$b\hat{P}(\lambda) = \frac{\lambda + a}{\lambda + c} \tag{7}$$

where $\hat{P}(\lambda)$ denotes the Laplace of P,

$$\hat{P}(\lambda) = \int_0^\infty e^{-\lambda s}P(s)ds$$

and

$$a = I\hat{C}(S + I) + SI\hat{C}'(S + I) - g'(S)$$

$$b = s\hat{C}(S + I) + SI\hat{C}'(S + I)$$

$$c = -g'(S).$$

Then the following result is applicable.

Theorem 1. [Hethcote, Stech, & van den Driessche (1981)] *If $a > c$ and $b\tau \leq 1$, then all roots of (7) have negative real part, but if a, c with c negative and $|c|$ sufficiently large, there may be roots with positive real part.*

It is not difficult to verify with the aid of (3) and (4) that $a > c$, $0 < b\tau \leq 1$. Thus if $g'(S) < 0$ all roots of (7) have negative real part but if $g'(S) > 0$ there is a possibility of roots of (7) with positive real part. However if $P(s) = e^{-s/\tau}$ all roots of (7) have negative real part regardless of the value of $g'(S)$ [Brauer (1990a)], while if $P(s) = 1$ $(0 \leq s \leq \tau)$, $P(s) = 0$ $(s > \tau)$, there can indeed be roots with positive real part [Hethcote, Stech, & van den Driessche (1981)]. The significance of this result is that for a fatal disease modelled by the system (1) the stability of the endemic equilibrium may depend on the distribution of

infective periods if the contact number is high enough and if the population dynamics permit an equilibrium with $g'(S) > 0$.

3 Long Infective Period

In order to model a disease with an infective period long enough to allow natural deaths during the infective period, we replace the hypothesis (H1) by a different assumption on the population dynamics. We continue to assume (H2) and (H3) but instead of (H1) we assume: (H1*) There is a birth rate $B(S)$ per susceptible and a death rate $D(N)$ per member of the population, $N = S + I$. All births are in the susceptible class and the death rate in each class is proportional to the size of the class. The population has a carrying capacity K,

$$B(K) = D(K),\ B'(K) < D'(K),\ B(N) \le D(N) \text{ if } N \ge K. \tag{8}$$

In addition, we assume

$$\begin{aligned} B'(S) \le 0,\ [SB(S)]' = B(S) + SB'(S) \ge 0 \\ 0 \le D'(N) \le D(N)/N. \end{aligned} \tag{9}$$

The assumption (H1*) implies that in unit time there are $SD(N)$ deaths in the susceptible class and $ID(N)$ (natural) deaths in the infective class. If $z(t)$ denotes the number of members who became infective at time x who have not died of natural causes by time t, then $z'(t) = -z(t)D\{N(t)\}$, and this implies

$$z(t) = z(x)e^{-\int_x^t D\{N(y)\}dy}.$$

Thus the fraction of the members who became infective at time x and who have not died either of natural causes or from disease is

$$P(t-x)\exp\left(-\int_x^t D\{N(y)\}dy\right).$$

This leads us to the model

$$\begin{aligned} S'(t) &= S(t)B\{S(t)\} - S(t)D\{N(t)\} - \hat{C}\{N(t)\}S(t)I(t) \\ I(t) &= \int_0^t \hat{C}\{N(x)\}S(x)I(x)e^{-\int_x^t D\{N(y)\}dy}P(t-x)dx. \end{aligned} \tag{10}$$

Here it is convenient to use N in the model to denote $S + I$.

The conditions for an asymptotic equilibrium (S, I) of (10), with $N = S + I$, are

$$\begin{aligned} SB(S) &= SD(N) + \hat{C}(N)SI \\ I &= \hat{C}(N)SI\int_0^\infty e^{-D(N)s}P(s)ds. \end{aligned}$$

Then either $I = 0$, which implies $SB(S) = SD(N)$, so that $S = N = K$, or

Then either $I = 0$, which implies $SB(S) = SD(N)$, so that $S = N = K$, or

$$\hat{C}(N)S\int_0^\infty e^{-D(N)s}P(s)ds = 1.$$

We define

$$Q(s) = e^{-D(N)s}P(s)$$

and

$$\hat{Q}(\lambda = \int_0^\infty e^{-\lambda s}Q(s)ds,$$

so that

$$\hat{Q}(0) = \int_0^\infty Q(s)ds.$$

Then the conditions for an endemic equilibrium are

$$\begin{aligned} B(S) &= D(N) + \hat{C}(N)I \\ S\hat{C}(N)\hat{Q}(0) &= 1. \end{aligned} \tag{11}$$

The existence of an endemic equilibrum requires $S < K$, or $K\hat{C}(N)\hat{Q}(0) > 1$. If there is an endemic equilibrium we have

$$\frac{1}{\hat{Q}(0)} = S\hat{C}(N) < N\hat{C}(N) = C(N) \leq C(K) = \beta K$$

by (3) and (5). Thus the existence of an endemic equilibrium requires $\beta K\hat{Q}(0) > 1$. The same argument as that used for the model of Section 2 [Brauer (1990a)] shows that the disease-free equilibrium $(K, 0)$ of (10) is asymptotically stable if and only if

$$\beta K\hat{Q}(0) < 1.$$

In particular, an endemic equilibrium exists if and only if the disease-free equilibrium is unstable.

The linearization of the system (10) about an endemic equilibrium (S, I) is

$$\begin{aligned} u' = {} & [B(S) + SB'(S) - D(N) - SD'(N) - SI\hat{C}'(N) - I\hat{C}(N)]u \\ & - [SD'(N) + SI\hat{C}'(N) + S\hat{C}(N)]v, \end{aligned}$$

$$\begin{aligned} v(t) = {} & \int_0^t [I\hat{C}(N) + SI\hat{C}'(N)]Q(t-x)u(x)dx \\ & + \int_0^t [S\hat{C}(N) + SI\hat{C}'(N)]Q(t-x)v(x)dx \\ & - \int_0^t SID'(N)\hat{C}(N)Q(t-x)\left[\int_x^t \{u(y) + v(y)\}dy\right]dx. \end{aligned}$$

A complicated calculation gives the characteristic equation at the endemic equilibrium

$$\hat{Q}(\lambda) = \frac{\lambda^2 + d\lambda + \hat{Q}(0)c}{a\lambda^2 + b\lambda + c} \tag{12}$$

with

$$\begin{aligned} a &= \nu S \\ b &= [D(N) + SD'(N) - B(S) - SB'(S)]\nu S - \mu I S D'(N) + \gamma \\ c &= \gamma[D(N) + \mu I - \nu S - B(s) - SB'(S)] \\ d &= \mu I + D(N) + SD'(N) - B(S) - SB'(S) + \gamma\hat{Q}(0) \end{aligned} \tag{13}$$

where

$$\begin{aligned} \mu &= \hat{C}(N) + S\hat{C}'(N) \\ \nu &= \hat{C}(N) + I\hat{C}'(N) \\ \gamma &= \hat{C}(N)SID'(N). \end{aligned} \tag{14}$$

From the condition (3) it is easy to deduce that

$$\begin{aligned} 0 \le -I\hat{C}'(N) \le \mu \le \hat{C}(N) \\ 0 \le -S\hat{C}'(N) \le \nu \le \hat{C}(N). \end{aligned} \tag{15}$$

In the particular case when $D(N)$ is a constant, so that $D'(N) = 0$, we have $\gamma = 0$ and $c = 0$. In this case the characteristic equation (12) reduces to

$$a\hat{Q}(\lambda) = \frac{\lambda + d}{\lambda + \frac{b}{a}},$$

which is of the form (7) and can be analyzed by Theorem 1. It is easy to show using (15) that

$$0 < a\hat{Q}(0) \le 1 \tag{16}$$

and that

$$d - \frac{b}{a} = \beta I + ID'(N) = [\beta + D'(N)]I > 0,$$

where $\beta = \hat{C}(K) = \hat{C}(N)$, so that $d > b/a$. If the contact number exceeds 1, so that there is an endemic equilibrium, but is close to 1, then I is close to zero and N and S are close to K. For such contact numbers

$$\begin{aligned} b &\approx \beta K[D(K) + KD'(K) - B(K) - KB'(K)] \\ &= \beta K^2[D'(K) - B'(K)] > 0, \end{aligned}$$

using (8). Then Theorem 1 shows that the endemic equilibrium is asymptotically stable for contact numbers close enough to 1, but may become unstable for large contact numbers with some choices of $P(s)$. In other words, the qualitative behavior of the model (10) is the same as that of the model (1) in the special case $D'(N) = 0$.

4 Analysis of the General Case

In the general case $D'(N) \neq 0$, the analysis of the characteristic equation (12) is considerably more complicated. The hypothesis (9) implies $\gamma > 0$. We have the following result, whose proof may be found in the appendix.

Theorem 2 *Under the conditions*

$$c < 0,\ 0 < a\hat{Q}(0) \leq 1,\ 0 < b\hat{Q}(0) \leq d \tag{17}$$

and

$$\int_0^\infty sQ(s)ds < \frac{d - b\hat{Q}(0)}{-c} \tag{18}$$

all roots of the characteristic equation (12) have negative real part.

For a general Q we can analyze the stability of the endemic equilibrium only for contact numbers close to 1. If the contact number is close to 1, so that $S \approx K$, $N \approx K$, $I \approx 0$, we have

$$\hat{C}(N) \approx \beta,\quad \hat{Q}(0) \approx \frac{1}{\beta K} \tag{19}$$

because of (5) and (11). For such contact numbers we also have

$$\mu \approx \beta + K\hat{C}'(K) > 0,\ \nu \approx \beta,\ \frac{\gamma}{I} \approx \beta K D'(K). \tag{20}$$

Using (19) and (20) we expand in powers of I to obtain

$$\begin{aligned} a &\approx \beta K > 0 \\ b &\approx K[D'(K) - B'(K)]\beta K - K^2\hat{C}'(K)D'(K)I > 0 \\ c &\approx -\beta K D'(K)I[\beta K + KB'(K)] < 0 \\ d &\approx K[D'(K) - B'(K)] + [\beta + K\hat{C}'(K) + D'(K)]I > 0. \end{aligned} \tag{21}$$

In (21) we have retained the terms in I in the approximations for b, c, d because they will be needed in the application of Theorem 2 to the model (10). Then

$$\begin{aligned} d - b\hat{Q}(0) &\approx \left[\beta + K\hat{C}'(K) + D'(K) + \frac{K\hat{C}'(K)D'(K)}{\beta}\right] I \\ &= [K\hat{C}'(K) + \beta]\left[1 + \frac{D'(K)}{\beta}\right] I \geq 0. \end{aligned} \tag{22}$$

Because of (16) we have now established that all conditions in (17) are satisfied for contact numbers close to 1.

Again using (21) along with (22) we have

$$\frac{d - b\hat{Q}(0)}{-c} \approx \frac{[K\hat{C}'(K) + \beta][1 + \frac{D'(K)}{\beta}]}{\beta K D'(K)[\beta K + KB'(K)]}. \tag{23}$$

We can not hope to establish the inequality (18) for arbitrary $\hat{C}(N)$ because while $K\hat{C}'(K)+\beta \geq 0$ in general, the choice $C(N)=\lambda$, so that $\hat{C}(N)=\frac{\lambda}{N}$ and $\lambda=\beta K$, gives $K\hat{C}'(K)+\beta=O$ and

$$\frac{d-b\hat{Q}(0)}{-c} \approx O.$$

The choice $\hat{C}(N)=\beta$ gives

$$\frac{d-b\hat{Q}(0)}{-c} \approx \frac{D'(K)+\beta}{\beta K D'(K)[\beta K + KB'(K)]}. \tag{24}$$

Integration by parts gives

$$\begin{aligned}\int_0^\infty sQ(s)ds &= \int_0^\infty se^{-D(N)s}P(s)ds \\ &= -\frac{1}{D(N)}[sP(s)e^{-D(N)s}]_0^\infty + \frac{1}{D(N)}\int_0^\infty e^{D(N)s}[sP'(s)+P(s)]ds \\ &= \frac{1}{D(N)}\int_0^\infty e^{-D(N)s}[sP'(s)+P(s)]ds \\ &\leq \frac{1}{D(N)}\int_0^\infty e^{-D(N)s}P(s)ds = \hat{Q}(0)/D(N),\end{aligned}$$

using $P'(s) \leq 0 \quad (0 \leq s < \infty)$. Now, for contact numbers close to 1, we can estimate $\int_0^\infty sQ(s)ds$ by $1/\beta K D(K)$. Since $B'(K) \geq 0, D'(K) \leq D(K)/K$ by the hypothesis (8), (24) gives

$$\frac{d-b\hat{Q}(0)}{-c} > \frac{\beta}{\beta^2 K^2 D'(K)} > \frac{1}{\beta K D(K)}.$$

From this we see that (18) is satisfied and thus that Theorem 2 is applicable. We now have the following result.

Theorem 3. *If the function $\hat{C}(N)$ is constant, then the endemic equilibrium of the model (10) is asymptotically stable at least for contact numbers sufficiently close to 1.*

The question of stability of the endemic equilibrium for more general $\hat{C}(N)$ is open. It is reasonable to conjecture that while the choice $\hat{C}(N)=\beta K/N$ must be excluded there is a class of non-constant functions $\hat{C}(N)$ for which stability can be established.

5 A Fox Rabies Model

A model for fox rabies has been proposed [Anderson et al (1981)] which is of the form (10) except for the incorporation of an exposed period and has an exponential distribution of infective periods. This model exhibits instability of the endemic equilibrium for high contact numbers, but if the model did not include an exposed period the endemic equilibrium would be asymptotically stable for all contact numbers.

We shall examine the special case of (10) with $P(s) = e^{-s/\tau}$ for which the model (10) reduces to the system of ordinary differential equations

$$\begin{aligned} S' &= SB(S) - SD(N) - \hat{C}(N)SI \\ I' &= \hat{C}(N)SI - ID(N) - \frac{1}{\tau}I. \end{aligned} \tag{25}$$

The linearization of (25) about an equilibrium (S, I) has coefficient matrix

$$M = \begin{bmatrix} B(S) + SB'(S) - D(N) - SD'(N) - \mu I & -SD'(N) - \nu S \\ [\mu - D'(N)]I & \nu S - D(N) - \frac{1}{\tau} - ID'(N) \end{bmatrix}$$

The endemic equilibrium satisfies

$$S\hat{C}(N) = D(N) + \frac{1}{\tau}, \quad B(S) = D(N) + \hat{C}(N)I$$

and this enables us to rewrite this coefficient matrix as

$$M = \begin{bmatrix} SB'(S) - SD'(N) - SI\hat{C}'(N) & -DS'(N) - \nu S \\ [\mu - D'(N)]I & [S\hat{C}'(N) - D'(N)]I \end{bmatrix}.$$

The endemic equilibrium is asymptotically stable if and only if tr $M < 0$, det $M > 0$. We have

$$\begin{aligned} \text{tr } M &= SB'(S) - SD'(N) - SI\hat{C}'(N) + SI\hat{C}'(N) - D'(N)I \\ &= SB'(S) - ND'(N) < 0, \end{aligned}$$

because of (9). Also,

$$\begin{aligned} \frac{\det M}{SI} &= [B'(S) - D'(N) - I\hat{C}'(N)][S\hat{C}'(N) - D'(N)] \\ &\quad + [D'(N) + \nu][\mu - D'(N)] \\ &= B'(s)S\hat{C}'(N) - B'(S)D'(N) \\ &\quad + [\hat{C}(N)]^2 + S\hat{C}(N)\hat{C}'(N) + I\hat{C}(N)\hat{C}'(N) \\ &= B'(S)[S\hat{C}'(N) - D'(N)] + \hat{C}(N)[\hat{C}(N) + N\hat{C}'(N)] \end{aligned}$$

and this is positive because of the assumptions (3), (4), (9). This establishes the asymptotic stability of the endemic equilibrium of (25) for all contact numbers. We thus have the same situation observed for the simpler model (1): For some

choices of birth and death rates, destabilization of the endemic equilibrium may depend on the distribution of infective periods.

Another possible cause of destabilization of the endemic equilibrium is an exposed period, as in the rabies model of Anderson et al (1981). Models with nonlinear population dynamics and with exposed and infective periods of fixed length have been formulated as delay equations with two delays [Brauer (1989)]. The formulation of models with arbitrarily distributed exposed and infective periods leads to integral equations whose kernel is the convolution of the exposed and infective kernels [Hethcote & Tudor (1980)]. Models with nonlinear population dynamics and arbitrarily distributed exposed and infective periods remain to be formulated and analyzed.

Another direction of generalization would be a model for a disease from which a fraction of infectives recover. Such a model would have to generalize the model (10) of Section 3 by allowing natural deaths in each class rather than the model (1) of Section 2. It should also have a birth rate of susceptibles depending on the recovered class size as well as the susceptible class size. If the fraction of infectives who recover is p $(0 \leq p \leq 1)$, a model would be

$$S' = (S+R)B(S+R) - SD(N) - \hat{C}(N)SI$$

$$I(t) = \int_0^t \hat{C}\{N(x)\}S(x)I(x)e^{-\int_x^t D\{N(y)\}dy}P(t-x)dx \tag{26}$$

$$R'(t) = -p\int_0^t \hat{C}\{N(x)\}S(x)I(x)e^{-\int_x^t D\{N(y)\}dy}P'(t-x)dx - R(t)D\{N(t)\}.$$

The model (10) is the special case $p = 0$ of (26). The case $p = 1$ of (26) would be an S-I-R model with recovery, for which the endemic equilibrium is always asymptotically stable [Brauer (1990b)]. Thus as p varies from 0 to 1 there may be a transition from instability and oscillation about the endemic equilibrium to stability, and the dependence of the behavior on the recovery fraction p is of interest.

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APPENDIX

In order to prove Theorem 2, we begin with a general but simple lemma.

Lemma a *Suppose that f and g are analytic in an open set containing the right half plane $\mathcal{R}\lambda \geq 0$ with $f(\bar{z}) = \overline{f(z)}$, $g(\bar{z}) = \overline{g(z)}$ and assume that*

(i) $f(0) = g(0) > 0$

(ii) $|f(iy)| < |g(iy)|, \quad 0 < y < \infty$

(iii) g has a single zero in $\mathcal{R}\lambda > 0$

(iv) $f'(0) > g'(0)$.

Then, except for a simple root at $\lambda = 0$, all roots of $f(\lambda) = g(\lambda)$ satisfy $\mathcal{R}\lambda < 0$.

Proof .. We consider the equation $rf(\lambda) = g(\lambda)$ with r varying from 0 to 1. For $r = 0$ there is a single root in $\mathcal{R}\lambda > 0$ and roots depend continuously on r. No root crosses the imaginary axis for $0 \leq r < 1$ because a crossing would require that either $\lambda = 0$ or $\lambda = iy$ is a root for some value of r, impossible since

$$|rf(0)| < |f(0)| = |g(0)|$$

$$|rf(iy)| < |f(iy)| < |g(iy)|.$$

There is a root $\lambda(r)$ with $\lambda(1) = 0$ because of (i). Implicit differentiation of $rf\{\lambda(r)\} = g\{\lambda(r)\}$ gives

$$\lambda'(r) = \frac{f\{\lambda(r)\}}{-rf'\{\lambda(r)\} + g'\{\lambda(r)\}}$$

and letting $r \to 1-$ we obtain

$$\lambda'(1) = \frac{f(0)}{g'(0) - f'(0)} < 0.$$

Thus the root $\lambda(r)$ approaches zero from the right half plane and $\lambda(0)$ must be the zero of g in the right half plane. This leaves no roots of $f(\lambda) = g(\lambda)$ in $\mathcal{R}\lambda \geq 0$.

To prove Theorem 2 we apply this lemma with

$$f(\lambda) = \hat{Q}(\lambda), \quad g(\lambda) = \frac{\lambda^2 + d\lambda + \hat{Q}(0)c}{a\lambda^2 + b\lambda + c}.$$

Then $f(0) = g(0) = \hat{Q}(0) > 0$ if $c \neq 0$ and g has a single zero with $\mathcal{R}\lambda > 0$ if $a > 0, c < 0$. Because

$$f'(0) = -\int_0^\infty sQ(s)ds, \quad g'(0) = \frac{d - b\hat{Q}(0)}{c},$$

the condition (iv) is satisfied if

$$-\int_0^\infty sQ(s)ds > \frac{d - b\hat{Q}(0)}{c}.$$

If $d > 0, c < 0$, this is equivalent to

$$\int_0^\infty sQ(s)ds < \frac{d - b\hat{Q}(0)}{-c}.$$

The verification of the hypothesis (ii) is more complicated. It is easy to calculate $|f(iy)| \leq \hat{Q}(0)$ and

$$|g(iy)|^2 = \frac{y^4 + \{d^2 - 2\hat{Q}(0)c\}y^2 + \{\hat{Q}(0)\}^2c^2}{a^2y^4 + (b^2 - 2ac)y^2 + c^2}.$$

We wish to minimize $|g(iy)|^2$ over $0 \leq y < \infty$. The sign of the derivative of $|g(iy)|^2$ with respect to y for $y = 0$ is the same as the sign of $\{d^2 - b^2[\hat{Q}(0)]^2\} - 2\hat{Q}(0)c\{1 - a\hat{Q}(0)\}$ and this is positive if $b > 0$, $c < 0$, $d > 0$, $b\hat{Q}(0) \leq d$, $a\hat{Q}(0) \leq 1$. If the function $|g(iy)|^2$ has no critical points, the minimum of $|g(iy)|^2$ must be $[\hat{Q}(0)]^2$, attained for $y = 0$. If $|g(iy)|^2$ has only one critical point, this critical point is a relative maximum and the minimum of $|g(iy)|^2$ is the smaller of $|g(0)|^2 = [\hat{Q}(0)]^2$ and $\lim_{y\to\infty} |g(iy)|^2 = \frac{1}{a^2}$. If $a\hat{Q}(0) \leq 1$, this minimum is again $[\hat{Q}(0)]^2$. In either case, $|f(iy)| < |g(iy)|$ for $0 < y < \infty$, and thus it remains to show that $|g(iy)|^2$ has at most one critical point.

We let $z = y^2$, $h(z) = |g(i\sqrt{2})|^2$ for $0 < z < \infty$, so that

$$h(z) = \frac{z^2 + \{d^2 - 2\hat{Q}(0)c\}z + [\hat{Q}(0)]^2c^2}{a^2z^2 + (b^2 - 2ac)z + c^2}.$$

If $a > 0$, $c < 0$, the denominator does not vanish for $0 \leq z < \infty$. The derivative of $h(z)$ has the sign of

$$(s - ar^2)z^2 + 2c^2\{1 - a^2[\hat{Q}(0)]^2\}z + \{rc^2 - [\hat{Q}(0)]^2c^2s\}$$

with

$$r = d^2 - 2\hat{Q}(0)c, \quad s = b^2 - 2ac.$$

Both r and s are nonnegative if $a > 0$, $c < 0$. If $a\hat{Q}(0) \leq 1$ and $|b|\hat{Q}(0) \leq d$, we have

$$rc^2 - [\hat{Q}(0)]^2c^2s = \{d^2 - b^2[\hat{Q}(0)]^2\} - 2\hat{Q}(0)c\{1 - a\hat{Q}(0)\} \geq 0.$$

Thus if $s - ar^2 \geq 0$ the derivative of $h(z)$ has no positive zero and if $s - ar^2 < 0$ the derivative of $h(z)$ has a single positive zero; in either case $|g(iy)|^2$ has at most one positive critical point.

We have now completed the verification of the hypotheses (i)-(iv) and may apply the lemma to yield the desired result:

Theorem 2. *If* $c < 0$, $0 < a\hat{Q}(0) \leq 1$, $0 < |b|\hat{Q}(0) \leq d$ *and if*

$$\int_0^\infty sQ(s)ds < \frac{d - b\hat{Q}(0)}{-c},$$

then all roots of the equation

$$\hat{Q}(\lambda) = \int_0^\infty e^{-\lambda s}Q(s)ds = \frac{\lambda^2 + d\lambda + \hat{Q}(0)c}{a\lambda^2 + b\lambda + c},$$

where $Q(0) = 1$, Q *is non-increasing and* $\int_0^\infty Q(s)ds < \infty$, *have negative real part.*