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Synthetic Seismograms Computation by Gaussian Beams Method

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SYNTHETIC SEISMOGRAMS COMPUTATION BY GAUSSIAN BEAMS METHOD

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Here we consider the Gaussian beams approach for evaluating theoretical seismograms in a media with slow vatiation of its parameters. We shall begin from the Helmholtz equation in 2D acoustic media and then discuss the procedure of surface wave seismograms calculation in 3D elastic media with slow lateral inhomogeneties. The formulas, describing the Gaussian beams, will be obtained by an extansion of the main results for the geometrical ray method. It will be shown, that unlike the geometrical ray method, the Gaussian beams approach gives regular results in the vicinity of caustics, lets transition from the lighted zone to the shadow and does not require finding a ray, passing exactly through source and receiver. In our account we will follow mainly the results of Cerveny et. al. (1982), Yomogida & Aki (1985), Yomogida (1988) and Ben-Menahem & Beydoun (1985).

1. The geometrical ray approximation.

Let us begin from the homogeneous wave equation in 2D media

$$\nabla^2 u(\vec{x},t) - \frac{1}{C^2(\vec{x})}\frac{\partial^2}{\partial t^2} u(\vec{x},t) = 0, \qquad (1)$$

where u is the displacement potential or preassure as a function of space coordinates $\vec{x}\ (\vec{x} \in R^2)$ and time t is the wave velocity. Applying to (1) the Fourier transform with respect to time t, we get the homogeneous Helmholtz equation:

$$\nabla^2 U(\vec{x},\omega) + \frac{\omega^2}{c^2(\vec{x})} U(\vec{x},\omega) = 0 \tag{2}$$

Solution of (2) will be constructed by means of the asymptotic ($\omega \to \infty$) ray series expansion (Babich & Buldyrev, 1972; Cerveny et. al., 1977):

$$U(x,\omega) = \exp[i\omega\tau(\vec{x})] \sum_{K=0}^{\infty} U_K(\vec{x})(-i\omega)^{-K}. \tag{3}$$

We shall determine the series solution by setting the coefficients of each power of ω separately equal to zero. The highest power is two, and for any nonzero choice of U_0 this coefficient will be zero, if we choose τ to satisfy the so-called eikonal equation

$$(\nabla\tau)^2 = c^{-2} \tag{4}$$

Setting the coefficient of ω equal to zero, yields the equation

$$(\nabla^2\tau)U_o + 2\nabla\tau\cdot\nabla U_o = 0, \tag{5}$$

which is called the transport equation. When just the first leading term in (3) is keeping, the constructed solution

$$U(x,\omega) \simeq \exp[i\omega\tau(\vec{x})]\, U_o(\vec{x}) \tag{6}$$

gets us the geometrical ray approximation.

2. The Hamilton - Jacobi system and solution of the eikonal equation.

The eikonal equation (4) is the first order nonlinear partial differential equation, belonging to the class of Hamilton - Jacobi equations:

$$H(\vec{p},\vec{q}) = 0, \qquad p_j = \frac{\partial\Psi}{\partial q_j} \tag{7}$$

where H is the Hamiltonian, vector $\vec{q}$ consists of the generalized coordinates and $\vec{p}$ is the vector of conjugate momentums $\Psi(\vec{q})$ is a function we are looking for. The Hamilton - Jacobi equations are solved by the method of characteristics, that reduses the partial differential equation to the system of ordinary differential equations.

At the hypersurface $H(\vec{p},\vec{q})$ in the phase space $(\vec{p},\vec{q})$ the following condition is satisfied :

$$dH = \sum_j \left(\frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial q_j} dq_j\right) = 0 \tag{8}$$

In particular, $\vec{q}$ and $\vec{p}$ will be belonged to the hypersurface $H(\vec{p},\vec{q}) = 0$, if

$$\frac{dq_j}{\partial H/\partial p_j} = -\frac{dp_j}{\partial H/\partial q_j} = ds, \tag{9}$$

where s is an independent variable. From (9) we get the so-called characteristic equations:

$$\frac{dq_j}{ds} = \frac{\partial H}{\partial p_j}, \qquad \frac{dp_j}{ds} = \frac{\partial H}{\partial q_j}. \tag{10}$$

Now, as

$$\frac{d\psi}{ds} = \sum_j \frac{\partial\psi}{\partial q_j}\frac{dq_j}{ds} = \sum_j p_j \frac{\partial H}{\partial p_j}, \tag{11}$$

the function ψ we are looking for can be found by integration along characteristic:

$$\psi = \psi_0 + \int_{s_0}^{s} \Big(\sum_j p_j \frac{\partial H}{\partial p_j}\Big) ds. \tag{12}$$

So, to determine the value $\psi(\vec{q})$ from the initial conditions $\psi(\vec{q}_0) = \psi_0$ we need first of all to find characteristic, passing through the points $\vec{q}_0$ and $\vec{q}$ (solve two - point boundary value problem) and then the value $\psi(\vec{q})$ will be obtained by integration along this characteristic.

Now let us return to the task of solving the eikonal equation (4). Here we have the following parallel with the general case, considered above:

$$\psi \to \tau, \qquad \vec{q} \to \vec{x}, \qquad \vec{p} \to \nabla\tau$$

To get the characteristic equations we must choose the concrete form of Hamiltonian, for example

$$H(\vec{p}, \vec{x}) = \big[\|\vec{p}\| - 1/c(\vec{x})\big] = 0 \tag{13}$$

where

$\vec{p}$ - slowness vector, $\vec{p} = \nabla\tau$, $\|\vec{p}\| = (p_1^2 + p_2^2)^{1/2}$.

Now, using (13) and (10), we derive the so-called ray tracing equations:

$$\begin{aligned} \frac{d\vec{x}}{ds} &= c\vec{p}, \\ \frac{d\vec{p}}{ds} &= \nabla\Big(\frac{1}{c}\Big). \end{aligned} \tag{14}$$

From (9) we also have, that

$$|d\vec{q}| = \Big[\sum_j \Big(\frac{\partial H}{\partial p_j}\Big)^2\Big]^{1/2} ds, \tag{15}$$

and for our choice of Hamiltonian (13)

$$\Big[\sum_j \Big(\frac{\partial H}{\partial p_j}\Big)^2\Big]^{1/2} = 1 \tag{16}$$

So, s in equations (14) is the arclength along the ray path. From (12) we also get

$$\tau(s) = \tau_0 + \int_{s_0}^{s} \frac{1}{c(\zeta)} d\zeta, \tag{17}$$

and here we obtained, that $\tau(s)$ has the physical meaning of the travel time along a ray.

3. Solution of the transport equations.

If we multiply the transport equation (5) by the U_0, the result can be written in the following form:

$$div\,(U_0^2 \nabla\tau) = div\,(U_0^2 \frac{1}{c}\vec{\nu}) = 0, \tag{18}$$

where $\vec{\nu}$ is the unit vector perpendicular to the line of constant τ (wavefronts) and determining the direction of the ray path. Now, let us create unlimitly narrow ray tube by two adjacent rays (fig.1) and integrate (18) over the area between two infinitely close sections of these tube. Then, applying the Gauss' theorem, we obtain:

$$\frac{U_0^2}{c} J(s)\,\delta\gamma = const = \phi^2(\gamma)\,\delta\gamma, \qquad (19)$$

where γ is the ray parameter: if, for example, the wavefield is generated by a point source, γ can be the angle between some reference axis and the initial direction of the ray at the source point; $J(s)\delta\gamma$ is the cross - section of the ray tube and $J(s)$ is the so-called geometrical spreading. The vector $\frac{U_0^2}{c}\vec{\nu}$ in (18) is the flux energy vector for the acoustic medium, so the formula (19) describes the law of preserving the total energy, flowing through the section of the ray tube.

Finally, using (6), (17) and (19) we get, that the geometrical ray approximation yields to the following expression:

$$U(s,\omega) = \phi\sqrt{\frac{c(s)}{J(s)}}\; exp\left[i\omega\int_0^{\sigma}\frac{1}{c(\xi)}d\xi\right]. \qquad (20)$$

Up to now we have just derived, that $U(s,\omega)$ is inversely proportional to $J(s)$, but we have not received yet any procedure of J calculation.

The geometrical ray approach is valid for a media with slow variation of its parameters, or more precisely, variation of the slowness $1/c(\vec{x})$ must be small on the scale of the radius of the first Freznel zone (Chernov, 1960; Ben-Menahem & Beydoun, 1985). But from (20) it follows, that even for a such kind of media this method can't be applied for calculation the wavefield in the vicinity of caustics, where $J(s)$ is equal to zero.

4. Dynamic ray tracing system and paraxial ray approximation.

According to the geometrical ray approach, to construct a solution of the eikonal equation at any point of acoustic media, we need to find a ray, passing through this point, and then evaluate the travel time $\tau(s)$ by integration along this ray. Now let us expand the constructed solution in the neighborhood of the ray and to this end we will use the method, initially suggested by Popov and Psencik (1978). To solve the task, they introduced the ray-centered coordinates (s, q) : s corresponds to the arclength along the ray, q is a distance from the ray, measured along the axis $\vec{n}$, perpendicular to the ray path (fig. 2). Now, the expression for the travel time in the neighbourhood of the ray (paraxial travel times) can be found by τ Taylor series expansion with respect to q around the point $(s, 0)$:

$$\tau(s,q) = \tau(s) + \frac{1}{2}Mq^2, \qquad (21)$$

with $M = \frac{\partial^2\tau}{\partial q^2}$ and where we also have taken into account, that $\nabla\tau\cdot\vec{n} = 0$

Truncating the Taylor expansion of τ at the quadratic term of q means, that the wavefront around the point $(s, 0)$ is approximated by the parabola, and cM is the curvature of the wavefront at $(s, 0)$.

To find M in (21) we need to differentiate the ray tracing equations (14) with respect to the ray parameter γ and then project the results onto the $\vec{n}$ direction:

$$\frac{d}{ds}Q = cP, \qquad \frac{d}{ds}P = -c^{-2}\frac{\partial^2 c}{\partial q^2}Q \qquad (22)$$

Here we used the notations:

$$Q = \left(\frac{\partial \vec{x}}{\partial \gamma}\right)\cdot \vec{n}\,, \qquad P = \left(\frac{\partial \vec{p}}{\partial \gamma}\right)\cdot \vec{n}\,. \tag{23}$$

Now, as

$$P = \frac{\partial p_n}{\partial \gamma} = \frac{\partial^2 \tau}{\partial q\, \partial \gamma} = \frac{\partial^2 \tau}{\partial q^2}\,\frac{\partial q}{\partial \gamma} = MQ$$

we obtain for M :

$$M = P/Q \tag{24}$$

Also, from (19) we have:

$$\mathcal{J}(s)\,\delta\gamma = \frac{\partial q}{\partial \gamma}\,\delta\gamma, \quad \text{or} \quad \mathcal{J}(s) = \frac{\partial q(s)}{\partial \gamma} = Q(s)\,. \tag{25}$$

Thus, solving the system (22) of two linear differential equations of the first order, which is called the dynamic ray tracing system, we find in the same time the geometrical spreading term $\mathcal{J}(s)$.

Now, using the paraxial approximation (21), we are able to evaluate the wavefield not only along the central ray γ, but also in some vicinity of it:

$$U_\gamma(s,q) = \phi(\gamma)\sqrt{\frac{C(s)}{Q(s)}}\,\exp\left\{ i\omega\left[\int_0^{\infty}\frac{d\zeta}{c(\zeta)} + \frac{M(s)}{2}q^2\right]\right\} \tag{26}$$

To find the total solution at the point (s, q) we must sum contributions of all possible central rays, so

$$U(s,q) = \int_\Omega U_\gamma(s,q)\,d\gamma \tag{27}$$

where $U_\gamma(s,q)$ is determined by (26).

5. The Gaussian beams approach.

Following the notations of Cerveny et. al. (1982), let us denote by $\Pi(s)$ the fundamental matrix of linearly independent real solutions of equations (22) :

$$\Pi(s) = \begin{pmatrix} Q_1(s) & Q_2(s) \\ P_1(s) & P_2(s) \end{pmatrix} \tag{28}$$

The first solution $Q_1(s)$, $P_1(s)$ corresponds to the initial conditions $Q_1(s_0) = 1$, $P_1(s_0) = 0$, and that is a "plane wave" initial conditions, since keeping $P = \frac{\partial p_n}{\partial \gamma} = 0$ while changing $Q = \frac{\partial x_n}{\partial \gamma}$ generates paraxial rays, which are initially parallel to the reference ray. The second solution $Q_2(s)$, $P_2(s)$ corresponds to the initial conditions $Q_2(s_0) = 0$, $P_2(s_0) = 1/c(s_0)$ and that is a "point source" initial conditions, since the coordinates of the starting point for the paraxial ray and the central one are the same. Namely just these second initial conditions are used in conventional geometrical ray approximation techniques to find the geometrical spreading $\mathcal{J}(s)$ by solving the dynamic ray tracing equations.

Now any complex solution of (22) can be expressed by sum of two linearly independent real solutions:

$$Q(s) = \varepsilon_1 Q_1(s) + \varepsilon_2 Q_2(s) \tag{29}$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are complex valued constants. Following Cerveny et. al. (1982), let us rewrite the formula (26) in the next form:

$$U_\gamma(s,q) = \phi(\gamma)\sqrt{\frac{C(s)}{Q(s)}}\exp\left\{i\omega\left[\tau(s)+\frac{K(s)}{2C(s)}q^2\right]-\frac{q^2}{L^2(s)}\right\}, \quad (30)$$

where

$$K(s)=C(s)Re\left(\frac{P(s)}{Q(s)}\right),\ L(s)=\left[\frac{\omega}{2}Im\left(\frac{P(s)}{Q(s)}\right)\right]^{-1/2},\ \tau(s)=\int_{s_0}^{s}\frac{d\zeta}{C(\zeta)}.$$

Now we have received a Gaussian beam solution with an amplitude, exponentionally decaying with a distance from the central ray $K(s)$ denotes the curvature of the phase front of the beam and $L(s)$ is frequency - dependent effective half-width of the beam. Also from (29) we have

$$\frac{P(s)}{Q(s)} = \frac{\mathcal{E}P_1(s)+P_2(s)}{\mathcal{E}Q_1(s)+Q_2(s)}, \qquad (31)$$

with $\mathcal{E}=\mathcal{E}_1/\mathcal{E}_2$ (if $\mathcal{E}_2\neq 0$). So, instead of dealing with two complex constants we can be concentrated just on the parameter $\mathcal{E}$ (the quantity $\mathcal{E}_2^{-1/2}$ following from the amplitude term $Q^{-1/2}$, can be included without loss of generality into the amplitude term $\phi(\gamma)$). Now let us show, that if $Im(\mathcal{E})<0$, the following conditions are satisfied :

1) $Q(s)\neq 0$, that provides finite amplitudes at caustics;

2) $Im(P/Q)>0$, so the solution is concentrated close to the central ray.

To prove the first condition, let us mention, that

$$\det \Pi(s) = Q_1(s)P_2(s)-P_1(s)Q_2(s) = const = C^{-1}(s_0), \quad (32)$$

so $Q_1(s)$, $Q_2(s)$ can never be both equal to zero at the same point s (caustics for plane waves and for a point source are at different places). The expression (32) is checked by its differentiating with respect to s and using the dynamic ray tracing equations (22).

To prove the second condition, let us note, that

$$Im\left(\frac{P}{Q}\right) = \frac{Im(PQ^*)}{QQ^*} =$$

$$\frac{Im(\mathcal{E})}{QQ^*}\left(P_1Q_2-P_2Q_1\right) = -\frac{Im(\mathcal{E})}{C(s_0)QQ^*}$$

where we used formula (32).

In their work Cerveny et. al. (1982) suggested the following expression for $\mathcal{E}$:

$$\mathcal{E} = -i\left|\frac{Q_2(s_r)}{Q_1(s_r)}\right| \qquad (33)$$

In the homogeneous media $L(s)$ is a hyperbola and the choice (33) provides, that $L(s)$ has a minimum value at $s=s_0$ (source position) and the beam width is as narrow as possible at the receiver position $s=s_r$.

Now the last step we need to determine the Gaussian beams solution (30) is to find the amplitude factor $\phi(\gamma)$ According to Cerveny et. al. (1982), this factor can be evaluated by comparizon of high - frequency asymptotic solution of wave equation for a line source in homogeneous media

$$U(\vec{x}) = -\frac{A(\omega)}{4}\left(\frac{2C_o}{\pi\omega r}\right)^{1/2} \exp\left\{i\frac{\omega r}{C_o} - i\frac{\pi}{4}\right\}, \quad (34)$$

and the steepest descent approximation of the integral (27), where $U_\gamma(s,q)$ is determined by (30) :

$$U_{st}(\vec{x}_r) = A(\omega)\phi(\gamma_r)\, C_o\left(\frac{2\pi}{\varepsilon\omega r}\right)^{1/2} \exp\left\{i\frac{\omega r}{C_o} - i\frac{\pi}{4}\right\}, (35)$$

here γ_r is an angle between the reference axis and direction from the source to the receiver point $\vec{X}_r$; $A(\omega)$ is the result of Fourier transformation of the source time function; r is the distance from the source to the receiver. Equating (35) to (34) (it is supposed, that the media model is homogeneous in some vicinity of the source), we finally get :

$$\phi(\gamma) = -\frac{i}{4\pi}\left(\frac{\varepsilon}{C_o}\right)^{1/2} \quad (36)$$

with ε evaluated, for example, by (32).

6. Surface wave seismograms calculation by the Gaussian beams method.

Here we present the formulas, describing the procedure of surface wave seismograms calculation by the Gaussian beams method in 3D media with weak and smooth lateral inhomogeneties. As for surface wave the ray trajectories are constrained on the free surface (where $C(\vec{x})$ is distribution of phase velocity for a given frequency), the procedure here is quite similar to 2D acoustic case, considered in the previous

section. Due to this similarity we will not give here the derivations of all required formulas. For details see, for example, Yomogida and Aki (1987), Yomogida (1988).

For surface wave problem the results similar to (26), (27) have the following form:

$$\vec{U}(s,q,z) = \int_0^{2\pi} \phi(\gamma)\vec{U}_\gamma(s,q,z)\,d\gamma, \quad (37)$$

$$\vec{U}_\gamma(s,q,z) = \frac{\vec{O}}{[U(s)I_1(s)Q(s)]^{1/2}} \exp\left\{i\omega\left(\tau(s) + \frac{M(s)}{2}q^2\right)\right\}, (38)$$

with

$$\vec{O} = (\vec{n} - C(s)M(s)q\vec{t}\,)\,\ell_1(s,z) \quad (39)$$

- for Love waves,

$$\vec{O} = (\vec{t} + C(s)M(s)q\vec{n}\,)\,r_1(s,z) + i\vec{z}\,r_2(s,z) \quad (40)$$

- for Rayleigh waves. In the formulas (38)-(40) $\vec{t}$, $\vec{n}$ are the unit vectors, corresponding to the ray-centered coordinates (s,q) respectively, while $\vec{z}$ is a unit vertical vector; $\ell_1(s,z)$, $r_1(s,z)$, $r_2(s,z)$ are the "local" Love and Rayleigh waves eigenfunctions. "Local" means, that the eigenfunctions at any point on the surface are corresponded to the laterally homogeneous media, defined by the vertical structure at this point; $C(s)$ and $U(s)$ are "local" phase and group velocities; $I_1(s)$ is a kinematic energy integral : $1/2\int_0^\infty \rho\,\ell_1^2\,dz$ for

Love waves and $\frac{1}{2}\int_0^\infty \rho \left(r_1^2 + r_1^2\right) dz$ for Rayleigh waves; the value $\tau(s)$ is determined by integration along the ray (17), while $M(s)$, $Q(s)$ by solution of dynamic ray tracing equations (22). Solving the ray tracing equations (14) and dynamic ray tracing equations (22), we replace the 2D wave velocity function $C(\vec{x})$ by phase velocity distribution at the considered frequency.

To complete the formulas, we also need to present an expression for the amplitude factor $\phi(\gamma)$ We suppose, that the wavefield is generated by a point source with the moment tensor $M_{ij}\ (i,j = x,y,z)$ (e.g., Aki & Richards 1980, chapters 3 and 7) and that a media structure is laterally homogeneous is some vicinity of the source. In that case, $\phi(\gamma)$ is expressed by the following way (Yomogida & Aki, 1987) :

$$\phi(\gamma) = \frac{1}{8\pi C(s_0)} \sqrt{\frac{Q(s_0)}{U(s_0)\, I_1(s_0)}} \left\{ \qquad \right\}, \qquad (41)$$

where for the Love waves

$$\left\{ \quad \right\} = \left\{ i K \ell_1(h) \left[M_{xx} \sin\gamma \cos\gamma - M_{yx} \cos^2\gamma + M_{xy} \sin\gamma - M_{yy} \sin\gamma \cos\gamma \right] - \left.\frac{d\ell_1}{dz}\right|_h \left[M_{xz} \sin\gamma - M_{yz} \cos\gamma \right] \right\}, \qquad (42)$$

and for the Rayleigh waves

$$\left\{ \quad \right\} = \left\{ K r_1(h) \left[M_{xx} \cos^2\gamma + (M_{xy} + M_{yx}) \sin\gamma \cos\gamma + M_{yy} \sin^2\gamma \right] + i \left.\frac{dr_1}{dz}\right|_h \left[M_{xz} \cos\gamma + M_{zy} \sin\gamma \right] + \left.\frac{dr_2}{dz}\right|_h M_{zz} \right\}. \qquad (43)$$

In the formulas (42), (43) K is the wavenumber, $K = \omega C$, while h is the source depth.

Up to now in this section we considered just one mode of Love or Rayleigh waves. Dealing with several modes, we must sum the results (37) over each individual mode.

Now let us discuss some practical aspects, arizing when implementing the Gaussian beams technique for evaluating the surface wave seismograms. First of all we need to form the media model. We use the layered structure model, where the layers parameters vary slowly with horizontal coordinates. To define this model the free surface of the investigated region is covered by rectangular grid anf for each grid point we form a vector of vertical structure parameters. At each frequency the distribution of phase velocity $C(\vec{x})$ is determined by cubic splines interpolation of phase velocity values at the grid points (using the cubic splines, we satisfy the requirements of continuity of phase velocity function $C(\vec{x})$ with its first and second derivatives). The values of phase velocity at the grid points are found by sum of phase velocity for basic horizontally layered model and perturbation term, calculated by variational formulas. The integral (37) is replaced by summing with some step $\Delta\gamma$ while the ray tracing equations and dynamic ray tracing equations are solved by Runge-Kutta scheme of the fourth order. After evaluating the results in the frequency domain, we use the FFT technique to obtain the surface wave seismograms. The described procedure is quite effective and can be implemented, for example, on PC with modest resources.

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