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WORKSHOP ON EARTHQUAKE SOURCES & REGIONAL LITHOSPHERIC STRUCTURES FROM SEISMIC WAVE DATA

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Propagation in Inhomogeneous Media
Ray Theory

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by

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Preliminary version

A - Introduction

Our object of investigation is the Earth at different scales : global, regional and local scales. We are interested in the propagation of seismic signals in the complex media which is the Earth. Records have been obtained for the Moon and for Mars (?), while projects are under investigation for Venus. In the future, we might increase our interest for "planetaryquakes" and see differences and common aspects of these different celestial bodies. Although these notes will be essentially concentrate on methodological and technical aspects of the propagation, applications of the ray tracing theory at different scales will be mentioned in conclusion. Previous lectures have also illustrated many applications of high frequency theory in our understanding of earthquakes and structures.

The Earth is a mechanical body whose behaviour is complex and depends, in first approximation, on the time scale that one looks at the Earth and on the characteristic length related to this time scale by an appropriate velocity. For a characteristic time of milliard of years, the Earth behaves nearly as a drop of water. We progressively go through a visco-elastic behaviour for a time of millions of years for the crust and for a time of ten thousands of years for the mantle. For shorter periods between few days and fractions of seconds which corresponds to the seismic window, the Earth behaves as an elastic body with a noticeable attenuation which must be taken into account. Except from the source area where complex rheologies might take place in a few seconds, the response of the Earth is linear which reduces the complexity of the different approaches we might consider. Let us emphasize that the seismic window is a very large window with more than 7 orders of magnitude in time. There are few domains in physics where such a widespread spectrum is valid for investigating a single object.

Our knowledge of the Earth interior since the beginning of the century has increased greatly. From a rather imprecise and poetic picture (figure A-1), the vertical structure has been precized from a global understanging (figure A-2) to a more quantified picture (figure A-3) during the first half of this century. Only recent accumulation of data has allowed an investigation of lateral variations (figure A-4).

Seismological data were crucial for this quantification : seismology is a very powerful tool for our knowledge of the Earth interior, because seismic waves go from one side to the other bringing to the surface the information they felt during their propagation. Other quantities as magnetic, electric or gravimetric fields allow different reconstruction of the Earth, but none has the resolution of seismic waves which is the reason of the importance of the seismic tool in the oil search in

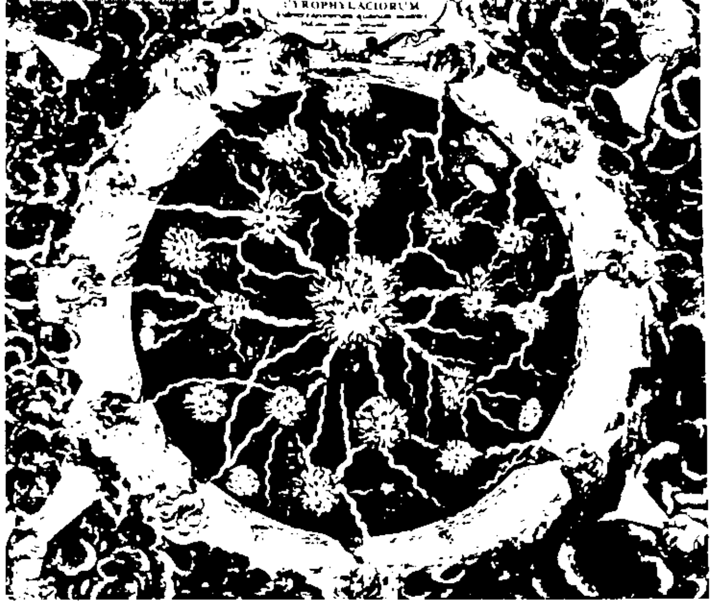


Figure 1.1 An early view (about 1800) of the Earth's interior. The writer conceived of Earth as a ball of solid material fissured by tubes of magma, connecting pockets of eruptive gases to volcanic vents on the Earth's surface.

Figure A-1

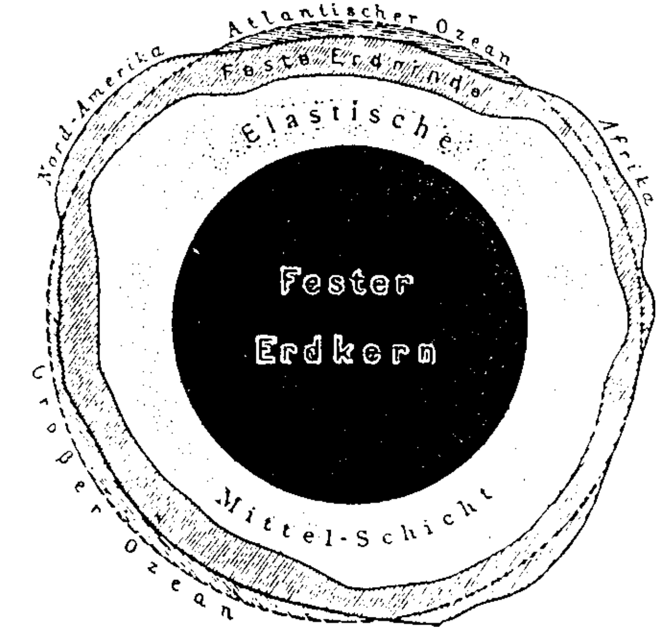


Figure 1.2 Sketch of the Earth's interior published in Berlin in 1902 (H. Kraemer). The Earth has three shells: a solid crust (Feste Erdrinde) supported by an elastic mantle (Elastische Mittel-Schicht) wrapped around a solid central core (Fester Erdkern). The change from Figure 1.1 reflects an improved physical understanding, but the model is still limited by lack of seismological data.

Figure A-2

FROM "Inside the Earth" by Bruce A. Bolt (1982

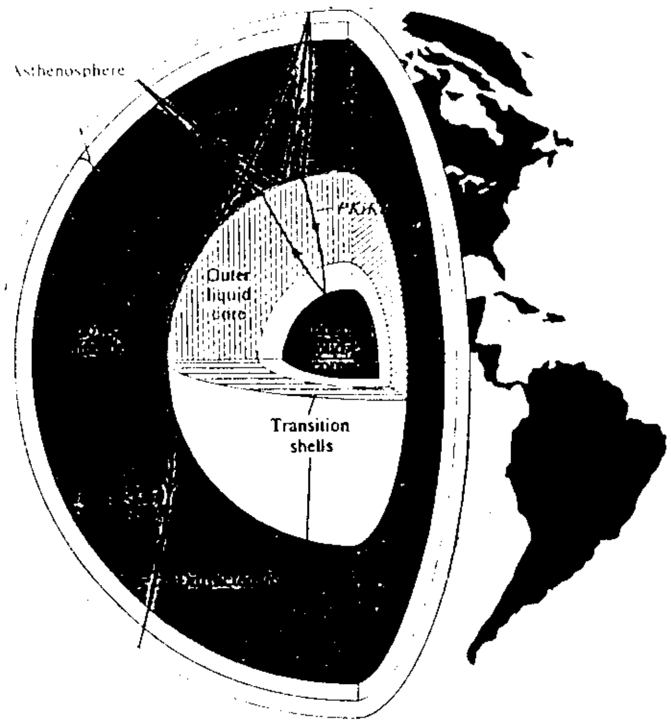


Figure A-3

Figure 1.3 A cross section of the Earth based on the most recent seismological evidence. The outer shell consists of a rocky mantle that has structural discontinuities in its upper part and at its lower boundary that are capable of reflecting or modifying earthquake waves. Below the mantle an outer fluid core surrounds a solid kernel at the Earth's center; between the two is a transition shell. The paths taken by three major kinds of earthquake waves are shown. The waves reflected from the outer liquid core are designated *PcP*, the waves reflected from the inner solid core are *PKiKP*; and the waves that creep around the liquid core are diffracted *P*. [From Bruce A. Bolt, "The Fine Structure of the Earth's Interior." Copyright © 1973 by Scientific American, Inc. All rights reserved.]

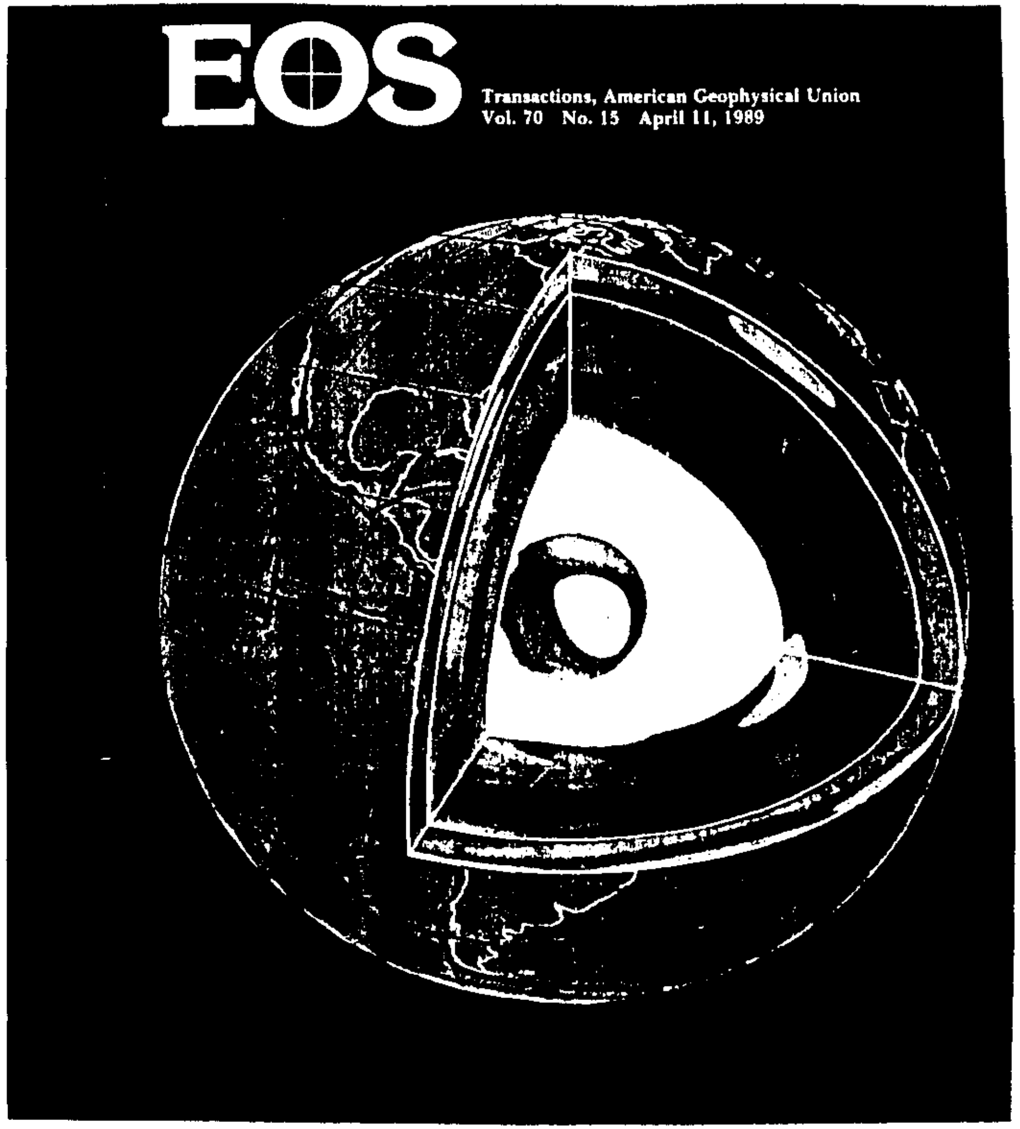


Figure A-4: A cross section of the Earth showing a composite image synthesized from seismic tomographic mapping down to the core. Lateral variations are

FROM cover of "EOS" 1989.

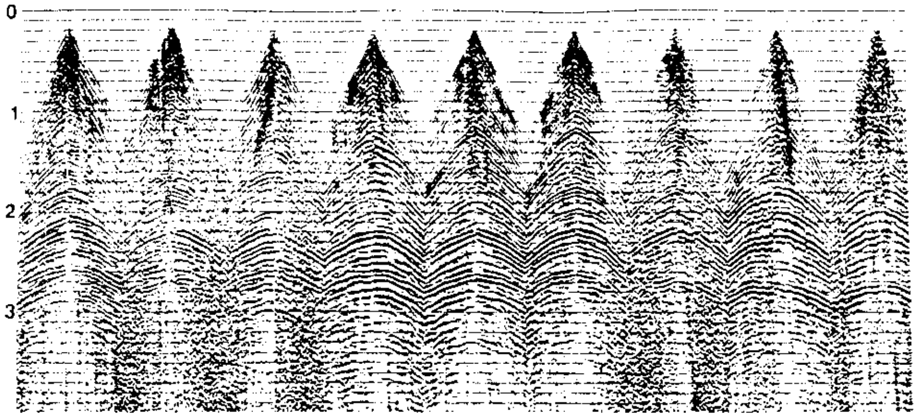


FIG. 2-81. Common-shot gathers with no deconvolution (vibroseis source). Geometric spreading correction and trace balancing have been applied.

Figure A-6: Please note the two time scales in these differents traces.

FROM "Seismic data processing" by ÖZDOĞAN YILMAZ (1983)

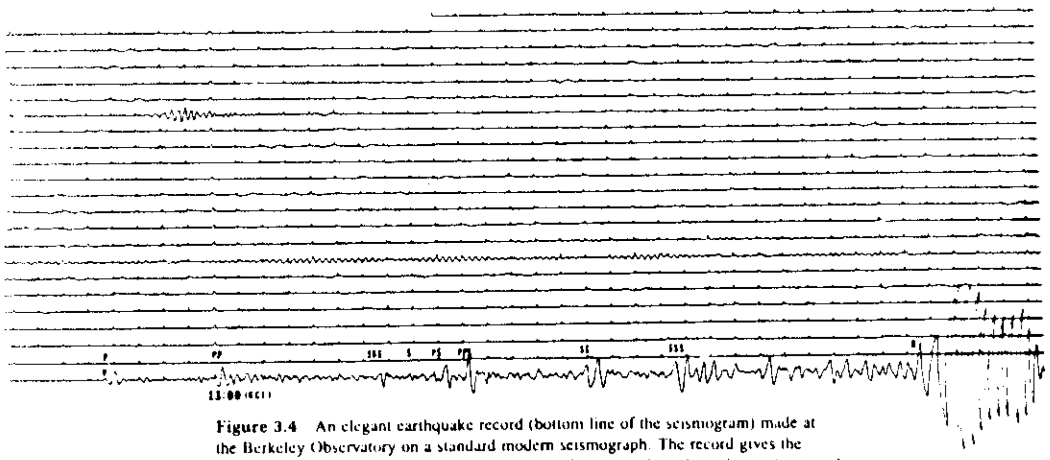


Figure 3.4 An elegant earthquake record (bottom line of the seismogram) made at the Berkeley Observatory on a standard modern seismograph. The record gives the vertical motion of the ground surface. The interval between the tick marks on the record corresponds to 1 minute, and time increases from left to right. This earthquake occurred near Borneo at a distance of 11,000 km from Berkeley. The onset of *P* waves is clearly seen, together with the *PP* reflection. This is followed by the onset of *SKS* and *S* waves and the reflections *PS*, *PPS*, *SS*, and *SSS*. At the end of the bottom trace can be seen the Rayleigh wave train, starting with long but decreasing periods (an example of wave dispersion). The record is not complete because the wave motion was interrupted by the operator inadvertently changing the seismogram.

Figure A-5: Please note the time scale associated to propagation and the time scale associated to source signal.

spite of a significant increase in cost. Electromagnetic waves are mainly diffused and not transported, while the gravimetric field suffers from the duality between the distance and the importance of the anomaleous structure. This resolution power has a counter part with the difficulty for the seismologist to interpret rather complex and deformed signals. This is why seismology, from my point of view, is so interesting.

In seismograms (as one call time records for the global and regional scale) or traces (as one call records for the local scale), two caracteristic times appear : the time associated to the source signal which has a content of few seconds down to milliseconds and the time associated with the propagation which goes from hundred of seconds down to few seconds (figure A-5 and figure A-6). For an opposite view, let us mention that recordings of the sea motion do no exhibit this so nice feature. One must exploit deeply this advantage seen on seismograms : This is the reason why we should look carefully to the ray tracing theory which is based explicitly on this decoupling between two time scales.

Nor the relative simplicity of the ray theory neither its computer efficiency make ray theory an often used technic in seismology, but its capacity for seismic interpretation. Let us underline that global models of Gutenberg and Jeffreys elaborated around 1940 are based on travel-time computed by ray tracing. The presence of a core has also been demonstrated by Oldman in 1906 using rays, just after the beginning of the modern seismology and the inner core was discovered by Lehmann in 1936 from ray tracing interpretation of travel-time arrivals. At the global and regional scales, ray tracing is used in daily earthquake locations, in polarizations studies and in tomographic pictures of the Earth interior. Seismic profils for oil research - diagramme x^2 - t^2, normal move-out, deep move-out, reflection hyperbole - exploit in an every day practical approach ray tracing results. Reflected tomography, migration techniques are techniques often based on ray theory.

This rather lengthy introduction shows, hopefully, that understanding the ray theory worths the required intellectual effort. Arguments in this lecture will show that seismology is an area where sophisticated tools based on ray theory have been designed for specific applications with no direct equivalence in other fields of physics. Acoustic wave propagation in oceans have different ranges of approximation, electromagnetic propagation in the ionosphere assumes with good accuracy layer approximation while optical ray tracing is often made in homogeneous media. For ray tracing inside the Earth, we are on our own.

B - WAVE PROPAGATION

In order to set up our notations, we shall discuss briefly wave propagation equations without complete discussions. Let us consider an heterogeneous elastic linear medium and a displacement $\mathbf{u}$ in a point M of cartesian coordinates (x, y, z). One can define two tensors of order 2 (just a matrix) at this point : the stress tensor σ and the strain tensor ϵ. These tensors, called infinitesimal, are local. The strain tensor is expressed with the help of the displacement vector of the point M. In a cartesian coordinate system, the expression

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad (b-1)$$

precises this relation. The coma in subscript indicates a partial derivative with respect to the following coordinate. The stress tensor comes with the elastodynamic equation

$$\sigma_{ij,j} + f_i = \rho u_{i,tt}, \qquad (b-2)$$

with body forces acting f_i on point M. The stress tensor is related to the acceleration on the point M. ρ is the volumic density. An additional relation is needed in order to connect the stress and the deformation. This relation defines the rheological behaviour of the material and is often a simple linear relation :

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl} \qquad (b-3)$$

where c_{ijkl} are elastic coefficients. Because of symmetrical relations, these 81 coefficients reduce to 21 independent ones. We find the linear elastodynamic equations under a general form for an anisotropic medium :

$$[c_{ijkl}u_{k,l}]_{,j} + f_i = \rho u_{i,tt}. \qquad (b-4)$$

where the function c_{ijkl} is differentiable and continuous, as well as its first derivative. The second derivative must exist and be continuous by step. We shall assume the same thing for the volumetric density.

For an istropic, elastic and linear body, the elastic coefficients can be expressed with only two independent ones as the Lamé parameters λ and μ for example :

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \qquad (b-5)$$

Other choices are possible as the Young's modulus E or the Poisson coefficient ν or any linear expression well adapted to the problem at hand. In isotropic medium, we obtain the following elastodynamic linear equation :

$$(\lambda + \mu)u_{j,ij} + \mu u_{i,jj} + \lambda_{,i}u_{j,j} + \mu_{,j}[u_{i,j} + u_{j,i}] + f_i = \rho u_{i,tt}. \qquad (b-6)$$

A further simplification is possible for material, as the water, where only compression takes place and the shearing is not possible. There is only one independent parameter and the incompressibility modulus K is often the most suited one. The pressure variation P verifies the scalar equation :

$$\frac{1}{K(\mathbf{x})}P_{,tt}(\mathbf{x},t) - \frac{1}{\rho(\mathbf{x})}P_{,i})_{,i}(\mathbf{x},t) = S(\mathbf{x},t) \qquad (b-7)$$

where S is an explosive source with a pression variation in space and in time. For an homogeneous volumetric density, we simply obtain the wave equation :

$$\Delta P(\mathbf{x},t) - \frac{1}{c^2(\mathbf{x})}P_{,tt}(\mathbf{x},t) = -S(\mathbf{x},t) \qquad (b-8)$$

where Δ denotes the laplacian at the point M and c is the wave speed.

Terminology is rather confusing between the wave equation, the scalar wave equation, the vectorial wave equation, the acoustic wave equation, the elastodynamic equation. For my point of view, the wave equation involves only the displacement while the elastodynamic equation takes into account the stress as well. To be aware of this slight ambiguity about the vocabulary is the best way to solve the difficulty often met about these equations.

Before closing the section, I would like to present a simple case which is interesting for the ray theory later on : this is the solution in a homogeneous medium of the wave equation. One can say that a so simple example does not require a complete discussion. As far as I know, obtaining this solution is difficult with careful analysis of opposite singularities at the source position : it is far from obvious.

Let us first consider a laplacian of a scalar quantity g which is singular inside a small volume of radius ϵ around the source. The singularity is such that the ponctual source is impulsive, i.e.

$$\int\int\int \Delta g(r)dv \to -\int\int\int \delta(\mathbf{x})dv = -1, \qquad (b-9)$$

We apply the Gauss theorem for reducing the volume integral down to a surface integral and then we take into account the symmetry of the geometry to integrate this equation. We find the following equations

$$\int\int\int \Delta g(r)dv = \int\int \mathbf{grad}(g)\cdot\mathbf{dS},$$

$$\int\int\int \Delta g(r)dv = (\frac{dg}{dr})_{r=\epsilon}4\pi\epsilon^2,$$

By comparison with equation (b-9), we find that the term dg/dr behaves as $1/r^2$, and, consequently, that g behaves as $1/r$ when r goes to zero.

For the acoustic wave equation in a three-dimensional homogeneous medium with an impulsive ponctual source,

$$\Delta P(\mathbf{x},t) - 1/c^2 P_{,tt}(\mathbf{x},t) = -\delta(\mathbf{x})\delta(t), \qquad (b-10)$$

an heuristic argument is to consider the laplacian as the dominant term compared to the time derivative since the laplacian involves the second derivatives of a three dimensional function $\delta(\mathbf{x})$. A more rigourous approach can be found in Morse et Feshbach (1953, p 838), solution which does not introduce new ideas compared to what I present here. The behaviour of P would be when r goes to zero

$$P(r,t) --> \delta(t)/4\pi r \qquad (b-11).$$

We look for a solution of the equation without the left hand side which verifies this condition (b-11). Because the medium is homogeneous and the source has a spherical symmetry, the pressure field P will only depend on the radius r. The expression of the laplacian in spherical coordinates gives :

$$1/r^2 \frac{\partial}{\partial r}(r^2 \frac{\partial P}{\partial r}) - 1/c^2 \frac{\partial^2 P}{\partial t^2} = 0 \qquad (b-12)$$

which reduces to the equation

$$\frac{\partial^2 (Pr)}{\partial r^2} - 1/c^2 \frac{\partial^2 (Pr)}{\partial t^2} = 0, \qquad (b-13)$$

the fundamental solution P of which is composed of two arbitrary functions h et k in the following expression :

$$P = [h(t - r/c) + k(t + r/c)]/r. \qquad (b-14)$$

From the condition (b-11), only the functions $\delta(t-r/c)/r$ or $\delta(t+r/c)/r$ are allowed as well as any combination. The second solution can be eliminated because the impulsion given at time zero must be felt at the position r only later on. The elementary solution in a three-dimensional homogeneous medium

$$P(r,t) = \frac{1}{4\pi}\delta(t - r/c)/r \qquad (b-15)$$

is the impulsive function weighted with a geometrical decrease $1/r$. This perturbation propagates at speed c as damped concentric shells. For a general source $S(\mathbf{r}_0, t)$, we consider the convolution product of an impulsive function with the

source function and we find the well known function called delayed potential in physics

$$P(\mathbf{r},t) = \frac{1}{4\pi}\int_{V_0} dv_0 \frac{S(\mathbf{r}_0, t - R/c)}{R}, \qquad (b-16)$$

where R is the distance between the point $\mathbf{r}$ and the integration point $\mathbf{r}_0$ over the source zone V_0.

For the two dimensional case where the distribution of sources is along the z axis, an integration of the solution (b-15) along this axis gives the following elementary solution

$$P(r,t) = \frac{1}{2\pi}\frac{H(t - r/c)}{\sqrt{(t^2 - r^2/c^2)}}, \qquad (b-17)$$

with a typical tail after the wavefront. By another integration, one one can deduce the elementary solution for a one-dimensional medium :

$$P(r,t) = \frac{c}{2} H(t - r/c). \qquad (b-18)$$

A given point stays at rest until it is reached by a constant pressure at time $t = r/c$. The time derivative of the pressure may be a better quantity with an impulsive shape. By anticipating on the definition of the Fourier transformation, we must underline that the solution (b-15) has the following Fourier transformation

$$\frac{1}{4\pi r} e^{i\omega r/c}, \qquad (b-19)$$

while the Fourier transformation of the two-dimensional solution (b-17) is the Bessel function singular at the origin, i.e. the Hankel function of order zero :

$$\frac{i}{4} H_0^1(\omega r/c). \qquad (b-20)$$

Finally, the Fourier transformation of the derivative of the pressure for an one dimensional medium is

$$\frac{1}{4\pi} e^{i\omega r/c}. \qquad (b-21)$$

These solutions are very important for the normalization of the asymptotic solutions of the ray theory. I show that their construction is far from obvious, but I am sure that they will be quite natural and intuitive to the reader after a while. We often use them in the ray theory.

C - Resolution methods of elastodynamic equation

Construction of solutions for the wave equation or the elastodynamic equation is a difficult task. In these notes, I shall mention only different methods in order to situate the ray theory among them in a global frame.

One can consider three main groups of methods with possible relations between two particular techniques.

The first approach deals with the space $(\mathbf{x}, t)$ where numerical methods solve directly the differential equations. These methods as finite difference or finite element methods are brute force methods and are very well suited to computers with specific architecture. Variations in order to estimate partial derivatives are based on interpolating function : Fourier methods, also called pseudo spectral methods, go to the Fourier space in order to evaluate partial derivatives, while differential equations are still verified in the space $(\mathbf{x}, t)$. Integral equations in the real space are also an alternative to the resolution of differential equations. These integral equations have the advantage to reduce by one dimension the problem to be solved at the expense of a linear system with full matrices while the finite element method deals often with sparse matrices.

The second group manages a transformation towards a new space $(\mathbf{k}, \omega)$, where the resolution of the transformed equations are expected to be simpler. These methods, called spectral methods, have many variations as the reflectivity method, discret wave number method or integral equations. Depending on the spatial variations of the medium properties, the separation of partial derivatives will be partial or complete. For partial separation, one can select the space $(\mathbf{x}, t)$ in the unsolved direction and use previous methods in order to obtain the solution. Transformation back to the real space are always required at the final stage.

The major disadvantage of the two already mentioned group is a great difficulty for the interpretation of synthetic seismograms.

A third class of methods assumes an asymptotic behaviour at high frequency as the ray theory, the WKBJ method for stratified media or the MASLOV method available for laterally varying media. Techniques of beam summations as the gaussian beam summation GBS has increased the interest of the seismological community for these asymptotic methods. The main advantage of these methods is not only their computer efficiency but their capacity of physical interpretation of computed results.

D - High frequency approximation (acoustic case)

We want to construct an high frequency approximation of the solution of the wave equation and, in order to do so, we start from an assumed expression of its Fourier transformation. At high frequency, this expression is related to the wavefront notion which is not destroyed by the heterogeneity of the medium. The use of the Fourier transformation is not strictly necessary as long as the wavefront meaning is kept. We shall not discuss this space-time ray theory in these notes.

Ansatz of the ray theory

For the introduction of the high frequency approximation, we need a definition of the Fourier transformation of the function f and we assume the following expression :

$$f(\omega) = \int_{-\infty}^{+\infty} f(t)\, e^{i\omega t}\, dt, \qquad (d-1)$$

with the inverse transformation

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\omega)\, e^{-i\omega t}\, d\omega. \qquad (d-2)$$

We shall keep the same notation for the function and its transformation : the argument of the function will tell us in which space we are. Let us recall the acoustic wave equation :

$$\nabla^2 P(\mathbf{x}, t) - \frac{1}{c^2(\mathbf{x})} \frac{\partial^2 P(\mathbf{x}, t)}{\partial t^2} = -S(\mathbf{x}, t), \qquad (d-3)$$

where P is the pressure in $\mathbf{x}$ at time t and S is the source function. The propagation velocity $c(\mathbf{x})$ may vary continuously in space. Initial conditions are requested and a zero pressure is the simplest condition as well as a zero temporal derivative in every point of the space at time zero. In the frequency domain, the wave equation becomes the Helmholtz equation :

$$\nabla^2 P(\mathbf{x}, \omega) + \frac{\omega^2}{c^2(\mathbf{x})} P(\mathbf{x}, \omega) = 0, \qquad (d-4)$$

outside the source area. The wave number is denoted by $k = \omega/c$. The solution of this equation may be rather complex, but physical considerations allow us to assume a particular form of the solution. An intuitive argument comes from the solution in a three dimensional homogeneous space

$$\frac{1}{4\pi r} e^{i\omega T}, \qquad (d-5)$$

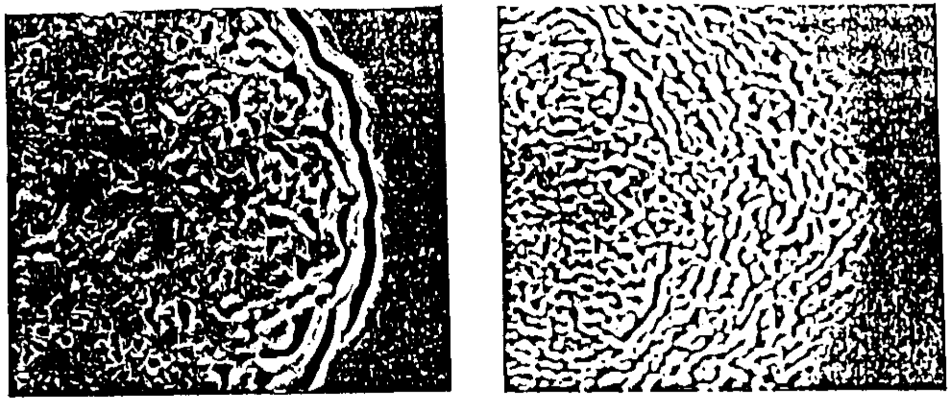


Figure D-1 : On the left, initial wavefronts are still present, while on the right the initial wavefronts have disappeared.

FROM FRANKEL and CLAYTON, JGR, 1986, p 6465 with an entirely different meaning here.

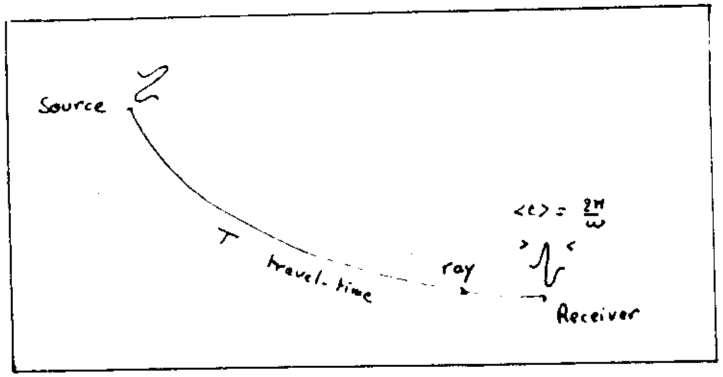


Figure D-2 : Schematic diagram with travel-time and source time

with two distint terms : the first one defines the amplitude while the second locates the wavefront. The travel time T is r/c in a homogeneous medium. The coherence of the wave front might be preserved in heterogeneous media with a travel time and an amplitude defined locally. Wave fronts are deformed but still exists. Figure (D-1) shows an example on the left where wavefronts are still visible in spite of the heterogeneity of the medium and an example on the right where the heterogeneity variation destroy entirely the wavefront coherence. Formally, the ansatz of the solution will be :

$$P(\mathbf{x},\omega) = S(\omega)A(\mathbf{x},\omega)e^{i\omega T(\mathbf{x})} \qquad (d-6)$$

where the function $S(\omega)$ is defined by initial conditions (source description or screen excitation, for example...). The asymptotic approximation assumes that the function $A(\mathbf{x},\omega)$ has the following form :

$$A(\mathbf{x},\omega) = \sum_{k=0}^{k=\infty} \frac{A_k(\mathbf{x})}{(-i\omega)^k}, \qquad (d-7)$$

which splits the spatial dependence and the frequencial or temporal dependence. From the practical point of view, we are interested only in the zero order approximation , i.e. $A(\mathbf{x},\omega) = A_0(\mathbf{x})$. The zero-order solution becomes

$$P(\mathbf{x},\omega) = S(\omega)A_0(\mathbf{x})e^{i\omega T(\mathbf{x})} . \qquad (d-8)$$

In the time domain, the solution has an elegant analytical time dependence

$$P(\mathbf{x},t) = A_0(\mathbf{x})S(t-T), \qquad (d-9)$$

which demonstrates that the source signal S propagates without distorsion at high frequency with a travel-time T. We shall assume that the source function spectra is zero for frequencies lower than ω_m. Moreover, we consider only positive frequencies. We shall see later on how to take into account negative frequencies.

We have now the explicit demonstration of two time scales in the seismic signal : the travel-time T and the characteristic time of the source $< t >$ as shown in figure D-2. This time of the source defines the spectral bandwidth of the emitted energy by the relation $< t >= 2\pi/\omega$ and the high-frequency approximation will be valid if $\omega T >> 1$. If this is not the case, we have interferences betweeen the propagated source signal and the medium. Diffraction and distorsion effects happen and the serie (d-7) might partially overcome this difficulty at the expense of additional difficulties.

One must underline that exact solutions for a point source in three dimensional and one dimensional media have the adequat expression for the ray theory where the signal propagates without deformation. For the two-dimensional case, the exact solution is not a delayed potential : we must look for its high frequency approximation. Going through the Fourier transform which is the Hankel function of zero order H_0^1, one might write the asymptotic form of H_0^1 which is a plane wave

$$P(r,\omega) = \frac{i}{4}H_0^1(\omega r/c) \simeq \frac{i}{4}\sqrt{\frac{2c}{\pi\omega r}}e^{i\omega r/c}e^{-i\pi/4}. \qquad (d-10)$$

This plane wave has a compatible form with the ray theory :

$$P(r,\omega) = [\frac{1}{4\pi}\sqrt{\pi/\omega}e^{i\pi/4}][\sqrt{2\frac{c}{r}}]e^{i\omega r/c}. \qquad (d-11)$$

The final approximation in time domain can be written as

$$P(r,t) = \frac{1}{2\pi}\sqrt{c/2r}\,\frac{H(t-r/c)}{\sqrt{t-r/c}} \qquad (d-12)$$

which is a good approximation of the exact solution near the wave front where the time can be estimated to r/c in the following expression

$$\sqrt{t^2-\frac{r^2}{c^2}} \sim \sqrt{2r/c}\sqrt{t-\frac{r}{c}}. \qquad (d-13)$$

Let us go back to the asymptotic serie. Inserting the ansatz (d-6) as well as the serie (d-7) in the Helmholtz equation (d-4) and ordering terms in power of the frequency ω, we find with the help of the two following equalities :

$$s(\omega)\nabla P = s(\omega)\nabla A e^{i\omega T} + i\omega\nabla T A e^{i\omega T}$$

$$s(\omega)\nabla^2 P = s(\omega)\nabla^2 A e^{i\omega T} + [i\omega 2\nabla A\cdot\nabla T + A\nabla^2 T]e^{i\omega T} - \omega^2(\nabla T)^2 A e^{i\omega T}$$

the cascade of equations

in ω^2: $((\nabla T)^2 - 1/c^2)A_0 e^{i\omega T} = 0$

in $-i\omega$: $(2\nabla A_0\nabla T + \nabla^2 T A_0)e^{i\omega T} = 0$

in $(-i\omega)^k$: $(\nabla^2 A_{k-1} + 2\nabla A_k\nabla T + (\nabla T)^2 A_{k-2})e^{i\omega T} = 0$ for $k > 1$

The equations for the two first powers of ω attract our attention. The first equation, called eikonal (from picture in greek),

$$(\nabla T)^2 = \frac{1}{c^2}, \qquad (d-14)$$

takes into account only the travel time, while the second equation

$$2\nabla A_0\cdot\nabla T + \nabla^2 T A_0 = 0. \qquad (d-15)$$

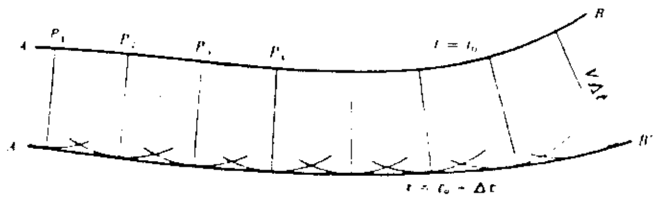


Fig. 4.11 Using Huygens' principle to locate new wavefronts.

Figure D-3: Construction of a new wavefront at time $t_0+\Delta t$ from the wavefront at time t_0. The length perpendicular to the wavefront is proportionnal to the speed V times the time increment

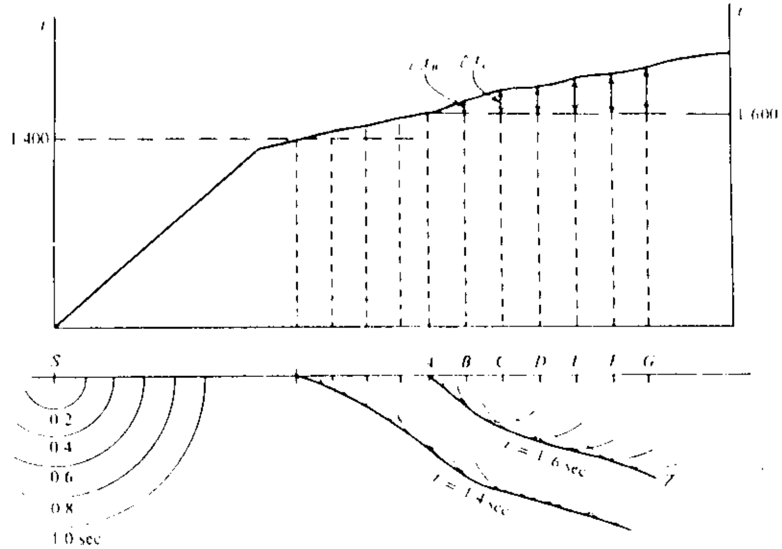


Fig. 4.122 Reconstruction of wavefronts.

Figure D-4: Reconstruction of wavefronts from refracted profils

FROM "Applied Geophysics" by W.M. TELFORD, L.P. GELDART, allows computation or transport of the amplitude at time T. We call it the transport equation. The other equations allow to estimate formally higher terms of the serie.

The convergence of the serie is not analyzed. We only need a formal identity with the wave equation : higher terms can be far from negligeable quantities, as for critical reflections, creeping waves and some conversions of compressive waves into shearing waves.

Eikonal equation

The eikonal equation

$$(\nabla T(\mathbf{x}))^2 - \frac{1}{c^2(\mathbf{x})} = 0. \qquad (d-16)$$

is the basic equation which controles the evolution of wavefronts. We shall see that it is also true for elastic medium for each kind of wave. Solving this equation is related to the kinematic propagation of wavefronts defined as equal phase surface $T(\mathbf{x}) = T$. Looking for the evolution of the function $T(\mathbf{x})$ is more general than the simple signification of the high frequency approximation. : wavefronts exist even when a medium has rapid variations. One can construct wavefronts at time $t+dt$ knowing the wave front at time t : it is enough to use the Huyghens principle for the geometrical construction. A constant length is taken away for the initial wave front such that the modulus of the gradient ∇T is equal to $1/c(\mathbf{x})$ at the current point as shown graphically (figure D-3). This technique has been extensively exploited in a graphical approach of the wave propagation and for graphical interpration of refracted profils (figure D-4). Because computer memories become rather inexpensive, this method, which requires important memory capacities, is now an attractive alternative to ray tracing for specific applications. Its computer implementation solves the eikonal equation by finite differences on a regular grid for the first travel-time. We shall find in a near future interesting applications of this method.

Instead of looking for wavefronts, we might focus our attention on orthogonal trajectories to wavefronts at each point (figure D-5). These trajectories are rays which have not the obvious physical support that they have in optics. The morphologic structure of our eye is responsable of this fact. Tracing rays, instead of wavefronts, is the standard way to solve ordinary differential equations, approach re going to set up. This approach is called the characteristic method. Let us consider the implicit equation of a ray $\mathbf{x}(s)$ where s is the curvilinear abscisse. The tangent is defined by

$$\mathbf{t} = \frac{d\mathbf{x}}{ds} \qquad (d-17)$$

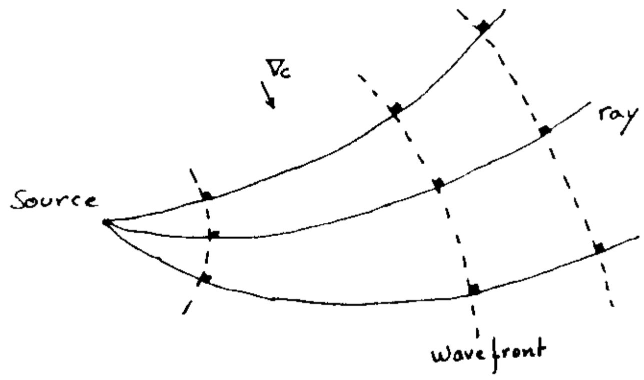


Figure D-5: rays are orthogonal trajectories to wavefronts.

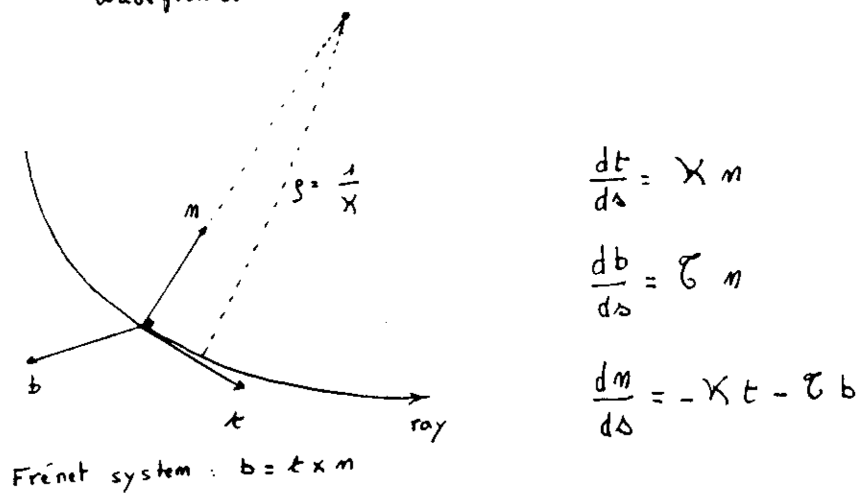


Figure D-6: Frénet system with curvature χ and torsion τ with a modulus equal to one by definition of the curvilinear abscissa. From ray definition as orthogonal trajectories, the tangent is parallel to ∇T. From the eikonal equation, one can deduce the following equation

$$\frac{d\mathbf{x}}{ds} = c\nabla\mathbf{T}. \qquad (d-18)$$

We often say that $| d\mathbf{x} | / c$ is the optical distance. After this tangent evolution of rays, we must study the normal evolution which comes with the evaluation of

$$\frac{d\nabla\mathbf{T}}{ds} = \frac{d}{ds}\left(\frac{1}{c}\frac{d\mathbf{x}}{ds}\right). \qquad (d-19)$$

Knowing that the derivative with respect to the curvilinear abscissa s is the projection of the vectorial gradient on the tangent, we obtain the following operator applied to each component of a vectorial quantity :

$$\frac{d}{ds} = \mathbf{t}\cdot\nabla = c\nabla\mathbf{T}\cdot\nabla, \qquad (d-20)$$

where we use the operator notation for vectorial gradient. We have successively and formally the following equalities

$$\begin{aligned}\frac{d\nabla\mathbf{T}}{ds} &= c\nabla\mathbf{T}\cdot\nabla(\nabla\mathbf{T})\\ &= \frac{c}{2}\nabla(\nabla T)^2\\ &= \frac{c}{2}\nabla\left(\frac{1}{c}\right)^2\\ &= \nabla\left(\frac{1}{c}\right)\end{aligned} \qquad (d-21)$$

from which we can deduce with the help of (d-19) the equation describing the evolution of the normal part of ray trajectories.

$$\frac{d}{ds}\left(\frac{1}{c}\frac{d\mathbf{x}}{ds}\right) = \nabla\left(\frac{1}{c}\right). \qquad (d-22)$$

this equation is also called the curvature equation because differentiation implies

$$\nabla\left(\frac{1}{c}\right) = \mathbf{t}\frac{d}{ds}\left(\frac{1}{c}\right) + \frac{1}{c}\frac{d\mathbf{t}}{ds} \qquad (d-23)$$

and, using the definition of curvature $\mathcal{K}$ in the Frénet system as mentioned in figure D-6

$$\frac{d\mathbf{t}}{ds} = \mathcal{K}\mathbf{n}, \qquad (d-24)$$

we find

$$\nabla\left(\frac{1}{c}\right) = \mathbf{t}\frac{d}{ds}\left(\frac{1}{c}\right) + \frac{\mathcal{K}}{c}\mathbf{n}. \qquad (d-25)$$

This relation is better written explicitly with the slowness u.

$$\nabla u = \mathbf{t}\frac{du}{ds} + \mathcal{K}\, u\, \mathbf{n}. \qquad (d-26)$$

The scalar product of this equation with the normal $\mathbf{n}$ controles the evolution of the curvature, which gives explicitly

$$\begin{aligned}\mathcal{K} &= \frac{1}{u}\,\mathbf{n}\cdot\nabla\mathbf{u}\\ &= \frac{1}{u}\frac{\partial u}{\partial n}\\ &= -\frac{1}{c}\frac{\partial c}{\partial n}\end{aligned} \qquad (d-27)$$

The curvature increases in the opposite direction of the velocity gradient perpendicular to the ray. The figure (D-7) indicates the vectorial construction deduced from equation (d-26) and allows the introduction of the angle i between the gradient of the slowness and the tangent to the ray with the simple relation

$$\mathcal{K} = \frac{1}{u}\,|\nabla u|\, \sin i. \qquad (d-28)$$

The torsion $\mathcal{T}$ is defined as the scalar product between $d\mathbf{n}/ds$ and the binormal $\mathbf{b}$. From the expression of the normal

$$\mathbf{n} = \frac{1}{\mathcal{K}u}\left[-\frac{du}{ds}\,\mathbf{t} + \nabla u\right], \qquad (d-29)$$

one can deduce that the only contribution to the derivative of $\mathbf{n}$ with respect to the curvilinear abscissa s along the binormal comes from the following term

$$\frac{1}{\mathcal{K}u}\frac{d\nabla u}{ds}. \qquad (d-30)$$

The torsion $\mathcal{T}$ interpreted in figure D-8 given by

$$\begin{aligned}\mathcal{T} &= \frac{1}{\mathcal{K}u}\frac{d\nabla u}{ds}\cdot\mathbf{b}\\ &= -\frac{1}{\mathcal{K}c}\frac{d\nabla\mathbf{c}}{ds}\cdot\mathbf{b}\end{aligned} \qquad (d-31)$$

is essential for the elastic case. For the moment, it completes the geometrical analysis of ray tracing. After these considerations, we may trace rays with the help

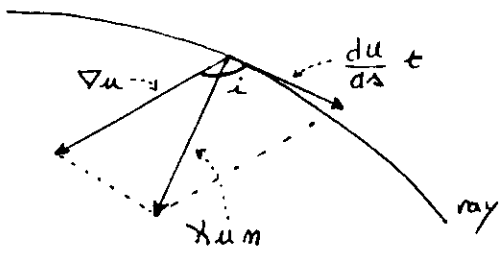


Figure D.7 : geometrical interpretation of equation (d-26) : rays are bent towards the gradient of the slowness u

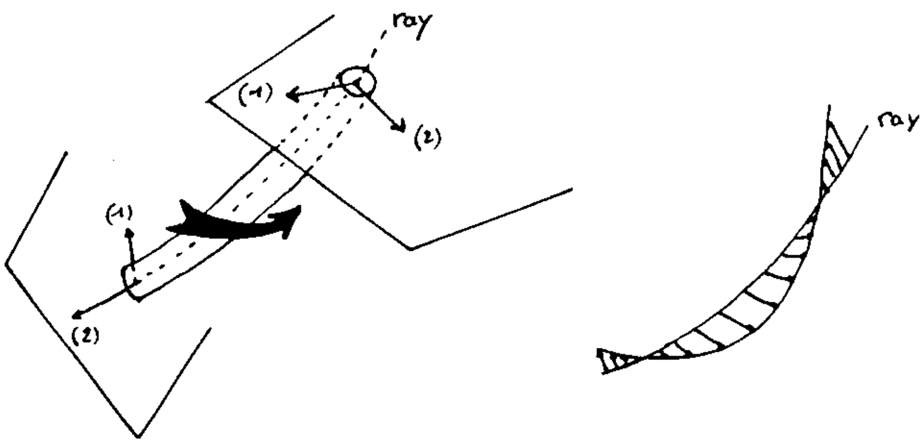


Figure D.8 : Two schematic diagrams for geometrical interpretation of torsion $\mathcal{T}$.

of the curvature equation which is a second-order ordinary differential equation O.D.E.

It is natural to reduce the order of the differential system by introducing an additional variable which comes from the equation along the tangent. This variable, called slowness vector **p**, is defined by

$$\mathbf{p} = \nabla T = \frac{1}{c}\frac{d\mathbf{x}}{ds}. \qquad (d-32)$$

An important system of first-order differential equations is deduced

$$\frac{d\mathbf{x}}{ds} = c\,\mathbf{p}$$
$$\frac{d\mathbf{p}}{ds} = \nabla\left(\frac{1}{\mathbf{c}}\right) \qquad (d-33)$$

with the following constraint coming from the eikonal equation

$$\| p \| = \frac{1}{c}. \qquad (d-34)$$

The phase or travel-time is obtained by integration of the eikonal equation along the ray, i.e.

$$\frac{dT}{ds} = \frac{1}{c} = u. \qquad (d-35)$$

We shall study the coupled non-linear system (d-33) where the velocity c depends on position **x** instead of the non-linear equation (d-22) because the variable **p** does not only play the role of an auxiliary variable but has an importance equal to the one of the position. We might expect behaviours of O.D.E with the influence of initial conditions, with the geometrical structure of the solution coming from the catastroph theory, with stiffness, strange attractors and so on. This problem in itself is a difficult problem to solve and we shall spend part of our effort in order to analyze it. Before doing it, we shall examine the transport equation

Equation de transport

The transport equation which is written

$$2\nabla A_0 \nabla \mathbf{T} + \nabla^2 T A_0 = 0, \qquad (d-36)$$

might be converted into a simpler form by multiplying the equation (d-36) with the non-zero quantity A_0 in order to get a divergence

$$div(A_0^2 \nabla \mathbf{T}) = 0. \qquad (d-37)$$

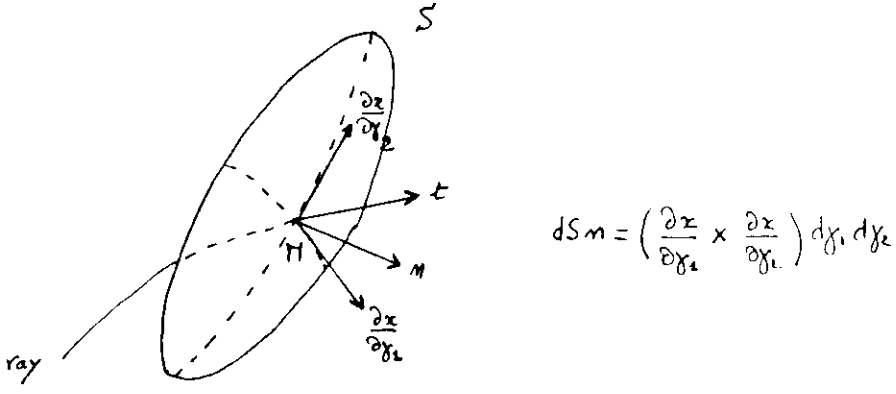
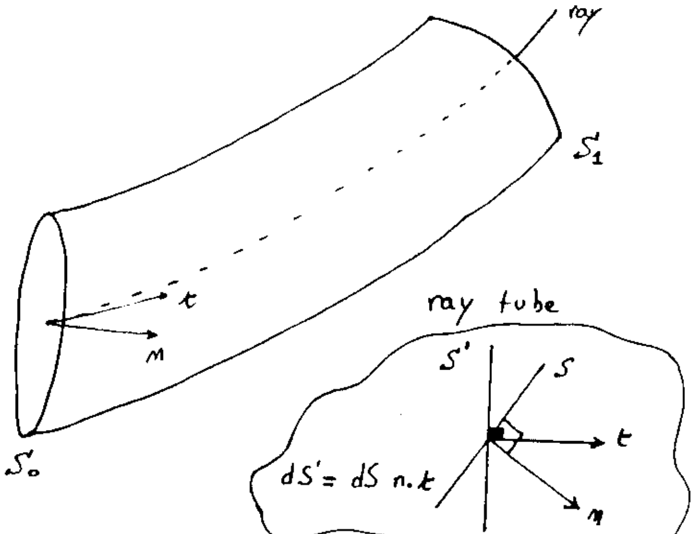


Figure D-9 : Parametrization of the surface S at the point M of the ray, as well as ray tube.



Let us consider around a ray an elementary cylindrical tube of volume V with a generatrice parallel to a segment of ray. The intersected surface by the cylindrical tube is described by two parameters γ_1 and γ_2 (figure D 9) such that the elementary surface is :

$$\mathbf{dS} = dS\mathbf{n} = \left(\frac{\partial \mathbf{x}}{\partial \gamma_1} \times \frac{\partial \mathbf{x}}{\partial \gamma_2}\right) d\gamma_1 d\gamma_2. \qquad (d\ 38)$$

The cross product is denoted by $\times$. The two elementary surfaces at curvilinear abscissa s_0 and s_1 are denoted by dS_0 et dS_1. The use of Ostrogradski theorem permits us to write

$$\int\int\int div(A_0^2 \nabla \mathbf{T}) dv = \int\int A_0^2\, \mathbf{n} \cdot \nabla \mathbf{T} dS = 0, \qquad (d\ 39)$$

which means that the flux of the field $A_0^2 \nabla \mathbf{T}$ is preserved during the propagation. Only the contribution of the two surfaces dS_0 and dS_1 exists. Knowing that $\nabla \mathbf{T}$ is parallel to $\mathbf{p}$, we can introduce two new surfaces dS'_0 et dS'_1 projections of surfaces dS_0 et dS_1 on the normal to the ray (figure D 9). We obtain the following equality

$$\frac{1}{c_0} A_0^2(s_0) dS'_0 = \frac{1}{c_1} A_0^2(s_1) dS'_1 \qquad (d\ 40)$$

where we have introduced the slowness a_0 et a_1 at positions s_0 and s_1. This equation (d 40) allows the amplitude computation at position s_1 from the amplitude at position s_0. This represents the energy flux averaged over a propagation time proportional to $1/c$ through the surface dS'. The zero-order approximation implies that the energy is preserved in the ray tube without any loss through lateral wedges. We find

$$A_0(s_1) = A_0(s_0) \sqrt{\frac{c_1 dS'}{c_0 dS'_1}}. \qquad (d\ 41)$$

The surface

$$dS' = dS\ \mathbf{n} \cdot \mathbf{t} = J\ d\gamma_1 d\gamma_2 \qquad (d\ 42)$$

is interpreted with the help of the jacobian

$$J = \begin{Vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial \gamma_1} & \frac{\partial x}{\partial \gamma_2} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial \gamma_1} & \frac{\partial y}{\partial \gamma_2} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial \gamma_1} & \frac{\partial z}{\partial \gamma_2} \end{Vmatrix} \qquad (d\ 43)$$

noted also

$$J = \left| \frac{\partial(x, y, z)}{\partial(s, \gamma_1, \gamma_2)} \right| \quad (\mathbf{t}, \gamma_1, \gamma_2). \qquad (d\ 44)$$

We obtain the generic formula for the acoustic case

$$A_0(s_1) = A_0(s_0) \sqrt{\frac{c_1 J_0}{c_0 J_1}} \qquad (d\ 45)$$

which can also be written

$$A_0(s_1) = A_0(s_0)\sqrt{\frac{u_0 J_0}{u_1 J_1}}. \qquad (d-16)$$

Often, one assume that the normal at the elementary surface is parallel to slowness vector $\mathbf{p}$ and identify surfaces S et S'. This subtle discussion has importance in designing ray tracing programs. We might be interested by surfaces S and not only S' during ray tracing. In any case, at the station point, we should go back to the surface S' in order to compute the amplitude. The use of arbitrary surfaces S is a freedom which makes easier the ray tracing with initial excitations on vibrating surface which are not wavefronts, interface projections ..., but surfaces S' determine the amplification modulation.

We are now able to describe the pressure observed at a given point coming from a source with temporal variation $S(\omega)$ and with a radiation pattern $\varphi(\gamma_1, \gamma_2)$. One can distinguish the excitation, the geometrical spreading and the propagation :

$$P(x,\omega) = S(\omega)\ \varphi(\gamma_1,\gamma_2)\ \sqrt{\frac{c}{J}}e^{i\omega T(\mathbf{x})}. \qquad (d-17)$$

How to estimate the intensity of the source φ? A possible solution is to select a particular example as an homogeneous medium of speed c_0. We look at a canonical problem in order to calibrate the high frequency solution with an already known solution. At the source itself, a singularity requires to move away at a distance of at least a wavelength. The complete high frequency Green function is given by

$$P(r,\omega) = S(\omega)\frac{1}{4\pi}\frac{1}{r}e^{i\omega T}. \qquad (d-48)$$

The jacobian is defined by $dS = J\ d\Omega$, which gives by integration on the sphere

$$4\pi R^2 = J\int d\Omega = 4\pi J. \qquad (d-49)$$

We write the solution under the asymptotic form

$$P(r,\omega) = S(\omega)\frac{1}{4\pi\sqrt{c_0}}\sqrt{\frac{c_0}{J}}e^{i\omega T}, \qquad (d-50)$$

that permits us to identify the radiation term of the isotropic and ponctual source :

$$\varphi(\gamma_1,\gamma_2) = \frac{1}{4\pi\sqrt{c_0}}. \qquad (d-51)$$

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CAUSTICS PATTERN

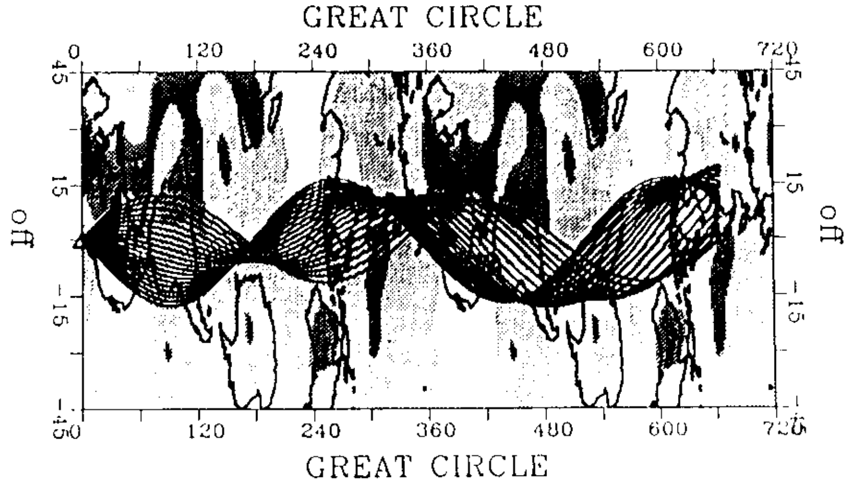


Figure D-10 : Caustics for surface-wave ray tracing on an hétérogeneous sphere : focal points are blurred out. Heterogeneous model velocity is represented on gray scale on a rotated map of the Earth where the equator is the great circle between the source and the receiver.

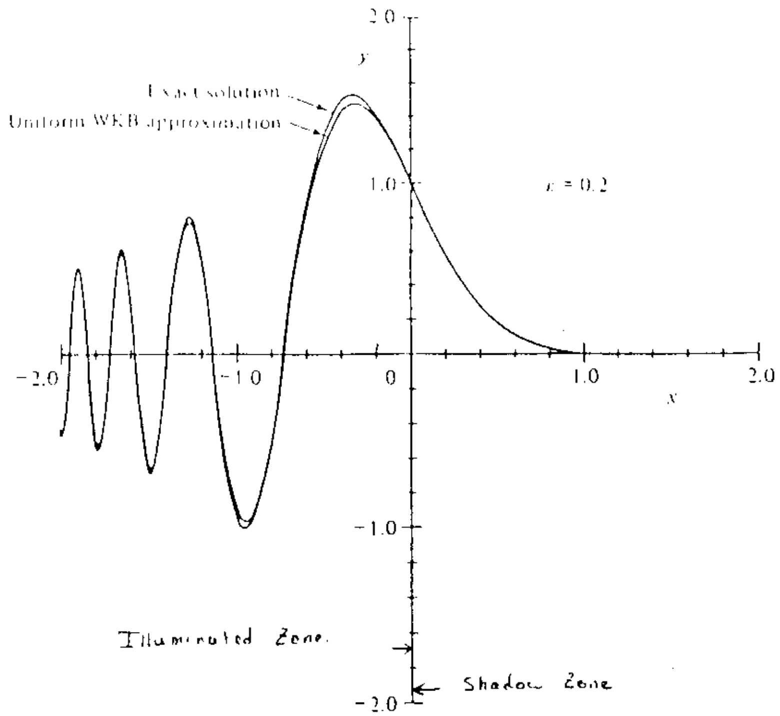


Figure 10.10 A comparison of the exact solution to $\varepsilon^2 y''(x) = \sinh x(\cosh x)^2 y(x)$ $y(+\infty) = 0]$, with the approximate solution from a one-turning-point WKB analysis. T approximate formulas are given in (10.4.14) and (10.4.15).

Figure D-11. Amplitude oscillation and decrease from different parts of the caustic

FROM 'Advanced mathematical methods for Scientists and Engineers' by C.M. BENDER and S.A. ORSZAG with a different interpretation here

We deduce the asymptotic solution for a variable velocity $c(\mathbf{x})$:

$$P(\mathbf{x},\omega) = S(\omega)\frac{1}{4\pi\sqrt{c_0}}\sqrt{\frac{c(\mathbf{x})}{J(\mathbf{x})}}e^{i\omega T(\mathbf{x})}, \qquad (d-52)$$

where c_0 is now the velocity at the source.

When the jacobian is strictly positive, the solution in the time domain can be constructed straigthforward by

$$P(\mathbf{x},t = \frac{1}{2\pi}\int P(\mathbf{x},\omega)e^{-i\omega t}\,d\omega \qquad (d-53)$$

which gives

$$P(\mathbf{x},t) = \frac{1}{4\pi}\sqrt{\frac{c(\mathbf{x})}{c_0 J(\mathbf{x})}}s(t - T(\mathbf{x})). \qquad (d-54)$$

Unfortunately, this simple expression is only valid at infinite frequency. For a finite frequency, any abrupt variation leads to different results. Moreover, a slight complication appears when the jacobian changes its sign or even when it vanishes. Situations giving this phenomenon are frequently met as shown for caustics in figure (D-10). When the jacobian goes to zero, the ray tube section degenerates into a zero section : we are on a caustic and the ray theory is no more valid because the amplitude of the signal will be infinite. Knowing the position of the caustic, it is possible to construct another asymptotic theory which takes into account the ondulatory aspect of the signal and allows a description of interferences near the caustic or, equivalently a description of the propagation depending on the frequency. In fact, on the illuminated side of the caustic, many rays go through the same point giving the oscillating aspect of the amplitude (figure D-11), while on the other side of the caustic, we find a exponential decay depending on the frequency. What we must learn is the inversion of sign of J when we go through the caustic and that the Airy theory permits us to connect the situation on each side of the caustic. We find the initial form

$$\sqrt{\frac{c}{J}} \Longrightarrow \sqrt{\frac{c}{|J|}}\,-\,i$$
$$\Longrightarrow \sqrt{\frac{c}{|J|}}e^{-i\frac{\pi}{2}} \qquad (d-55)$$

which introduces a phase shift when one goes through the caustic. The sign - is not given by the ray theory but by the Airy theory. This is the only contribution of the Airy theory considered here.

Formally we can write that phase shift in a suitable form

$$e^{-i\pi/2} = e^{-i\omega\frac{\pi}{2\omega}} \qquad (d-56)$$

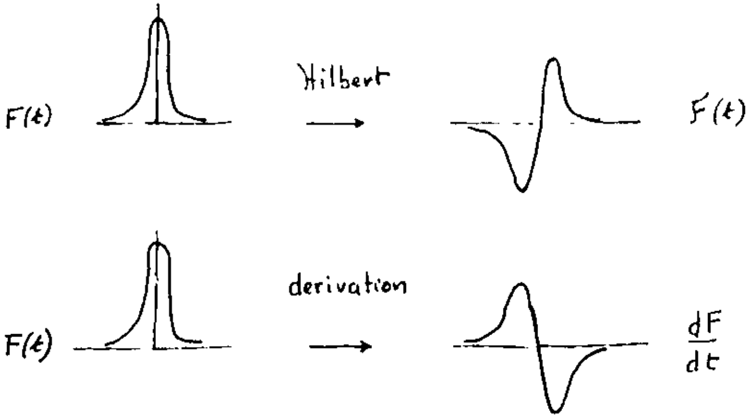


Figure D-12 : Comparison between Hilbert transformation and derivation.

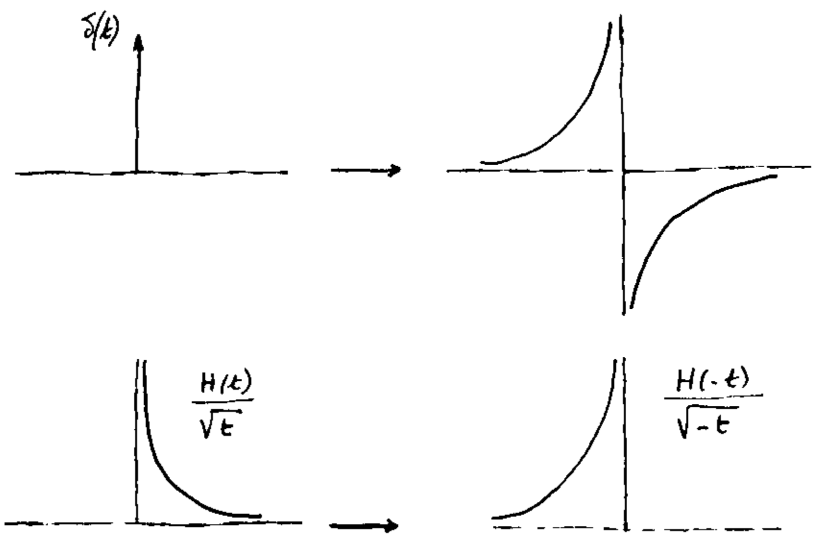


Figure D-13 : Geometrical description of two hilbert transformations.

by exhibiting a frequency dependence of the travel time : the wave arrives before expected. This phase shift induces the Hilbert transformation of the propagated signal which is written

$$P(\mathbf{x},t) = \frac{1}{4\pi}\sqrt{\frac{c(\mathbf{x})}{c_0\,|\,J(\mathbf{x})\,|}}\,\mathcal{S}(t - T(\mathbf{x})) \qquad (d-57)$$

where $\mathcal{S}$ is the Hilbert transformation of S.

Let us recall that the Hilbert transformation $\mathcal{F}(t)$ of a function $F(t)$ defined as the principal value of the following integral,

$$\frac{1}{\pi}\int \frac{F(\tau)}{(t-\tau)}d\tau, \qquad (d-58)$$

has a Fourier transformation $-i\ sgn(\omega)\ F(\omega)$. We find that a function and its Hilbert transformation have the same spectrum, that the Hilbert transformation acts in opposite phase with respect to the derivation on the signal (figure D-12). Convolution formula can be deduced :

$$\frac{d}{dt}\frac{H(t)}{\sqrt{t}} * \frac{H(t)}{t-\tau} = \pi\delta(t-\tau). \qquad (d-59)$$

Two Hilbert transformations must be noticed

$$\delta(t) \longrightarrow -\frac{1}{\pi t}$$
$$\frac{H(t)}{\sqrt{t}} \longrightarrow \frac{H(-t)}{\sqrt{-t}}, \qquad (d-60)$$

with a geometrical description given by the figure (D-13).

Of course, the sign inversion of the jacobian J might be repeated when the ray hits a new caustic. This phenomenon is often met in surface-wave ray tracing where, for geometrical reason, waves are focused on the antipode of the source, then on the source again when the sphere is homogeneous. In order to keep track of these intersections with caustics, we introduce the KMAH index (for Keller, Maslov, Arnold et Hormander) initially taken as zero and which increases by 1 for each caustic. Moreover, when the ray tube section reduces to a single point in a three dimensional medium, we count two crossing for the focal point. We have the complete expression

$$P(\mathbf{x},\omega) = S(\omega)\,\phi(\gamma_1,\gamma_2)\sqrt{\frac{c(\mathbf{x})}{|\,J(\mathbf{x})\,|}}\,e^{i\omega T(\mathbf{x})}\,e^{-i\frac{\pi}{2}\,sgn(\omega)\,KMAH}. \qquad (d-61)$$

We can write the pressure in the time domain under the general form :

$$P(\mathbf{x},t) = \Re\{\, P(\mathbf{x})\hat{S}(t - T(\mathbf{x}))\} \qquad (d-62)$$

where

$$P(\mathbf{x}) = \varphi(\gamma_1,\gamma_2)\sqrt{\frac{c(\mathbf{x})}{|J(\mathbf{x})|}}\, e^{-i\frac{\pi}{2}KMAH} \qquad (d-63)$$

and $\hat{S}$ is the analytic function associated to the function S. Practically, we only consider positive frequencies which introduce a modification of Fourier transformation of real functions which are such that the value at the frequency $-\omega$ is equal to the complex conjugate of the value at the frequency ω. We can therefore reduce the domain of integration and find

$$f(t) = \frac{1}{\pi}\Re\int_0^{\infty} f(\omega)e^{-i\omega t}\,d\omega. \qquad (d-64)$$

The following complex function

$$\hat{f}(t) = \frac{1}{\pi}\int_0^{\infty} f(\omega)e^{-i\omega t}\,d\omega \qquad (d-65)$$

has a Fourier transformation $2H(\omega)f(\omega)$. Its real part is $f(t)$. Its imaginary part F is written

$$F(t) = \frac{1}{\pi}\Re\{\int_0^{\infty} -if(\omega)e^{-i\omega t}\,d\omega\} \qquad (d-66)$$

and reduces to

$$F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} -isgn(\omega)f(\omega)e^{-i\omega t}\,d\omega \qquad (d-67)$$

which shows that F is the Hilbert transformation of the function f. Therefore, the analytic function $\hat{f}$ is the sum of the function and its Hilbert transformation. This is the justification of the use of the analytic function in the expression of the pressure when we consider only positive frequencies.

For the zero order approximation, we need now to estimate the ray trajectory, the travel-time T and the jacobian J at the point where we wish to evaluate the asymptotic solution. This task is technically difficult and we shall concentrate on it in two following sections. Before doing it, let us mention one additional point.

If we consider the next term of the serie A_1, we have the following equation :

$$2\nabla T\cdot\nabla A_1 + \nabla^2 T A_1 = -\nabla^2 A_0 \qquad (d-68)$$

or

$$\frac{div(A_1^2\nabla T)}{A_1} = -\nabla^2 A_0 \qquad (d-69)$$

which shows that fast variations of A_0 lead to amplitudes of the term A_1 often not a all negligeables. This is the case near the critical angle for refracted waves, for converted phases S near the free surface and the obtention of new solutions valid for these particular cases is a game enjoyed by many researchers.

E - EXAMPLES OF RAY TRACING

Let us first consider simple examples where tracing rays is rather straigthforward. We are looking for rays, travel time and geometrical expansion coefficient from the differential system

$$\begin{aligned}\frac{d\mathbf{x}}{ds} &= c(\mathbf{x})\,\mathbf{p}\\ \frac{d\mathbf{p}}{ds} &= \nabla(\frac{1}{c(\mathbf{x})}).\end{aligned} \qquad (e-1)$$

Homogeneous media

For an homogeneous medium $c(\mathbf{x}) = c_0$, the ray solution is simply straight segments. We find the following equations,

$$\begin{aligned}\mathbf{p}(s) &= \mathbf{p}_0\\ \mathbf{x}(s) &= \mathbf{x}_0 + c_0\,(s - s_0)\,\mathbf{p}_0,\end{aligned} \qquad (e-2)$$

where the position at s_0 is $\mathbf{x_0}$. The jacobian increases as r^2 in a three-dimensional medium and, consequently, the amplitude decreases as $1/r$. In a two dimensional medium, the amplitude exhibits a typical tail from a variation in $1/\sqrt{r}$, while the one dimensional propagation prevents any spreading and the amplitude is kept constant during the propagation.

Constant gradient of the velocity

Seismologists are often interested by a velocity structure which increases with depth. The simplest model one can think about is a linear increase with depth of the velocity : this does not mean that this leads to the simplest ray tracing. The following velocity structure

$$c(\mathbf{x}) = c_0 + Γz \qquad (e-3)$$

implies simple differential equations from equation (e-1) for the components p_x and p_y of the slowness vector : they are constant along the ray. The ray is in a vertical plane without torsion and one can assume that p_y is set to zero by an adequat selection of the coordinate system : the ray lies inside the plane (xoz). The horizontal component p_x is called the ray parameter and denoted sometimes p. Let us introduce the angle θ between the vertical and the ray (figure E-1). This angle is a good parameter for tracking the ray itself. By definition of the tangent,

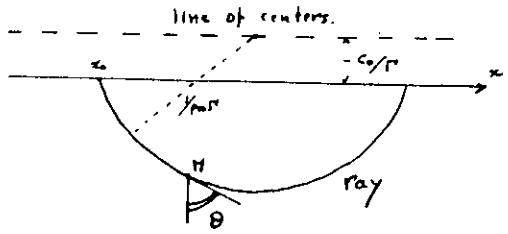


Figure E-1 : Geometry of rays in a medium with a constant gradient of velocity.

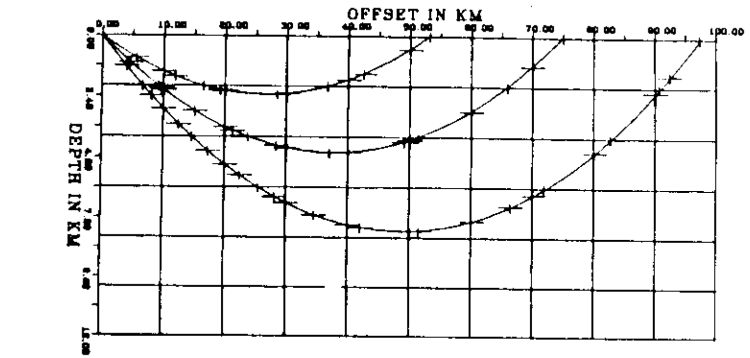


Fig. 4. A example of ray tracing in a triangular mesh. The medium has a velocity distribution, $v(z) = 3.00 + 0.02\, z\ km/s$. It is discretised using a triangular mesh, and rays are traced using our finite element technique. The continuous lines are the exact rays. The crosses indicate the points where the numerical rays intersect the sides of the elements.

Figure E-2 : Discretization in constant gradient of the square of slowness for a constant gradient of velocity : rays agree perfectly.

we have

$$p_x(s) = \frac{sin\theta(s)}{c(z)}$$
$$p_z(s) = \frac{cos\theta(s)}{c(z)}. \qquad (e-4)$$

The curvature K becomes constant through equation

$$K = \frac{1}{c(z)}\frac{dc}{dz}sin\theta = \Gamma\, p_x. \qquad (e-5)$$

which means that the ray is a portion of a circle of radius $1/\Gamma p_x$. From the geometrical definition of the curvature (figure D-6), one can deduce the evolution of the angle θ by

$$\frac{d\theta}{ds} = K = \Gamma\, p_x. \qquad (e-6)$$

With this parameter, one can integrate ray tracing equations

$$x - x_0 = \int_{x_0}^{x} dx' = \int_{\theta_0}^{\theta}\frac{dx}{ds}\frac{ds}{d\theta'}d\theta' = \frac{1}{\Gamma p_x}\int_{\theta_0}^{\theta} sin\theta' d\theta'$$
$$z - z_0 = \int_{z_0}^{z} dz' = \int_{\theta_0}^{\theta}\frac{dz}{ds}\frac{ds}{d\theta'}d\theta' = -\frac{1}{\Gamma p_x}\int_{\theta_0}^{\theta} cos\theta' d\theta' \qquad (e-7)$$

and obtains analytical trajectories starting from θ_0

$$x - x_0 = \frac{1}{\Gamma p_x}(cos\theta_0 - cos\theta)$$
$$z - z_0 = \frac{1}{\Gamma p_x}(sin\theta - sin\theta_0). \qquad (e-8)$$

With the help of the trigonometric expression $sin^2\theta + cos^2\theta = 1$, we find the equation of a circle

$$(x - x_0 - 1/\Gamma p_x cos\theta_0)^2 + (z - z_0 + 1/\Gamma p_x sin\theta_0)^2 = (1/\Gamma p_x)^2. \qquad (e-9)$$

If we assume x_0, z_0 (related to θ_0) as the system origin, one can find the simplified equation

$$(x - 1/\Gamma p_x cos\theta_0)^2 + (z + c_0/\Gamma)^2 = (1/\Gamma p_x)^2 \qquad (e-10)$$

which shows that the circle centers of rays for different shooting angles belongs to a straigth line $z + c_0/\Gamma = 0$ (figure E-1). The travel-time T is deduced by direct integration

$$T - t_0 = \int_{s_0}^{s} u ds' = \int_{\theta_0}^{\theta}\frac{1}{c}\frac{ds}{d\theta'}d\theta' = \frac{1}{\Gamma}\int_{\theta_0}^{\theta}\frac{1}{sin\theta'}d\theta' \qquad (e-11)$$

which gives the final analytical expression

$$T = T_0 + \frac{1}{\Gamma} Log[\frac{tan(\theta/2)}{tan(\theta_0/2)}].$$

For the geometrical spreading evaluation, one must express the coordinate x with respect to the initial angle at constant depth z through

$$cos\theta = \sqrt{1 - \frac{c^2}{c_0^2} sin^2\theta_0} \qquad (e\ 13)$$

which gives

$$x - x_0 = \frac{1}{\Gamma p_x}(cos\theta_0 - \sqrt{1 - \frac{c^2}{c_0^2} sin^2\theta_0}). \qquad (e\ 14)$$

The estimation of the jacobian

$$J(\theta) = |\frac{\partial x}{\partial \theta_0}|\ cos\theta \qquad (e\ 15)$$

allows the analytical computation of the amplitude for this simple velocity distribution.

Constant gradient of the square of the slowness

Another example which gives the simplest solution for ray tracing is a constant gradient of the square of slowness. The square of the slowness, related to the square of the refraction index widely used in scattering theory, has been disregarded by seismologists because this parametrization leads to a decrease of the velocity rather an increase with the depth. The simplicity of the solution might overcome the penalty of a dense discretization of the velocity structure in many elements (Figure E-2).

The velocity structure is defined by the square of the slowness

$$u^2 = u_0^2 + \gamma \cdot \mathbf{x} \qquad (e\ 16)$$

with a vectorial representation. The use of vectorial notation shows the separability property of the square of slowness between what happens along the x axis and along the z axis. This advantage avoids any rotation in order to align the gradient towards only one direction, rotation required for the previous example of the constant gradient of the velocity. Associated to this distribution, a new parametrization τ of the ray defined as $ds = ud\tau$ allows the simplification of the differential system (e 1) into

$$\frac{d\mathbf{x}}{d\tau} = \mathbf{p}$$
$$\frac{d\mathbf{p}}{d\tau} = \frac{1}{c(\mathbf{x})}\nabla_{\mathbf{x}}(\frac{1}{c(\mathbf{x})}) = \frac{1}{2}\nabla_{\mathbf{x}} u^2(\mathbf{x}) = \gamma. \qquad (e\ 17)$$

The analytical ray is a parabole given by

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{p}_0(\tau - \tau_0) + \frac{1}{4}\gamma(\tau^2 - \tau_0^2)$$
$$\mathbf{p} = \mathbf{p}_0 + \frac{1}{2}\gamma(\tau - \tau_0), \qquad (e\ 18)$$

while the travel time T is obtained by direct integration

$$T = T_0 + \int u^2 d\tau = t_0 + \int (u_0^2 + \gamma \cdot \mathbf{x}) d\tau$$
$$T = T_0 + (u_0^2 + \gamma \cdot \mathbf{x})(\tau - \tau_0) + . \qquad (e\ 19)$$

The jacobian is estimated at constant τ by

$$J = \mathbf{n} \cdot \frac{\partial \mathbf{x}}{\partial \theta_0} \qquad (e\ 20)$$

which becomes an expression increasing with τ

$$J = (\tau - \tau_0) p_0. \qquad (e\ 21)$$

Many other velocity distributions lead to analytical expressions and our purpose is not to investigate all of them. Let us just underline that the distribution of the velocity in $ch(z)$ relates ray tracing on a homogeneous sphere to ray tracing in a two-dimensional medium for this particular distribution. Even more, using a transformation based on the hyperbolic cosine, the body wave ray tracing programs can be adapted for surface-wave ray tracing. This distribution has also been studied intensively in fiber optics as an approximation to quadratic evolution of the refraction index away from the center of the optical fiber, the hyperbolic cosine giving analytical solutions.

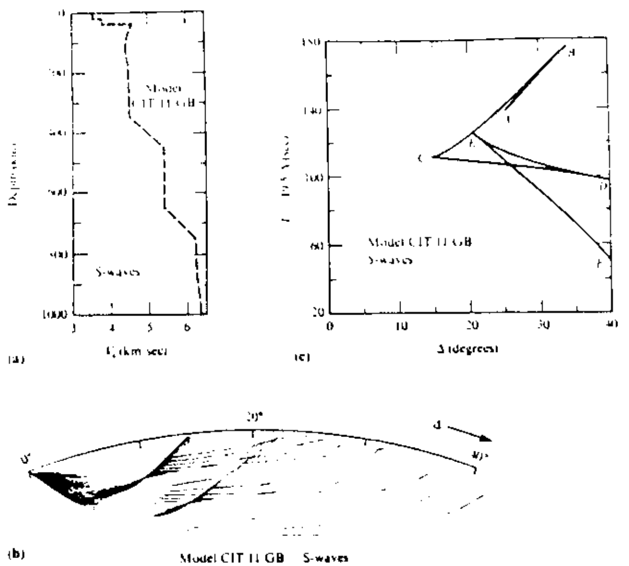


FIGURE 9.12

(a) The S-wave velocity for the upper mantle, taken from model CIT 11 GB. (b) Corresponding S-wave rays for a point source at the surface, calculated for take-off angles increasing from 28° to 50° in ½° increments. Note that distance between source and receiver in the Earth is measured by the angle Δ subtended at the Earth's center. (c) Corresponding reduced tr... -time curve. Point C is clearly identified with strong focusing of rays in (b) at Δ near ..., and amplitudes there will be large. Lines AB, BC, and CD together constitute a *triplication*, and each of the two triplications shown is associated with a major velocity increase (with depth) in the Earth model. [After Julian and Anderson, 1968.]

Figure F-1 : Depth-dependent velocity structure inside the Earth for S waves.

F - Arbitrary variation along one direction

Arbitrary variation along one direction is a natural extension from previous simple examples and has concentrated the attention of seismologists because the variation of the velocity structure inside the Earth is mainly along the depth direction (figure F-1). The slowness component $p_\perp$ perpendicular to this direction taken as z is constant because the velocity does not vary along this perpendicular direction :

$$\frac{dp_\perp}{ds} = \nabla_\perp\left(\frac{1}{c}\right) = 0. \qquad (f-1)$$

The ray lies inside a vertical plane that one can define as the plane (xoz). This argument follows the same line as the one presented for a constant gradient of the velocity and shows its generality. The differential system (e-1) reduces to the following system

$$\begin{aligned} \frac{dx}{ds} &= c(z)p_x \\ \frac{dz}{ds} &= c(z)p_z \\ \frac{dp_x}{ds} &= 0 \\ \frac{dp_z}{ds} &= \frac{d}{dz}\frac{1}{c(z)}. \end{aligned} \qquad (f-2)$$

Instead of using the curvilinear abscissa s as the parameter along the ray, one can use the coordinate z which controls the velocity variation. Because p_x is constant, the eikonal allows to deduce p_z from p_x. The evolution of x and p_x determines perfectly the ray and one has to solve an one-dimensional problem : this is the Landau reduction widely applied in physics. Because p_x is constant, we go down to a single equation

$$\frac{dx}{dz} = \frac{p_x}{p_z} = \frac{p_x}{\pm\sqrt{u^2(z) - p_x^2}} \qquad (f-3)$$

with the implicit equation $dp_x/dz = 0$ as associated equation. The unknown sign is deduced from initial conditions. We obtain for a ray pointing initially downwards $(z > 0)$ the following expression of the position X :

$$X(z, p_x) = X_0 + \int_{z_0}^{z} \frac{p_x}{\sqrt{u^2(z) - p_x^2}}\,dz \qquad (f-4)$$

At the depth z_p where the slowness parameter taken as the horizontal component of the slowness vector verifies $p_x^2 = u^2(z_p)$, the ray has a turning point and the

FROM "Quantitative Seismology" by Aki and Richards (1980).

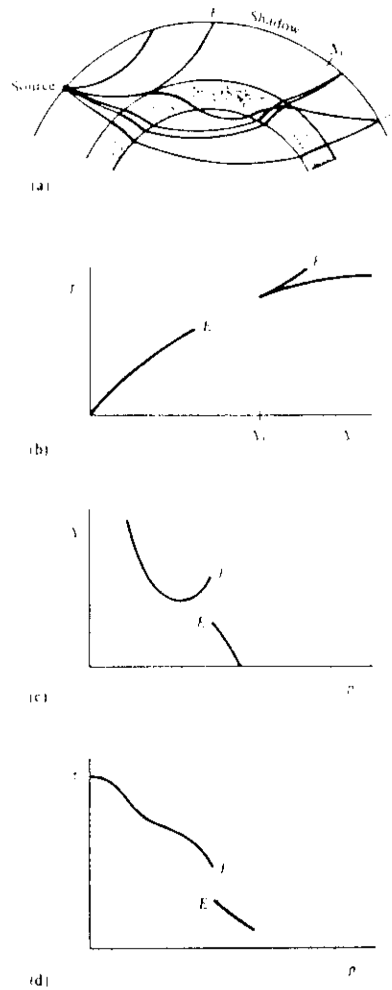


FIGURE 9.14

The behavior of T, X and τ for a velocity decrease with depth. (a) A low-velocity zone (within which dr/dz ... and there are no turning points) is shown shaded and a shadow within which no rays are received is observed at the surface. (b) The travel-time curve. The upper boundary of the low-velocity zone is the turning point for the ray emerging at point E. Point F has the same ray parameter but lies on a ray going through the low velocity zone itself. As ray parameter decreases slightly from its value at F, distance X decreases until a caustic is reached at X_c. (c) The values of $\Delta = \Delta(p)$. These show that the further boundary of the shadow is in fact a caustic. (d) Upper and lower boundaries of the low-velocity zone are turning points for rays that differ infinitesimally in their ray parameter. The turning-point radius is therefore a discontinuous function of p. This is also a discontinuity in $\tau(p) = T - p\Delta$ and in the gradient $d\tau/dp = -\Delta(p)$.

Figure F-2: Interpretation of travel-time T, station position Δ<->X and intersection time τ<->𝒯

integral (f-1) has a integrable singularity as long as du^2/dz does not vanish at this particular point. We end up with the final expression

$$X(z,p_x) = X_0 + \int_{z_0}^{z_p} \frac{p_x}{\sqrt{u^2(z) - p_x^2}} dz + \int_{z}^{z_p} \frac{p_x}{\sqrt{u^2(z) - p_x^2}} dz \qquad (f-5)$$

From the travel time expression $dT = u ds$, one deduce

$$\frac{dT}{dz} = \frac{1}{c^2(z) p_z} = \frac{u^2(z)}{\pm\sqrt{u^2(z) - p_x^2}} \qquad (f-6)$$

and the known expression

$$T(z,p_x) = t_0 + \int_{z_0}^{z_p} \frac{u^2(z)}{\sqrt{u^2(z) - p_x^2}} dz + \int_{z}^{z_p} \frac{u^2(z)}{\sqrt{u^2(z) - p_x^2}} dz. \qquad (f-7)$$

This expression is the basic formula for recovering the vertical velocity structure from travel time data and has been used in many inverse problem formulations. Combining expressions of the offset X and the travel time allows to cancel the singularity under the integral expression. The linear expression

$$\mathcal{T} = T - p_x X = T_0 - p_x x_0 + \int_{z_0}^{z} \sqrt{u^2(z) - p_x^2} dz \qquad (f-8)$$

is often called the intersection time from the graphical interpretation given by figure (F-2). This quantity taken as a function of p_x is a monotone decreased function which was not the case for the offset x because

$$\frac{d\mathcal{T}}{dp_x} = -X \qquad (f-9)$$

This intersection time appears naturally in analytical inverse formulations of the travel time and the new function

$$J(X_0, p_x) = \mathcal{T} + X_0 p_x \qquad (f-10)$$

has a derivative equal to zero for the point X_0. Rays rays arriving at point X_0 are the extrema of the function $J(X_0, p_x)$ (figure F-3).

When the source and the station are at the free surface of the Earth, we find basic formula from equations (f-4) and (f-7)

$$R(p) = 2\int_0^{z_p} \frac{p_x}{\sqrt{u^2(z) - p_x^2}}$$
$$T(p) = 2\int_0^{z_p} \frac{u^2(z)}{\sqrt{u^2(z) - p_x^2}} \qquad (f-10)$$

which have a great importance for the Earth structure. The extension in spherical coordinates does not change basic results. From an experimental curve $X(p_x)$ one can integrate equation (f-9) and construct the function $\mathcal{T}$ as a function of p_x which results in an unfolding of the triplication (figure F-4).

FROM "Quantitative Seismology" by Aki and Richards (1980)

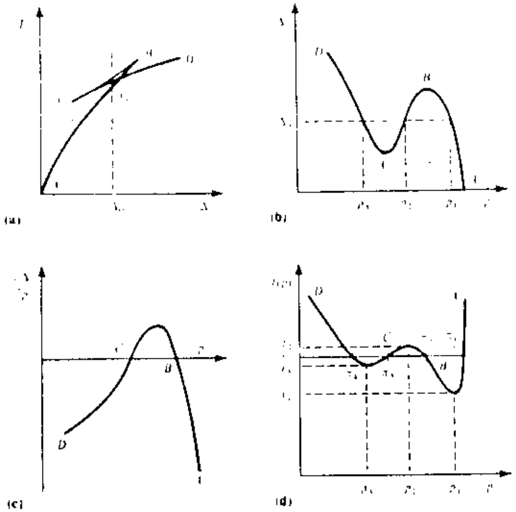


FIGURE 9.21

A travel-time curve with caustics at B and C and derived curves. Geometrical ray quantities are indicated for a distance Δ_0 in the triplication. (a) Travel-time curve with arrivals at t_1, t_2, t_3. (b) $\Delta = \Delta(p)$, with three solutions p_1, p_2, p_3 to $\Delta(p) = \Delta_0$. (c) Zeros in $\partial\Delta/\partial p$ occur at B and C. (d) The construction of solutions $\pi_i(t)$ to $t = J(p)$ is shown for a particular distance Δ_0. [After Chapman, 1976b; copyrighted by the American Geophysical Union.]

F-3: Analysis of a triplication with the function $J(X_0, p_x)$.

FROM "A simple method for the computation of body-wave seismograms" by S.K. DEY-SARKAR and C.H. CHAPMAN, BSSA, 1577-1593, 1978.

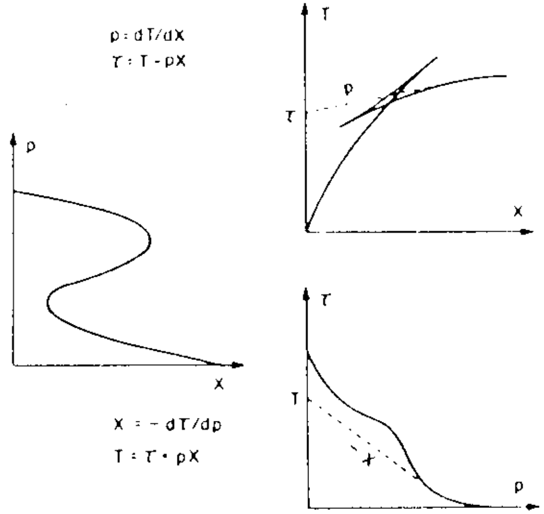


FIG. 1. A diagram of a typical travel-time curve with a triplication. The T versus X, p versus X, and τ versus p curves are shown.

Figure F-4: Unfolding the triplication by using function $T \Leftrightarrow \tau$. The function τ is a monotonic function of $p_x \Leftrightarrow p$.

G - Variational Approach

Another way to define the ray is based on variational arguments. The ray between a source and a receiver is the trajectory with the minimum travel time. In fact, one can state more precisely the Fermat principle for which the ray is the trajectory of travel time extrema. This approach is very interesting because it seems more adapted to the seismological problem where the ray must connect the source and the receiver while the problem solved previously is a problem with initial conditions : the ray leaves the source with a preselected direction.

Variational calculus demonstrates that, if the function f is such that the following integral

$$\int_1^2 f(\nu, \mathbf{x}, \dot{\mathbf{x}}) dt \qquad (g-1)$$

is extremum, the function verifies the local differential equation, called Euler equation,

$$\nabla_{\mathbf{x}} f - \frac{d}{d\nu} \nabla_{\dot{\mathbf{x}}} f = 0. \qquad (g-2)$$

Let us apply this variational principle to the travel time T which is an extremal function as stated by Fermat principle and which is defined as an integral

$$T(\xi, \mathbf{x}, \dot{\mathbf{x}}) = \int_1^2 u(\mathbf{x}) ds = \int_1^2 u[\mathbf{x}(\xi)] \|\dot{\mathbf{x}}\| d\xi \qquad (g-3)$$

where ξ is an independent parameter defined by $\|\dot{\mathbf{x}}\| d\xi = ds$. The curvilinear abscissa s is not an independent parameter because it is related to the total length L of the ray by

$$\int_1^2 ds = L.$$

The local differential equation comes from expression (g-2)

$$\|\dot{\mathbf{x}}\| \nabla_{\mathbf{x}} u = \frac{d}{d\xi} [(u(\mathbf{x}) \frac{\dot{\mathbf{x}}(\xi)}{\|\dot{\mathbf{x}}(\xi)\|}]. \qquad (g-4)$$

and might be converted into the curvature equation

$$\nabla_x u = \frac{d}{ds}(u \frac{dx}{ds})$$

by eliminating the variable ξ. The equivalence between ray tracing equations (local equations) and Fermat principle (global approach) is demonstrated. One

can exploit the lagrangian formalism in more details. The function f is often called Lagrangian $\mathcal{L}$ and is given by

$$\mathcal{L} = u[x(\xi)] \|\dot{\mathbf{x}}\| \qquad (g-5)$$

for the specified sampling parameter ξ. Other definitions are possible and might be more adapted to ray theory than the previous straigthforward definition. The lagrangian, whatever it is, is often split in two terms which separate the dependence in $\mathbf{x}$ and $\dot{\mathbf{x}}$ with the formula

$$\mathcal{L} = E_c(\nu, \dot{\mathbf{x}}) - E_p(\nu, \mathbf{x}) \qquad (g-6)$$

and the local equation becomes

$$-\frac{d}{d\nu} \nabla_{\dot{\mathbf{x}}} E_c = \nabla_x E_p \qquad (g-7)$$

In order to apply this separation to equation (g-4), we need to avoid the term $\|\dot{\mathbf{x}}\|$ on the left hand side and the term depending in $\mathbf{x}$ on the right end side. The simplest way is to absorb $u(\mathbf{x})$ in our choice of the sampling parameter which becomes $ds = u\, d\tau$ and gives the simple equation

$$-\frac{d\dot{\mathbf{x}}}{d\tau} = -u(\mathbf{x}) \nabla_{\mathbf{x}} u(\mathbf{x}) \qquad (g-8)$$

where $\dot{\mathbf{x}}$ stands now for $d\mathbf{x}/d\tau$. The expressions for E_c and E_p are the following

$$E_c = \frac{1}{2} \|\dot{\mathbf{x}}\|^2$$
$$E_p = -\frac{1}{2} u^2(\mathbf{x}) \qquad (g-9)$$

Of course, the choice of these functions is not unique and we are guided by the mechanical analogy of kinetic energy for E_c and potential energy E_p. We introduce the quantity $\mathbf{p} = \dot{\mathbf{x}}$ as an independent variable and we switch from the lagrangian formalism towards the hamiltonian formalism with this new variable. The Hamiltonian $\mathcal{H}$ is deduced from the Lagrangiana $\mathcal{L}$ by

$$\mathcal{H} = p\dot{\mathbf{x}} - \mathcal{L}(\mathbf{x}, \mathbf{p}) = \frac{1}{2}[p^2 - u^2(\mathbf{x})]. \qquad (g-10)$$

To this Hamiltonian is associated differential equations on which we concentrate our attention in a next section

H - Validity of the ray theory

Before going in more details in the ray tracing equations for arbitrary heterogeneous medium, let us go back to the high frequency approximation and discuss the domain of validity of this theory and when one expect it to break-down.

The high frequency approximation means that the wavelength λ of the propagating signal is such that is lower than any spatial scales L associated to heterogeneities. One can think about velocity variations or interface curvatures. Often the wavelength is comparable to the spatial scale. When irregularities exist, the high frequency approximation requires to be at a distance over a wavelength in order to avoid any interferences. Morevoer, the total distance D traveled by wavefronts should be significantly lower than L^2/λ. This condition prevents the ray theory to be applicable strictly speaking to surface-wave train R4 at period 165 sec, for example.

When these conditions are fulfilled, the source wavelet propagates without distorsion and only the amplitude is modulated. Unfortunately, irregularities are found and the ray theory fails. These failures can be classified in order to detect them and, when possible, to palliate them.

Classification

It is useful to describe standard situations we possibly meet during ray tracing : they are canonical ray problems already described in many papers.

(1) *Normal and turning rays*

This is the simplest case where the medium contains no discontinuities and the ray paths as well as amplitude variation vary smoothly (figure H-1). Ray theory in this case works fine.

(2) *Reversed rays*

If rays cross, then the amplitude becomes infinite and ray theory breaks down (figure H-2). Special methods are needed in the vicinity of this singularity called caustic. Nethertheless, ray theory can be used before and after the caustic provided we take care of the amplitude correctly as we already mention it.

(3) *Reflected and transmitted rays*

When an interface has a smooth variation and the reflection/transmission coefficient varies gently, ray theory works for reflected/transmitted rays. Amplitudes are modified by reflection/transmission coefficients and by the interface curvature (figure H-3). Total reflections are possible with complex coefficients.

These previously mentioned rays can be described by ray theory while the

FROM "Ray theory and its extension - WKBJ and Maslov seismograms" by C.H. CHAPMAN, Erice, 1984.

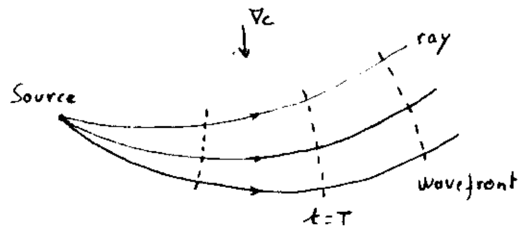


Figure H-1: Normal turning rays with wavefronts shown in dashed line.

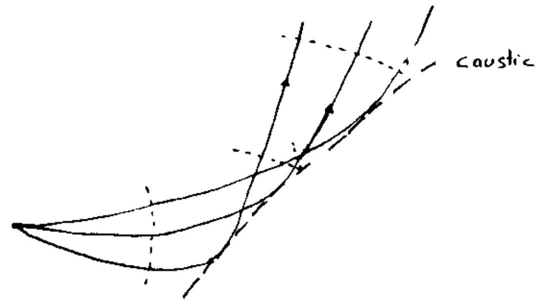


Figure H-2: Reversed rays after touching a caustic.

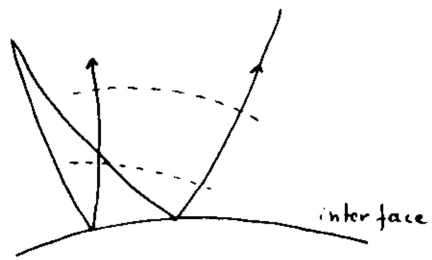


Figure H-3: Rays reflected from a curved interface.

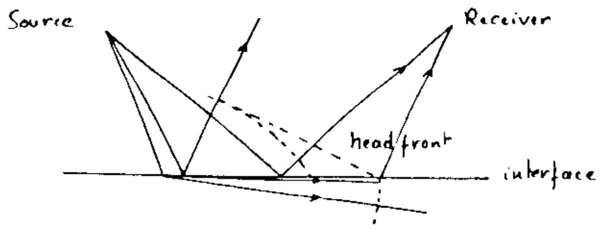


Figure H-4: A critical reflection and transmission, total reflection and head wave.

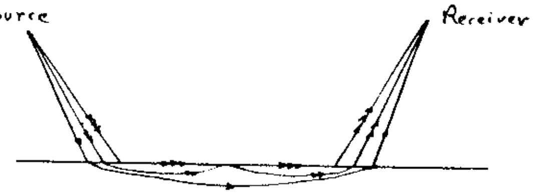


Figure H-5: Rays contributing to an interference head wave. Only one multiple refraction is shown and no attempt is made to include wavefronts.

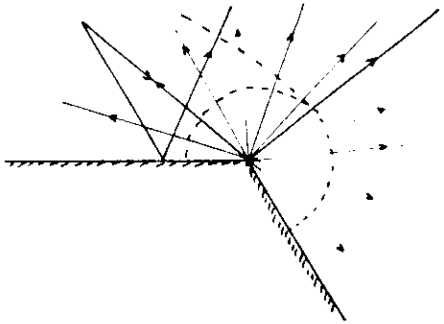


Figure H-6: Reflection and diffractions from a corner.

FROM "Ray theory and its extension - WKBJ and Maslov seismograms"

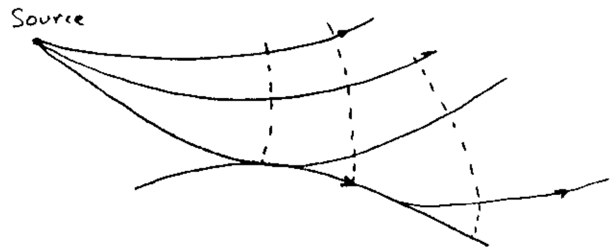


Figure H-7: Rays grazing on interface and diffracting along it.

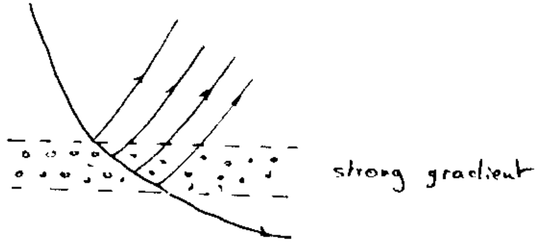


Figure H-8: A ray propagating through a region of high gradient and being reflected continuously from it.

FROM "Ray theory and its extension . WKBJ and Maslov seismograms" by C.H. CHAPMAN, Erice, 1984.

following rays are not handled by the ray theory in its simple form.
(4) *Critical rays and head waves*
At critical angle, reflection coefficient has a square root singularity and the grazing transmitted ray has a zero geometrical amplitude. The discontinuity in the reflected wavefront and the transmitted wavefront are connected by another wavefront, the head wave (figure II-4). Simple ray theory does not describe the critical region and the head wave. More elaborated high frequency theories are required.
(5) *Interference head waves*
A simple head wave rarely exists. Invariably, a velocity gradient or the curvature of the interface cause a turning ray with similar travel-time and creates interferences (figure II-5). Taking care of this interference will requires careful counting of rays by the theory. More appropriate approximations are required for the lower medium or for the interface interaction.
(6) *Airy caustics*
In the vicinity of the caustic, there is interference between normal rays and reversed rays (figure II-2). Taking care of this interference requires more elaborated ansatz as the Airy function in the frequency domain.
(7) *Edge and point diffractions*
If an interface is discontinuous, diffracted signals are generated at the corner (figure II-6). Geometrical diffraction theory can be used to model diffraction by the corner with frequency-dependent diffraction coefficient obtained by a local canonical problem.
(8) *Interface diffractions*
If a ray grazes an interface (figure II-7), a discontinuity in the wavefront is generated and an interface wave described the decay in the created shadow. It is necessary to solve boundary conditions for a grazing ray and an interface. The amplitude and the travel-time of the signal are frequency-dependent.
(10) *Gradient coupling*
When a strong gradient of velocity is present, reflected and transmitted waves are observed when the wavelength of the source signal is noticeable compared to the thickness of the area with strong gradient (figure II-8). Iterative methods handle this problem.

I - Ray tracing in heterogeneous media

Tracing rays requires to solve a non linear ordinary differential equation or a system of non-linear ordinary differential equations depending on the number of variables we consider. This problem is a rather simple one if we assume initial conditions compared to the problem with boundary conditions. Unfortunately, this is the second one which is the one we face in geophysics because we need rays arriving at stations.

Many formulations exist for the initial value problem : I select an approach based on hamiltonian formulation. The hamiltonian approach is not strictly necessary in this lecture but its relative simplicity and its elegance justify in itself this introduction. The power of this formulation is still a research investigation and I shall not mention the different alternatives on can think about.

The general hamiltonian we consider is related to the eikonal equation and is an extension to the one found in equation g-10. We consider the hamiltonian

$$H(\xi, \mathbf{x}, \mathbf{p}) = f(\mathbf{x})[p^2 - u^2(\mathbf{x})] \qquad (i-1)$$

which is equal to zero from the eikonal equation (d-16). The choice of the variable ξ, which is a sampling parameter, is related to the choice of the hamiltonian and, for example, the particular hamiltonian

$$H(\tau, \mathbf{x}, \mathbf{p}) = \frac{1}{2}[p^2 - u^2(\mathbf{x})] \qquad (i-2)$$

is related to the parameter τ defined by $ds = u(\mathbf{x})d\tau$. Whatever is the selected hamiltonian, characteristic equations are associated and have the universal form :

$$d\xi = \frac{d\mathbf{x}}{\nabla_{\mathbf{p}} H} = \frac{d\mathbf{p}}{-\nabla_{\mathbf{x}} H}. \qquad (i-3)$$

We call these equations canonical equations for their generality. The ray is defined by its canonical vector,

$$y(\xi) = \begin{pmatrix} \mathbf{x}(\xi) \\ \mathbf{p}(\xi) \end{pmatrix}, \qquad (i-4)$$

which verifies the following equations for the particular hamiltonian we have selected

$$\begin{aligned} \frac{d\mathbf{x}}{d\tau} &= \nabla_{\mathbf{p}} H = \mathbf{p} \\ \frac{d\mathbf{p}}{d\tau} &= -\nabla_{\mathbf{x}} H = \frac{1}{2}\nabla_{\mathbf{x}} u^2(\mathbf{x}) \end{aligned} \qquad (i-5)$$

The first equation is the definition of slowness vector if one remember the relation between z and s. The second equation is the curvature equation. This hamiltonian is the first one can think about starting from ray tracing equations. From the hamiltonian formulation we can argue that other choices fo hamiltonians lead to other differential equations but to the same trajectories which are rays ; we have the freedom to select the most suitable hamiltonian for the problem at hand.

By opposition to the lagrangian formulation, we have extented the space of variables to what is called phase space with 6 variables in a three-dimensional medium. Because the slowness vector is a very important quantity for rays, this extension is of practical interest for us.

The travel time is connected to the hamiltonian by a subtle relation which is not describe here. Let us mention only the relation needed to compute it in the general case

$$d\xi = \frac{dT}{\mathbf{p}\cdot\nabla_{\mathbf{p}} H}. \qquad (i-6)$$

which becomes for our particular hamiltonian

$$d\tau = \frac{dT}{p^2} = \frac{dT}{u^2}, \qquad (i-7)$$

an already mentioned equation.

The number of free parameters in the three-dimensional space is not the number of variables of the phase space, i.e. 6 variables. The eikonal equation reduces to five this number while the implicit parameter ξ reduces to four the number of degrees of freedom. One can think about inverting both equations of (i-4) and equating the variable ξ of these inverted equations which gives this extra condition for going down to four independent variables.

This reduction of variables could be exploited in the phase space itself in order to solve differential equations for the minimum of variables. Following this idea, one can introduce an hamiltonian from the previous one which has not the eikonal equation in it. A simple way is to select one cartesian variable as the sampling parameter and to write the new hamiltonian from the eikonal equation

$$H(z,x,y,p_x,p_y) = -p_z = -\sqrt{u^2(x,y,z) - p_x^2 - p_y^2} \qquad (i-8)$$

with equations

$$\begin{aligned}
\frac{dx}{dz} &= \frac{\partial H}{\partial p_x} = \frac{p_x}{\sqrt{u^2 - p_x^2 - p_y^2}} \\
\frac{dy}{dz} &= \frac{\partial H}{\partial p_y} = \frac{p_y}{\sqrt{u^2 - p_x^2 - p_y^2}} \\
\frac{dp_x}{dz} &= -\frac{\partial H}{\partial x} = -\frac{1}{2}\frac{1}{\sqrt{u^2 - p_x^2 - p_y^2}}\frac{\partial u^2}{\partial x} \\
\frac{dp_y}{dz} &= -\frac{\partial H}{\partial y} = -\frac{1}{2}\frac{1}{\sqrt{u^2 - p_x^2 - p_y^2}}\frac{\partial u^2}{\partial y}.
\end{aligned} \qquad (i-9)$$

If the medium is only dependent in z, we find again the results of section F. Unfortunately, one cartesian coordinate is emphasized compared to the two others and in a general heterogeneity, one would like to avoid the a-priori of selecting one coordinate. In order to do so, a new coordinate system must be introduced which is related to the sampling parameter s. This is the so-called centered ray coordinates system which is a curvilinear system. It has been proposed by the russian school and developped in Prague by different researchers. This system is interesting except in the complexity of the hamiltonian. Because it follows the ray, no difficulty is expected compared to the reduced hamiltonian in z. The hamiltonian is no more constant along the ray. It must be underlined that the opposite procedure is performed in physics : the hamiltonian for non isolated system varies. The associated phase space is embedded inside another one where the system is isolated and the hamiltonian constant. The reason for this transformation is the expected simplicity of the extended hamiltonian. We propose the same argument for using the complete hamiltonian for rays.

Once we have defined the non-linear system to solve, we have to look for very efficient solvers because the number of rays we often need is the order of thousands and the knowledge of the medium we have is not so accurate that we need very precise and stable solvers. Moreover, the variation of velocity we find in the Earth is not such that strange attractors are met with problems of stiffness of the differential equations.

Usual solvers as Runge Kutta or predictor-corrector schemes are suitable for tracing rays. Adaptative steps might increase the efficiency of the solver especially where the medium is homogeneous. We refer the reader to any numerical book where the details of these numerical schemes are explained. We find that a second order runge kutta is easy to program and give accurate results for most purposes in ray tracing.

Another alternative is a finite element method where the medium is described by elementary cells with a simple velocity structure inside each bloc in order to solve analytically the equations. The task is to compute intersection points between blocs which are an order of magnitude le than for numerical solvers. Of course, the difficulty is not solved and goes to the description in blocs of the medium. Often, the description in blocs of the medium is so crude that instabilities in amplitude estimation for a given ray are found.

Finally, perturbation methods use the solution of the medium divided by blocs and construct an approximate solution which is expected to be near the one obtained by numerical methods. This intermediate solution removes instabilities in amplitude. This is a current research line and we shall see in the near future whether or not these techniques will be more efficient than a dumb numerical solver.

J - Paraxial theory or linearization

Tracing rays is not enough : we need to estimate the amplitude in order to compute seismogram and also for many other aims as two point ray tracings, caustic detections... The equation we need to solve is the transport equation. In fact, this equation is related to the ray tube which is defined by rays. Solving the transport equation requires tracing rays in the vicinity of a given selected ray which drives us to ray tracing equations again. Solving directly transport equation is avoided.

Until recently, the ray tube was estimated by tracing a nearby ray independent of the considered ray. It was a strong weakness of the ray theory because any small perturbation seen by the nearby ray and not by the true ray induced instable behaviour of the amplitude estimation (figure J-1).

Paraxial rays

By a relatively distorded approach linked to gaussian beams, seismologists go back to elementary approach of perturbation theory which computes an infinitesimal nearby ray knowing an already traced ray. This is the usual linearization approach, called paraxial theory, where the reference axis is the studied ray. The linearization stabilizes the amplitude estimation because the nearby ray is only deviated by perturbations seen by the reference ray. Moreover, the linearization makes the computation very fast.

Let us assume that the nearby ray is defined by a position and a slowness vector

$$\begin{aligned} \mathbf{x}(\xi) &= \mathbf{x_0}(\xi) + \delta\mathbf{x}(\xi) \\ \mathbf{p}(\xi) &= \mathbf{p_0}(\xi) + \delta\mathbf{p}(\xi) \end{aligned} \qquad (J-1)$$

The reference ray is located by $\mathbf{x_0}$ and $\mathbf{p_0}$ at parameter ξ such that

$$\begin{aligned} \frac{d\mathbf{x_0}}{d\xi} &= \nabla_\mathbf{p} H(\xi, \mathbf{x_0}, \mathbf{p_0}) \\ \frac{d\mathbf{p_0}}{d\xi} &= -\nabla_\mathbf{x} H(\xi, \mathbf{x_0}, \mathbf{p_0}) \end{aligned} \qquad (J-2)$$

The following equations which must be verified by the nearby ray

$$\begin{aligned} \frac{d\mathbf{x}}{d\xi} &= \nabla_\mathbf{p} H(\xi, \mathbf{x}, \mathbf{p}) \\ \frac{d\mathbf{p}}{d\xi} &= -\nabla_\mathbf{x} H(\xi, \mathbf{x}, \mathbf{p}) \end{aligned} \qquad (J-3)$$

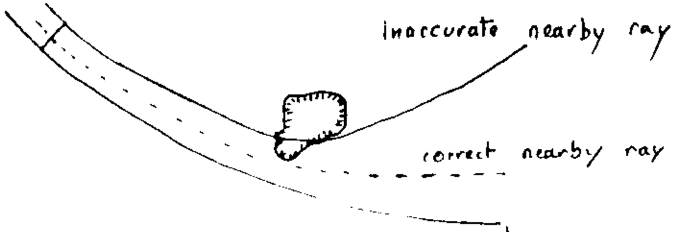


Figure J-1 : Estimation of the ray tubeby nearby rays.

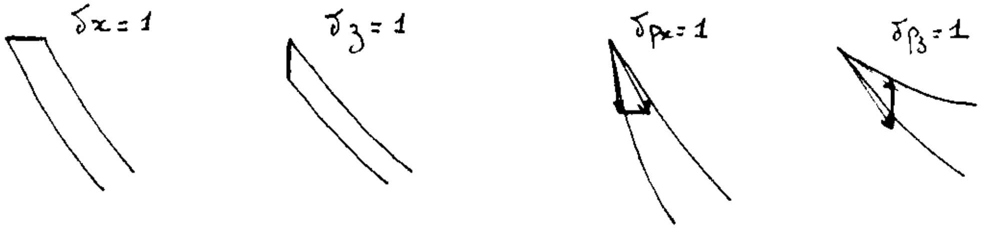


Figure J-2 : Geometrical illustration of elementary trajectories for a 2-D medium : they are not rays!

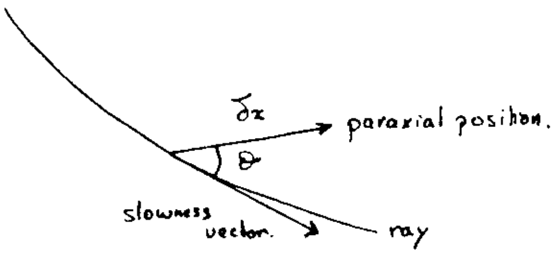


Figure J-3 : Geometry of paraxial position in order to estimate the jacobian and the amplitude.

are perturbated to first-order giving for the first equation following equalities

$$\begin{aligned}\frac{d\mathbf{x}_0}{d\xi} + \frac{d\delta\mathbf{x}}{d\xi} &= \nabla_{p_0+\delta p} H(\xi, \mathbf{x}_0 + \delta\mathbf{x}, \mathbf{p}_0 + \delta\mathbf{p}) \\ \frac{d\mathbf{x}_0}{d\xi} + \frac{d\delta\mathbf{x}}{d\xi} &= \nabla_{p_0} H(\xi, \mathbf{x}_0, \mathbf{p}_0) + \nabla_{p_0}\nabla_{p_0} H(\xi, \mathbf{x}_0, \mathbf{p}_0)\delta\mathbf{p} \\ &+ \nabla_{p_0}\nabla_{x_0} H(\xi, \mathbf{x}_0, \mathbf{p}_0)\delta\mathbf{x}_0 .\end{aligned} \qquad (j-4)$$

Further elimination of the evolution of the reference ray provides us the linear system

$$\begin{aligned}\frac{d\delta\mathbf{x}}{d\xi} &= \nabla_{p_0}\nabla_{x_0} H \; \delta\mathbf{x} + \nabla_{\mathbf{p}_0}\nabla_{\mathbf{p}_0} H \; \delta\mathbf{p} \\ \frac{d\delta\mathbf{p}}{d\xi} &= -\nabla_{x_0}\nabla_{x_0} H \; \delta\mathbf{x} + \nabla_{\mathbf{x}_0}\nabla_{\mathbf{p}_0} H \; \delta\mathbf{p}\end{aligned} \qquad (j-5)$$

where we have dropped the subscript zero for the reference ray because confusion is no longer possible. Let us introduce the paraxial canonical ray $\delta\mathbf{y}^t = (\delta\mathbf{x}, \delta\mathbf{p})^t$ which verifies the linear system

$$\frac{d\delta\mathbf{y}}{d\xi} = A \; \delta y \qquad (j-6)$$

where A is the following matrix deduced from equation (j 5)

$$A = \begin{pmatrix} \nabla_{\mathbf{p}}\nabla_{\mathbf{x}} H & \nabla_{\mathbf{p}}\nabla_{\mathbf{p}} H \\ -\nabla_{\mathbf{x}}\nabla_{\mathbf{x}} H & -\nabla_{\mathbf{x}}\nabla_{\mathbf{p}} H \end{pmatrix} \qquad (j-7)$$

computed on the reference ray, sometimes called central ray. The nearby ray computed by this paraxial equation is only sensitive to heterogeneities felt by the reference ray, providing us a stable estimation of the ray tube. The linear system (j 6) can be solved by the propagator technique and the solution at parameter ξ is deduced from the solution at parameter ξ_0 by

$$\delta y(\xi) = \mathcal{P}(\xi, \xi_0) \; \delta y(\xi_0) \qquad (j-8)$$

where $\mathcal{P}(\xi, \xi_0)$ is the propagator allowing to go from ξ_0 to ξ. From propagator theory, we have following properties

$$\frac{d\mathcal{P}}{d\xi} = A \; \mathcal{P} \;\; and \;\; \mathcal{P}(\xi_0, \xi_0) = I \qquad (j-9)$$

In order to distinguish what concerns the position and the slowness vector, we write the propagator in the standard form :

$$\mathcal{P}(\xi, \xi_0) = \begin{pmatrix} Q_1 & Q_2 \\ P_1 & P_2 \end{pmatrix} \qquad (j-10)$$

where Q_1, Q_2, P_1, P_2 are sub matrices whose dimensions depend on the differential system we have selected. From the Liouville theorem which states that a volume in phase space is incompressible, we have the important property of the propagator :

$$Trace(A) = 0 \leftrightarrow det\ \mathcal{P}(\xi, \xi_0) = 1. \qquad (j-11)$$

Other relations, often called Luneberg relations, coming from differential rules are

$$\begin{aligned} Q_1 Q_2^t - Q_2 Q_1^t &= 0 \\ Q_1 P_2^t - Q_2 P_1^t &= I \\ P_1 P_2^t - P_2 P_1^t &= 0 \\ P_2 Q_1^t - P_1 Q_2^t &= I \end{aligned} \qquad (j-12)$$

which implies immediately (j-11) as well as the inverse propagator

$$\mathcal{P}^{-1} = \begin{pmatrix} P_2^t & -Q_2^t \\ -P_1^t & Q_1^t \end{pmatrix}. \qquad (j-13)$$

Each submatrice is obtained by solving the propagator for particular initial conditions. By assuming $\delta y(\xi_0)^t = [I\ 0]$, we obtain submatrices Q_1 and P_1, while $\delta y(\xi_0)^t = [0\ I]$ gives the two other submatrices Q_2 and P_2. The linear combination of these solutions is also a solution.

In a two-dimensional medium with cartesian coordinate system, we may define four elementary solutions with initial paraxial canonical vector equal to zero except for one component δx or δz or δp_{x0} or δp_{z0} shown in figure (J-2). General solutions are obtained linear combination but they are not paraxial rays : They must verify the additional equation

$$\delta H = \nabla_{\mathbf{x}} H\ \delta\mathbf{x} + \nabla_{\mathbf{p}} H\ \delta\mathbf{p} = 0 \qquad (j-14)$$

which means that $\nabla_{\mathbf{x}} H\ \delta\mathbf{x} = 0$ when $\delta\mathbf{p} = 0$ and $\nabla_{\mathbf{p}} H\ \delta\mathbf{p} = 0$ when $\delta\mathbf{x} = 0$. Generally, elementary solutions are not paraxial rays.

It is very important to stress that the paraxial solution is always coordinate-dependent and that coordinate transformations provide other paraxial approximations. Only when coordinate transformations are linear, an equivalence between paraxial solutions can be obtained. In particular, this is true between ray centered coordinate system and a local cartesian coordinate system.

Let us consider a vertical dependence of the velocity : the reduced hamiltonian

$$\mathcal{H}(z, x, p_x) = -\sqrt{u^2(z) - p_x^2} \qquad (j-16)$$

is selected for a ray pointing downward. We find the following linear differential system

$$\frac{d\delta\mathbf{y}}{dz} = \begin{pmatrix} 0 & \frac{u^2}{(u^2 - p_x^2)^{3/2}} \\ 0 & 0 \end{pmatrix} \qquad (j-17)$$

which leads to a quadrature for the paraxial canonical vector

$$\begin{aligned} \delta p_x(z) &= \delta p_x(z_0) \\ \delta x(z) &= \delta x(z_0) + \delta p_x(z_0) \int_{z_0}^{z} \frac{u^2}{(u^2 - p_x^2)^{3/2}} dz \end{aligned} \qquad (j-18)$$

and a simple expression for the propagator

$$\mathcal{P} = \begin{pmatrix} 1 & Q_2 \\ 0 & 1 \end{pmatrix}. \qquad (j-19)$$

The term Q_2 identified by relation (j-18) is related to the position X by

$$Q_2 = \frac{dX}{dp_x} = \frac{d}{dp_x} \int_{z_0}^{z} \frac{p_x}{\sqrt{u^2 - p_x^2}} dz. \qquad (j-20)$$

which is linked to the jacobian estimation (e-15) by anticipation.

For a constant gradient of the square of slowness, it is a trivial matter to obtain the propagator as

$$\begin{pmatrix} 1 & (\tau - \tau_0) \\ 0 & 1 \end{pmatrix}. \qquad (j-21)$$

which gives the term Q_2 equivalent to the equation (e-21) by anticipation.

For the general case, we must use previously mentioned method for ray tracing as numerical solvers, but the linearity allows one to integrate with a rather important sampling parameter. The effort for computing many elementary solutions is the same as the one for one elementary solution because we need partial derivatives of the hamiltonian along the central ray which is the most time-consuming effort and which is performed only once

Beams of rays

Tracing paraxial rays with random initial conditions will give the whole space : we must select initial conditions in order to observe a plane wave or waves emitted by a point source. By a proper choice of initial conditions, we define a beam of paraxial rays. In order to preserve the linearity of solutions, we define initial conditions as an hyperplane in phase space,

$$a\ \delta\mathbf{p}(\xi_0) + b\ \delta\mathbf{x}(\xi_0) = 0 \qquad (j-22)$$

which often is written in the following form

$$\delta \mathbf{x}(\xi_0) = \epsilon\, \delta \mathbf{p}(\xi_0) \qquad (j-23)$$

with a possible infinite value for ϵ. When $\epsilon = 0$, we have the point source condition and when $\epsilon = \infty$, we have the plane wave conditions. From equation (j-8) and equation (j-23), a linear relation between paraxial position and paraxial slowness vector

$$\delta \mathbf{p}(\xi) = \mathcal{M}(\xi)\delta \mathbf{x}(\xi) \qquad (j-24)$$

introduces the matrix

$$\mathcal{M}(\xi) = \frac{\epsilon P_1 + P_2}{\epsilon Q_1 + Q_2} \qquad (j-25)$$

which is related to the curvature of the local wavefront. In a two dimensional medium, we have the radius of curvature R equal to

$$R = \frac{\delta n}{\delta \theta} = \frac{u\, \delta n}{\delta p_n} = u(\xi)\frac{1}{M(\xi)} \qquad (j-26)$$

as shown by figure (D-6). The curvature,

$$\mathcal{K} = c(\xi)\mathcal{M}(\xi), \qquad (j-27)$$

is therefore directly proportional to matrix $\mathcal{M}$ which justifies the notation. Finally, the continuity of the wavefront deduced from the paraxial theory is related to the spatial derivatives of the travel time, which is an information important in different applications as earthquake locations or travel time interpolations.

One might quote the different applications of the paraxial theory. The first application is the estimation of the jacobian and, consequently, the amplitude evaluation. The elementary surface dS' is given by

$$dS'(\xi) = det(\epsilon Q_1 + Q_2)\frac{sin\theta}{sin\theta_0}dS'(\xi_0) \qquad (j-28)$$

where θ is the angle between the slowness vector and the paraxial position as shown in figure (J-3). We have the following simple interpretation of the geometrical spreading of a plane wave with Q_1 and the geometrical spreading of a point source with Q_2. Another application is the estimation of travel-time under a parabolic approximation through the formula

$$T(\mathbf{x} + \delta \mathbf{x}) = T(\mathbf{x}) + \mathbf{p}\cdot\delta \mathbf{x} + \frac{1}{2}\delta \mathbf{x}^t \mathcal{M}\delta \mathbf{x}. \qquad (j-29)$$

Finally, the estimation of geometrical spreading allows by iteration to shoot at a given station by a Newton method. The two point ray tracing problem can be solved locally

In partial conclusion, one can say that this paraxial theory is a rather classical section of mechanical or optical textbooks and has been recently rediscovered by seismologists with the complexity of the heterogeneity we have in the Earth.

K - High frequency approximation (elastic case)

Until now, we have focused our attention on scalar acoustic equation, stating that the elastic case is not very different. We shall discuss here the high frequency solution for the elastic case. We start from a similar ansatz of the solution

$$u(\mathbf{x},\omega) = S(\omega)\mathbf{A}(\mathbf{x},\omega)e^{i\omega T(\mathbf{x})} \qquad (k-1)$$

where the amplitude term is a vector. A serie in power of ω is assumed for this vector

$$\mathbf{A}(\mathbf{x}\omega) = \sum_k \frac{\mathbf{A}^k(\mathbf{x})}{(-i\omega)^k} \qquad (k-2)$$

and is inserted in the elastodynamic equation

$$(C_{ijkl}u_{k,l})_{,j} + f_i = \rho u_{i,tt} \qquad (k-3)$$

for an anisotropic medium. We consider an anistropic medium simply because the elastodynamic equation has a compact form. Arranging the equation (k-3) in powers of ω, we find a cascade of equations. The term in ω^2 gives the following equation

$$C_{ijkl}T_{,j}T_{,l}A_k^0 - \rho A_i^0 = 0 \qquad (k-4)$$

which can be written as a relatively simple equation

$$\Gamma_{ik}A_k^0 - A_i^0 = 0 \qquad (k-5)$$

with

$$\Gamma_{ik} = \frac{C_{ijkl}T_{,j}T_{,l}}{\rho}. \qquad (k-6)$$

The matrix Γ is called the elastodynamic matrix of the ray theory and have very interesting properties which we do not discuss here as they have not direct applications in the following.

Let us look for a non-zero solution of (k-5) and, consequently, for the eigenvalues G_m and eigenvectors g_m of equation (k-5). The eigenvalues G_m are defined by the following determinant

$$D = det(\Gamma_{ik} - G_i\delta_{ik}) \qquad (k-7)$$

equal to zero. After a tedious manipulation, one can factor the determinant into two terms

$$D = (\frac{\mu}{\rho}T_{,i}T_{,i} - G_m)^2(\frac{\lambda + 2\mu}{\rho}T_{,i}T_{,i} - G_m) \qquad (k-8)$$

for an isotropic medium. We find two eigenvalues one of which has a double degeneracy. The associated eigenvectors verify the following equations

$$(\Gamma_{ik} - G_m \delta_{ik}) g_k^m = 0 \qquad (k-9)$$

The eigenvector $\mathbf{g}^3$ is associated to the single eigenvalue and is orthogonal to the two others $\mathbf{g}^1$ and $\mathbf{g}^2$ which can not be determined uniquely. The degeneracy of the second eigenvalue determines only the plane where they are lying. Let us denote two quantities with a notation which will be understood in a moment

$$\alpha = \frac{\lambda + 2\mu}{\rho} \qquad \beta = \frac{\mu}{\rho} \qquad (k-10)$$

The equation (k-5) requires eigenvalues equal to the unity which implies one of the two following equations

$$T_{,i}T_{,i} = \frac{1}{\alpha^2} \qquad T_{,i}T_{,i} = \frac{1}{\beta^2} \qquad (k-11)$$

which show that α and β are local phase velocities and that equations (k-11) are identical to the eikonal for the acoustic case. In other words, solving eikonal equations for the elastic case is the same as solving eikonal equations for the acoustic case.

For polarizations, the situation is slightly more complex and requires explicitly the isotropic case for equation (k-4). We have

$$\frac{\lambda + 2\mu}{\rho} \nabla T (\nabla T \cdot \mathbf{A}^0) + \frac{\mu}{\rho} (\nabla T)^2 A^0 - A^0 = 0 \qquad (k-12)$$

which gives the following equations by taking the scalar and cross products :

$$[\alpha^2 (\nabla T)^2 - 1](\mathbf{A}^0 \cdot \nabla T) = 0 \qquad [\beta^2 (\nabla T)^2 - 1](\mathbf{A}^0 \times \nabla T) = 0 \qquad (k-13)$$

For waves propagating at speed α, the following equation

$$\mathbf{A}^0 \times \nabla T = 0 \qquad (k-14)$$

should be verified and demonstrates that we have compressive waves called P waves parallel to ∇T or slowness vector $\mathbf{p}$. The amplitude is linearly polarized perpendicular to the wavefront

$$\mathbf{A}^0 = A_3 \mathbf{g}^3. \qquad (k-15)$$

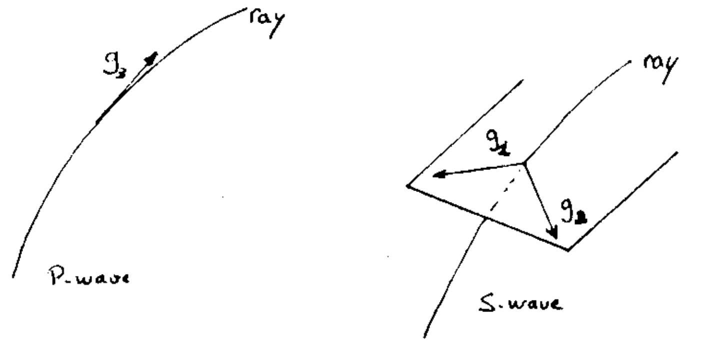


Figure K-1 : Polarization vectors in a 3-D medium.

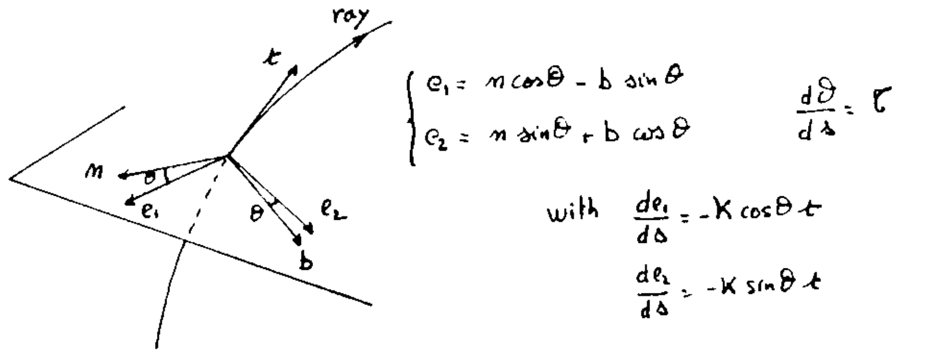


Figure K-2 : Geometrical relations between Frénet vectors and ray-centered coordinate system.

For waves propagating at speed β, the following equation

$$\mathbf{A}^0 \cdot \nabla T = 0 \qquad (K-16)$$

shows that the motion is perpendicular to the slowness vector and creates a shear wave called S waves. The amplitude is elliptically polarized from the general expression

$$\mathbf{A}^0 = A_1 \mathbf{g}^1 + A_2 \mathbf{g}^2. \qquad (K-17)$$

The figure K-1 summarizes the polarization of the two kinds of waves. The expression for the P wave displacement in the frequency domain is

$$\mathbf{u}(\mathbf{x},\omega) = S(\omega)\ \mathbf{t}\ A_3(\mathbf{x}) e^{i\omega T_p(\mathbf{x})} \qquad (K-18)$$

with an amplitude proportionnal to $1/\sqrt{\alpha\rho \mid J \mid}$. By going back to the time domain, we find a similar solution to the acoustic case

$$\mathbf{u}(\mathbf{x},t) = \mathbf{t}\ \varphi_3(\gamma_1,\gamma_2)\ \Re\{\frac{1}{\sqrt{\alpha\rho J}} \dot{S}(t - T_p(\mathbf{x}))\}. \qquad (K-19)$$

For S waves, an arbitrary selection of eigenvectors $\mathbf{g}_1$ and $\mathbf{g}_2$ leads to a coupling between the propagation of amplitude A_1 and amplitude A_2. For a specific set of eigenvectors such that

$$\frac{d\mathbf{g}^1}{ds} \propto \mathbf{t} \qquad \frac{d\mathbf{g}^2}{ds} \propto \mathbf{t} \qquad (K-20)$$

we obtain an independent propagation for quantities A_1 and A_2. These particular vectors, denoted $\mathbf{e}_1$ and $\mathbf{e}_2$ are those of the ray centered coordinates system (figure K-2) which provides a simple description of the propagation with a decoupling between A_1 and A_2. I think this is the most important contribution of this particular coordinate system. The S wave displacement is given finally in the time domain by

$$\mathbf{u}(\mathbf{x},t) = \mathbf{e}_1\ \varphi_1(\gamma_1,\gamma_2)\ \Re\{\frac{1}{\sqrt{\beta\rho J}} \dot{S}(t - T_s(\mathbf{x}))\} + \mathbf{e}_2\ \varphi_2(\gamma_1,\gamma_2)\ \Re\{\frac{1}{\sqrt{\beta\rho J}} \dot{S}(t - T_s(\mathbf{x}))\} \qquad (K-19)$$

We must follow the evolution of the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ during the propagation. This is the only added difficulty compared to the acoustic case, a remarkable result of the ray theory.

L - Interfaces

High frequency propagation assumes a smooth variation of physical properties of the medium, while the Earth sharp boundaries are often met. If the boundary is sharp enough in order to avoid any effect of a length scale, we can still apply the ray theory from one side to the other one of the discontinuity and check explicitly the "continuity" of the solution along the interface.

Starting with an incident wave denoted with subscript i, a reflected wave denoted with subscript r and a transmitted wave denoted with subscript t are generated at the interface position. The continuity of the phase of the wave field and the invariance with respect to time implies the equality of travel-times :

$$T_i = T_r = T_t, \qquad (L-1)$$

while the spatial tangential invariance along the interface implies the following equality

$$\mathbf{n} \times \nabla_{\mathbf{x}} T_i = \mathbf{n} \times \nabla_{\mathbf{x}} T_r = \mathbf{n} \times \nabla_{\mathbf{x}} T_t \qquad (L-2)$$

which is known to be the Snell-Descartes law. At the same time, we must require the continuity of displacements and stresses along the interface. We must evaluate the surface of ray tube intersected by the interface for the three kinds of rays as well as reflection and transmission coefficients. The incident pressure at position $\mathbf{x}_I$ on the interface

$$P_i(\mathbf{x}_I,\omega) = S(\omega)\ \phi(\gamma_1,\gamma_2)\sqrt{\frac{1}{u(\mathbf{x}_I) \mid J_i(\mathbf{x}_I) \mid}}\ e^{i\omega T_i(\mathbf{x}_I)} e^{i\frac{\pi}{2} KMAH} \qquad (L-3)$$

generates reflected pressure at position $\mathbf{x}$

$$P_r(x,\omega) = S(\omega)\ \phi(\gamma_1,\gamma_2) R \sqrt{\frac{\mid J_r(\mathbf{x}_I) \mid}{\mid J_i(\mathbf{x}_I) \mid}} \sqrt{\frac{1}{u(\mathbf{x}) \mid J_r(\mathbf{x}) \mid}}\ e^{i\omega T_r(\mathbf{x})} e^{i\frac{\pi}{2} KMAH} \qquad (L-4)$$

as well as transmitted pressure

$$P_t(x,\omega) = S(\omega)\ \phi(\gamma_1,\gamma_2) T \sqrt{\frac{u_t(\mathbf{x}_I) \mid J_t(\mathbf{x}_I) \mid}{u(\mathbf{x}_I) \mid J_i(\mathbf{x}_I) \mid}} \sqrt{\frac{1}{u(\mathbf{x}) \mid J_t(\mathbf{x}) \mid}}\ e^{i\omega T_t(\mathbf{x})} e^{i\frac{\pi}{2} KMAH}. \qquad (L-5)$$

Denoting θ_i the angle between the normal at the interface and the slowness vector of the incident field at the hitting point as well as the angle θ_r for reflected fields

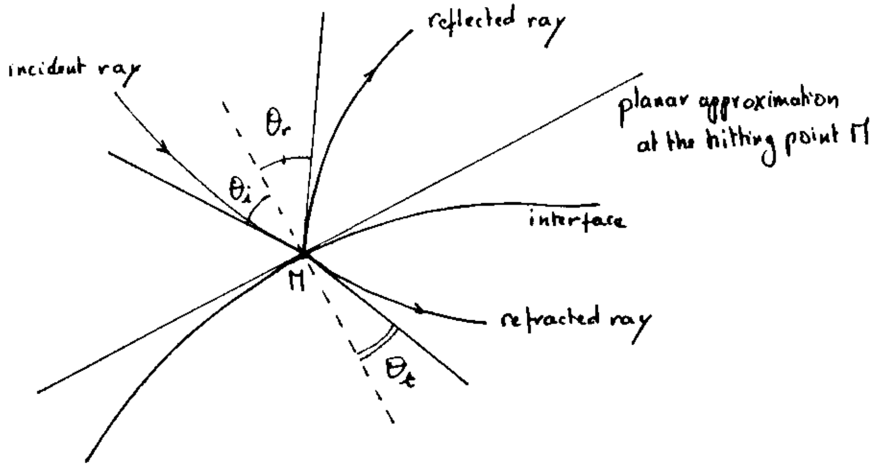


Figure L-1: Geometry at the interface between two media.

and the angle θ_t for transmitted fields (figure L-1), we found geometrically

$$\begin{aligned} J_r(\mathbf{x}_I) &= -J_i(\mathbf{x}_I) \\ J_t(\mathbf{x}_I) &= J_I(\mathbf{x}_I)\cos\theta_t \\ J_i(\mathbf{x}_I) &= J_I(\mathbf{x}_I)\cos\theta_i \end{aligned} \qquad (L-6)$$

and the continuity of the energy implies that J_I is continuous which means

$$\frac{J_t}{\cos\theta_t} = \frac{J_i}{\cos\theta_i} \qquad (L-7)$$

The reflected and transmitted coefficients R and T are those for plane waves hitting a planar interface as a valid approximation at the high frequency we are looking (figure L-1). Incorparating the effect of geometrical spreading in the coefficients, we found extended coefficients

$$\begin{aligned} t &= T\sqrt{\frac{u_t(\mathbf{x}_I) \mid \cos\theta_t \mid}{u_i(\mathbf{x}_I) \mid \cos\theta_i \mid}} \\ r &= R \end{aligned} \qquad (L-8)$$

The final solution for a ray which has undertaken different conversions at interfaces is

$$P(x,\omega) = S(\omega)\ \phi(\gamma_1,\gamma_2)\ \Pi \sqrt{\frac{1}{a(\mathbf{x}) \mid J(\mathbf{x}) \mid}}\, e^{i\omega T_t(\mathbf{x})}\, e^{i\frac{\pi}{2}KMAH}\,. \qquad (L-9)$$

where Π is the product of extended coefficients r and t along the ray. For the displacement, we have to modify slightly the final formula as shown in the elastic approach and we get

$$u(x,\omega) = S(\omega)\ \phi(\gamma_1,\gamma_2)\ \Pi \sqrt{\frac{1}{\rho(\mathbf{x})c(\mathbf{x}) \mid J(\mathbf{x}) \mid}}\, e^{i\omega T_t(\mathbf{x})}\, e^{i\frac{\pi}{2}KMAH}\,. \qquad (L-10)$$

Finally, the ray theory is valid for smooth variation of reflection coefficients and the interface curvature. Different strategies must be used when these properties are not fulfilled as we have seen in section II.

M - Synthetic seismograms

We have seen that, once one knows the ray arriving at a station, evaluating synthetic seismograms is a simple matter in the framework of the ray theory. Unfortunately, we need to sum up rays arriving at the selected station in a given time windows. The two-point ray tracing is a time-consuming task and missing a ray is always possible. Moreover, singularities are present in the ray theory because one assumes an high frequency approximation. For finite frequencies, we expect singularities to be smoothed out by diffusion : other area not crossed by rays contribute to seismograms. From a local point of view, we move to a more global approach which is linked to spectral methods where the contribution is entirely global.

Global and local points of view have their own difficulties. We present how to move from the spectral approach to the local one for very simple examples.

Let us start with a two-dimensional homogeneous medium of speed c. The pressure P in the spectral domain is given by

$$P(\mathbf{x},\omega) = \frac{i}{4} H_0^1(\frac{\omega r}{c}). \qquad (m-1)$$

The Hankel function is decomposed on exponential functions which gives the Weyl integral

$$P(\mathbf{x},\omega) = \frac{i}{4\pi} \int_{-\infty}^{\infty} e^{i\omega(px+qz)} \frac{dp}{q}. \qquad (m-2)$$

where

$$q^2 + p^2 = \frac{\omega^2}{c^2} \qquad (m-3)$$

for positive frequency ω and positive coordinate z. The quantity q needed in integral (m-2) requires the definition of the square root : we select q such that $Im(q) > 0$ in order to have a damping of waves when z is positive. When the square root is real, the equation (m-2) is a decomposition in plane waves and, when the square root is complex, we have contribution of inhomogeneous waves (figure M-1).

The solution in the time domain

$$P(\mathbf{x},t) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} e^{i\omega(px+qz-t)} \frac{i}{2} \frac{dp}{q}. \qquad (m-4)$$

can be evaluated either by integrating on p before integrating on ω - reflectivity method (real p) and full wave theory (complex p) - or by integrating first on

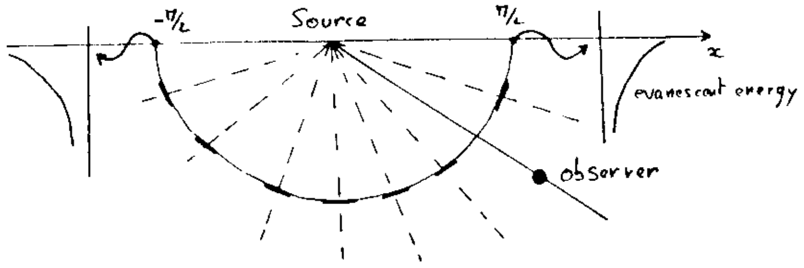


Figure M-1 : Summation of plane waves (progressive and evanescent waves) for a seismogram at the observer.

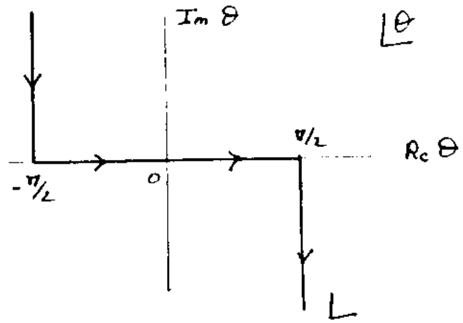
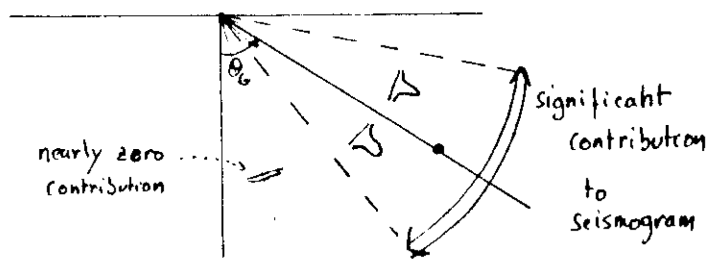


Figure M-2 : Integration path over the complex angle θ



ω and then on p - generalized ray method (complex p) or WKBJ method (real p) . The last method has a simple physical interpretation related to rays and we describe it now.

WKBJ seismograms

Changing the integration variable from p to angle θ by

$$p = \frac{sin\theta}{c} \quad and \quad q = \frac{cos\theta}{c} \qquad (m-5)$$

simplify even more the equation (m-4)

$$P(\mathbf{x},t) = \frac{1}{4\pi^2}\int d\omega \int_L e^{i\omega(T-t)}\,\frac{i}{2}d\theta. \qquad (m-6)$$

The contour L is given by figure (M-2) where angle θ has imaginary components. Integration is often along the real axis. The travel-time $T = \mathbf{p}\cdot\mathbf{x}$ is the travel time of a plane wave with a direction $\mathbf{p}$. This equation is exact with inhomogeneous waves along the z axis. These waves which are important near the source are often neglected at high frequency. Moreover, the principal contribution of the oscillating exponential term comes from the saddle point where the oscillation is the smallest. This saddle point is given by

$$\frac{\partial T}{\partial \theta} = \frac{1}{c}(xcos\theta - ysin\theta) \qquad (m-7)$$

which gives the angle of the geometrical ray arriving at the station (figure M-3)

$$tan(\theta_G) = \frac{x}{y} \qquad (m-8)$$

and a contribution to the pressure from the saddle point approximation

$$\frac{1}{4\pi}\sqrt{\frac{2c}{r}}\,\frac{H(t-\frac{r}{c})}{\sqrt{t-\frac{r}{c}}} \qquad (m-9)$$

equal to the high frequency approximation computed in section B. We extend the integral evaluation on the real contour of L and obtain by integration on ω

$$P(\mathbf{x},t) = \frac{1}{4\pi^2}\Re\Big[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{t-T(\theta)}d\theta\Big]. \qquad (m-10)$$

This integral has a singularity in $t = T$ which is removed by an adequat smoothing : a small imaginary part (often equal to $i\Delta t$ where Δt is the time sampling) is added to the travel-time T. The missing segments of contour L induce cut off phases because plane waves have equal importance in their contribution to the pressure, but the approximation is better than taking only the geometrical arrival which need not to be calculated : two point ray tracing is avoided.

The extension in a medium with arbitrary vertical variation of the velocity is straigthforward and one obtains

$$P(\mathbf{x},t) = \frac{1}{4\pi^2}\Re\int_L \sqrt{\frac{q(z_0,p)}{q(z,p)}}\,\frac{d\theta}{t-T(p,z,x)} \qquad (m-11)$$

where q is now depending in z and the travel time $T(p,z,x)$ is the sum of the travel time of the ray reaching the depth z and the horizontal travel time between x and the position of the ray $X(p,z)$ (figure M-3), i.e.

$$T(p,z,x) = T(p,z) + p(x - X(p,z)). \qquad (m-12)$$

The procedure to compute pressure P is done in three steps : (1) decomposition of the source in Snell waves (decomposition in p), (2) propagation of each Snell wave and (3) summation at the station of the different Snell waves with the travel time T and the geometrical spreading $1/\sqrt{q}$. An extension of the method called Maslov method allows to consider laterally variable medium.

The cut-off phases coming from neglecting inhomogeneous waves is the main drawback of this approach. It has been proposed to evaluate asymptotically these branches. Another technique is to make negligeable the contribution of these branches by deforming the plane wave decomposition in order to have more local decomposition around the geometrical arrival.

Gaussian beam summation

The geometrical spreading in vertically varying medium is given by

$$q(z,p) = \frac{cos\theta}{c(z)}Q_1 \qquad (m-13)$$

for a plane wave. We write the decomposition (m-11) in a more explicit expression

$$P(\mathbf{x},t) = \frac{1}{4\pi^2}\Re\int_L \sqrt{\frac{c(z)cos\theta_0}{c(z_0)cos\theta Q_1}}\,\frac{d\theta}{t-T(p,z,x)} \qquad (m-14)$$

and generalize the factor Q_1 to a factor Q which includes the effect of the plane wave as well as the effect of the point source. The geometrical spreading can be defined by

$$Q = Q_1 + \epsilon^{-1}Q_2 \qquad (m-15)$$

as seen in the paraxial theory section : The plane wave with a zero courbure of the wavefront is deformed into a wave with a curvature $\mathcal{M}$ given by

$$\mathcal{M} = \frac{P_1 + \epsilon^{-1} P_2}{Q_1 + \epsilon^{-1} Q_2} \qquad (III-16)$$

and related to a wavefront defined by

$$T(p,x,z) = T(p,z) + p(x - X(p,z)) + \frac{1}{2}[x - X(p,z)]^t \mathcal{M}[x - X(p,z)]. \qquad (III-17)$$

The good selection of ϵ is still an open question and is related to the completeness of the decompositon of the pressure in these local waves. For a fixed arbitrary error in the initial pressure, a decomposition in gaussian waves can be performed. For Gaussian Beam Summation, the parameter ϵ is complexe, while, for the Maslov method, the parameter ϵ is real. Moreover, for the Maslov method, the parameter ϵ is such that one obtains plane waves at the receiver.

These extension of the ray theory for synthetizing seismograms have participated to the renewal of the ray theory in seismology and have increased their domain of application.

X - Conclusion

The ray theory allows many interpretations of propagation inside the Earth. In these notes, we have not considered dissipation or dispersion which are often met during propagation. The anisotropy has been a subject we have neglected. Also the lamelled structure which is questionable for the deep crust has not been considered here. Many extensions of the ray theory are possible and the futur will show us how rich of consequences is this theory as it has already been in the past

For applications, let us quote an obvious list. Locating earthquakes requires ray tracing between the expected position of the earthquake and stations at the Earth surface. The different slowness vectors at the source position allows to move the source towards a more accurate position which minimizes travel-time residues. At the same time, these initial slowness vectors gives the position onto the focal sphere of different stations : focal mechanisms can be deduced. Routine programs use mainly the layered approximation which is often a crude approximation for many local networks. Synthesizing seismograms is also very important for the interpretation of seismic profils along complex geological structures. Travel time tomography with very sophisticated inversion schemes needs efficient ray tracing in order to give the most accurate image of the Earth interior. Going to diffraction tomography where the amplitude is also analyzed is a further step where ray theory brings its efficiency and its capacity of interpretation. These different subjects, which have been presented in other lectures, are those which should interest you for different applications of ray theory.

preliminary end.......

references will be given later on.