INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

SMR/534-11

ICTP/WMO WORKSHOP ON EXTRA-TROPICAL AND TROPICAL
LIMITED AREA MODELLING
22 October - 3 November 1990

"Semi-Lagrangian Techniques & Variable Mesh"

M. HORTAL
ECMRWF
Reading
U.K.

Please note: These are preliminary notes intended for internal distribution only.

Fully reduced grid T106

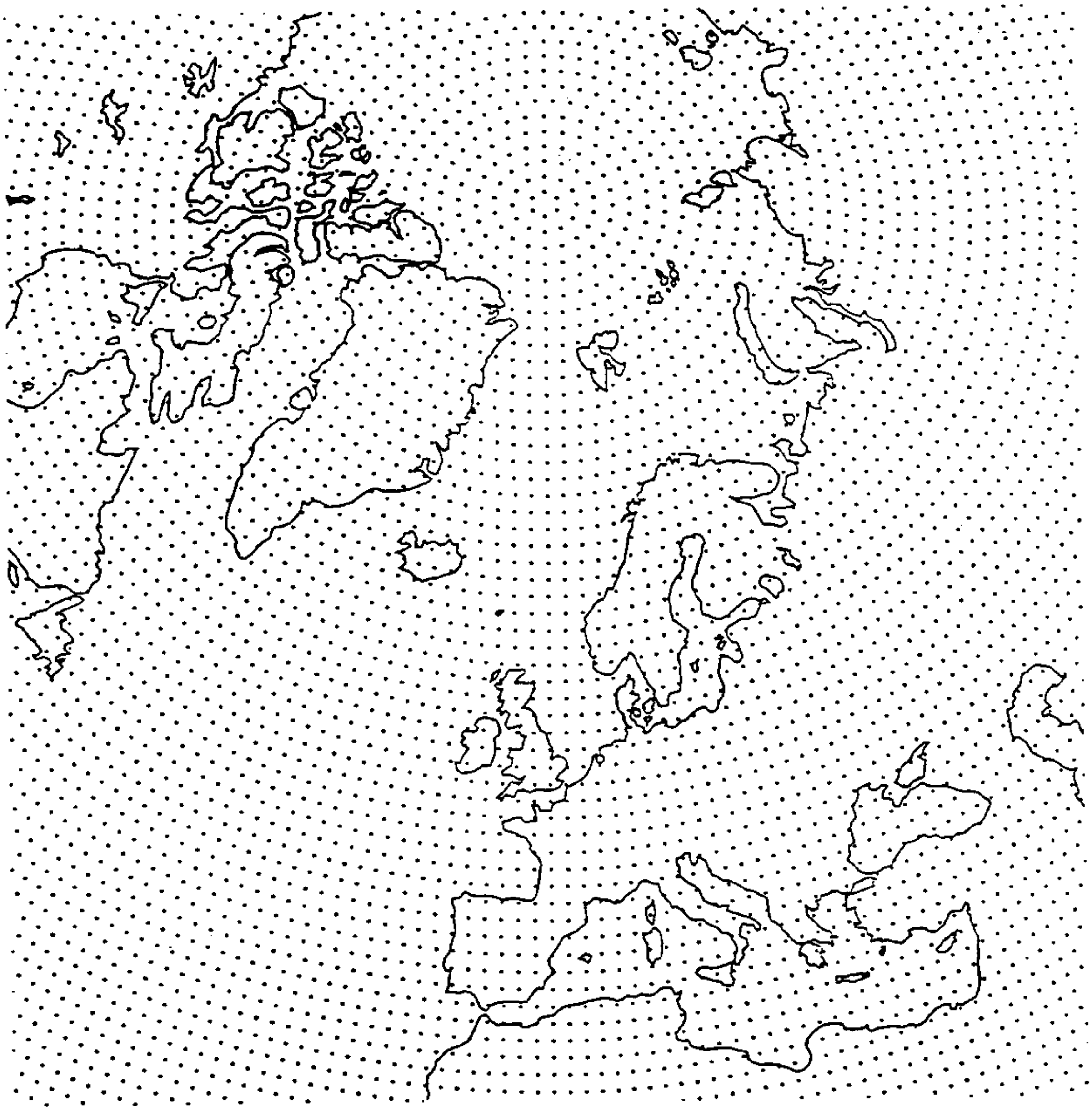


Fig 3

The need to increase the time step

T106 model $\xrightarrow{\text{explicit}}$ $\Delta t < 2$ min

$\downarrow$

$\lambda > 270$ Km at 45°

$\Delta t_{stability} \ll$ time scale of evolution

$\Delta t_{accuracy}$

- Implicit treatement of adjustment terms (linear part of)

  Robert's "semi-implicit"

  Δt limited by the advection terms

  Limited possibility of an implicit treatment of the advection terms because of the linearization

  multigrid methods ???

The cause of the instability

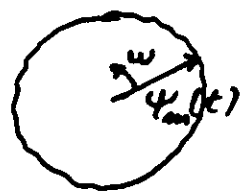
Linear advection equation (1D)

$$\frac{\partial \psi}{\partial t} + c\,\frac{\partial \psi}{\partial x} = 0$$

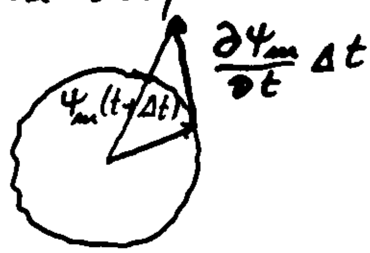
$\downarrow$

Eulerian
tendency (observer at rest)

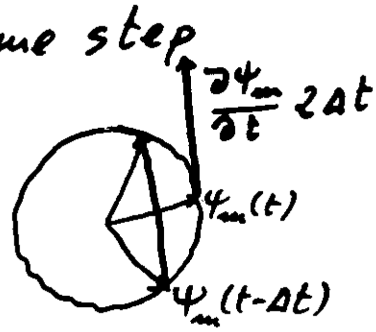
- Spectral model
  - Analytical solution



  - Forward time step

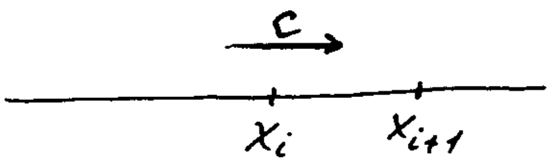


  - Centered time step

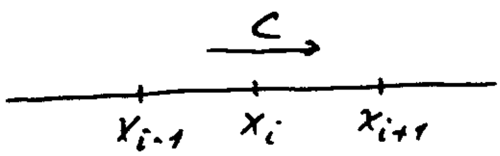


- Fix: convert the tendency into an arc
  - It works in a barotropic model
  - It fails in primitive equation models because the equations are too far from being linear

- Grid point models



downwind $\quad \dfrac{\psi_i^+ - \psi_i}{\Delta t} = -c\,\dfrac{\psi_{i+1} - \psi_i}{\Delta x}$ (unstable)



Leapfrog $\quad \dfrac{\psi_i^+ - \psi_i^-}{2\Delta t} = -c\,\dfrac{\psi_{i+1} - \psi_{i-1}}{\Delta x}$

C.F.L. limit $\quad \dfrac{c\Delta t}{\Delta x} \leq 1$

Meaning: the particle arriving at x_i at time $t+\Delta t$ must come from $[x_{i-1}, x_{i+1}]$ at time t

Otherwise we are extrapolating

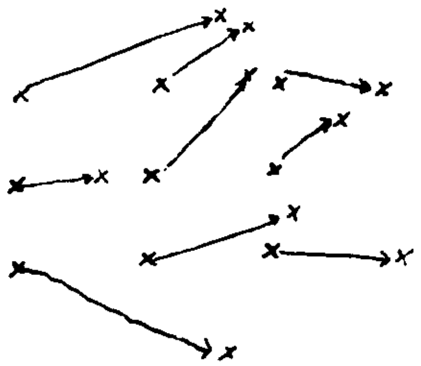
- Fix: follow the particles (Lagrangian view)

$$\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial x}\frac{dx}{dt} \longrightarrow \frac{d\psi}{dt}$$

The semi-Lagrangian technique

$$\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} \equiv \frac{d}{dt}$$

material derivative or time evolution along a trajectory



From a regular array of points we end up after Δt with a non-regular distribution

Semi-Lagrangian: tracking back

Solution of the one-dimensional advection equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 = \frac{du}{dt} \longrightarrow u_j^{n+1} = u_*^n$$

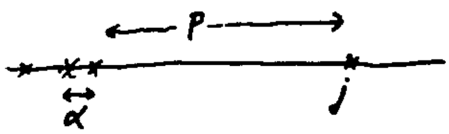
origin point

• Stability in one dimension

Linear advection equation

$$\frac{d\varphi}{dt} \equiv \frac{\partial \varphi}{\partial t} + U_0 \frac{\partial \varphi}{\partial x} = 0$$

origin of parcel at j : $x_* = x_j - U_0 \Delta t$



$U_0 \Delta t = (p+\alpha)\Delta x$ p: integer

Linear interpolation

$\varphi_*^n = (1-\alpha)\varphi_{j-p}^n + \alpha\, \varphi_{j-p-1}^n$ $\varphi_j^{n+1} = (1-\alpha)\varphi_{j-p}^n + \alpha\, \varphi_{j-p-1}^n$

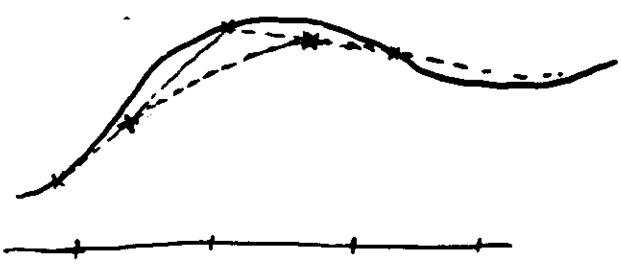
using von Neumann $\varphi_j^n = \varphi_0 \lambda^n e^{ikj\Delta x}$

α is not the CFL number except when p=0 (then ⇒ upwind scheme)

$$\lambda = \left[1 - \alpha\left(1 - e^{-ik\Delta x}\right)\right] e^{-ipk\Delta x}$$

$$|\lambda|^2 = 1 - 2\underbrace{\alpha(1-\alpha)}_{\text{max val} = 0.25}\left[1 - \cos(k\Delta x)\right]$$

$|\lambda| \leq 1$ if $\underline{0 \leq \alpha \leq 1}$ (interpolation from two nearest points)



(damping)

Quadratic or cubic interpolation ⟶ less damping

Quasi-Lagrangian schemes in 2-D

$$\frac{\partial X}{\partial t} + U\frac{\partial X}{\partial x} + V\frac{\partial X}{\partial y} \equiv \frac{dX}{dt} = L\cdot X + N(X)$$

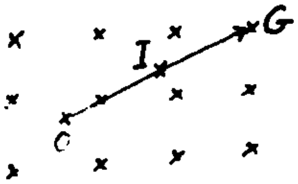
L: linear operator

N: non-linear function

3 time levels schemes

- Interpolating

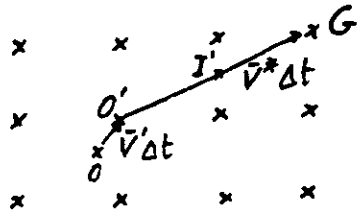
$$\frac{X_G^{t+\Delta t} - X_O^{t-\Delta t}}{2\Delta t} = L\cdot\frac{X_G^{t+\Delta t} + X_O^{t-\Delta t}}{2} + [N(X^t)]_I$$



Two interpolations needed

- Ritchie

$$U = U^* + U' \qquad V = V^* + V'$$



$$\frac{X_G^{t+\Delta t} - X_{O'}^{t-\Delta t}}{2\Delta t} = L\cdot\frac{X_G^{t+\Delta t} + X_{O'}^{t-\Delta t}}{2} + [N(X^t)]_{I'} - \left(U'\frac{\partial X}{\partial x} + V'\frac{\partial X}{\partial y}\right)_{I'}^t$$

- Non-interpolating

$$X_G^{t+\Delta t} - X_{O'}^{t-\Delta t} = L\cdot\frac{X_G^{t+\Delta t} + X_{O'}^{t-\Delta t}}{2} + \frac{1}{2}\Big\{[N(X^t)]_G + [N(X^t)]_{O'} -$$

$$- \left[\left(U'\left(\frac{\partial X}{\partial x}\right)_G^t + V'\left(\frac{\partial X}{\partial y}\right)_G^t + U'\left(\frac{\partial X}{\partial x}\right)_{O'}^t + V'\left(\frac{\partial X}{\partial y}\right)_{O'}^t\right]\Big\}$$

- Stability

  - In the linear advection equation all three are stable if $\alpha \leq 1$
  - In the linear shallow water eq the stability limit is

$$\Delta t \cdot f < 1 \qquad \text{Coriolis term}$$

  - In the two non-interpolating schemes also

$$(kU' + lV')\Delta t \leq 1$$

which can be shown to be allways true

$$r = \frac{\omega_{Num}}{\omega_{Anal}} \quad (dispersion)$$

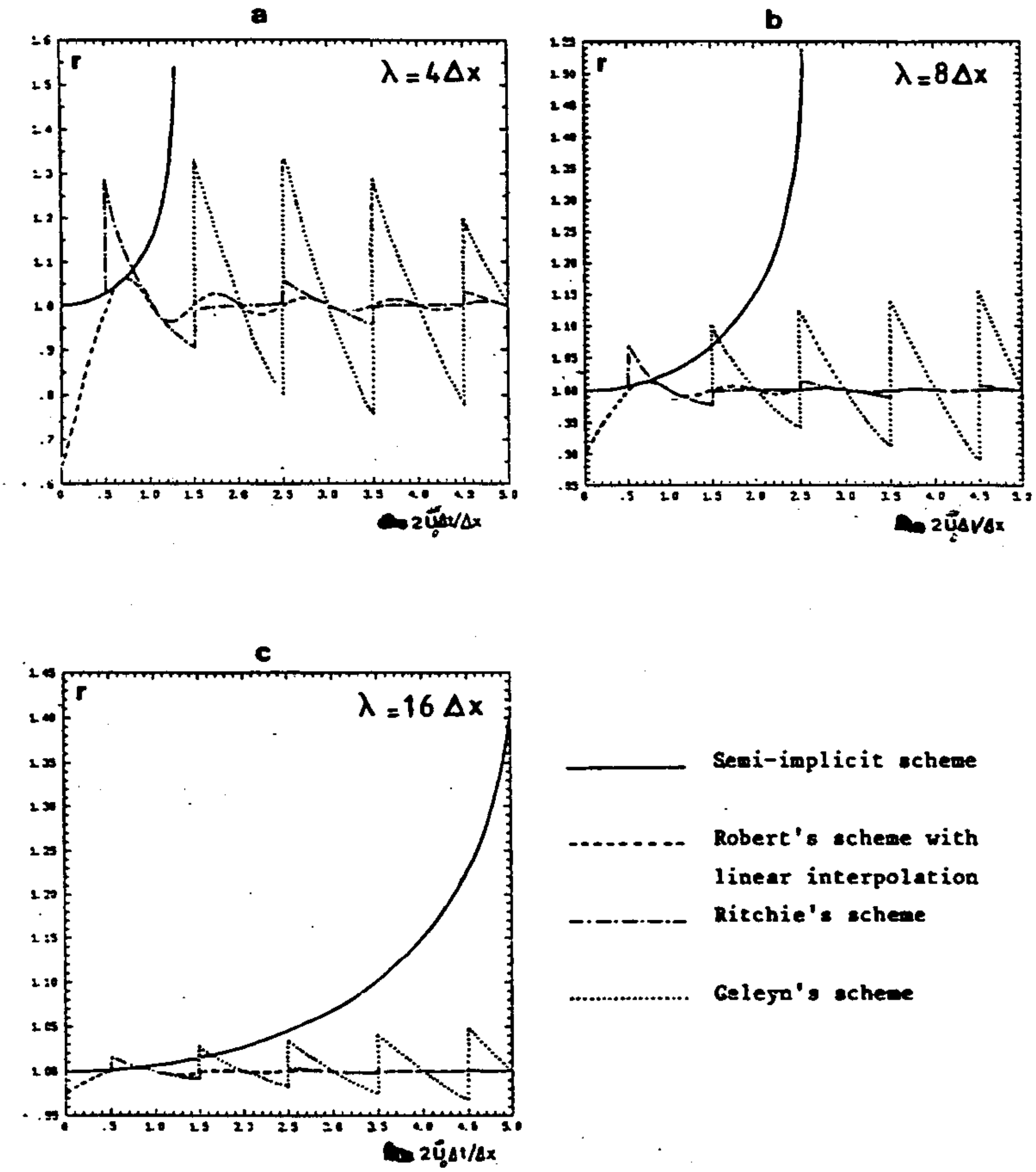


Figure 5. Effect of the time integration on the slow wave for various values of the wavelength.

Two time levels semi-Lagrangian

- In three time levels, trajectory begins at

$$(\underset{\sim}{x}_G - 2\underset{\sim}{\alpha}, t - \Delta t)$$

and ends at

$$(\underset{\sim}{x}_G, t + \Delta t)$$

$\underset{\sim}{\alpha}$ corresponding to $\underset{\sim}{x}_G$ is calculated

$$\underset{\sim}{V}(\underset{\sim}{x}_G - \underset{\sim}{\alpha}, t) = \frac{2\alpha}{2\Delta t} \qquad \text{(implicit)} \quad O(\Delta t^2) \quad O(\Delta x^2)$$

- In two time levels we can
  - Calculate $\underset{\sim}{\alpha} = \Delta t\, \underset{\sim}{V}(\underset{\sim}{x}_G, t)$ (explicit) $O(\Delta t)$ $O(\Delta x)$

    trajectory is then first order accurate in Δt
  - Extrapolate

    $$\underset{\sim}{V}(\underset{\sim}{x}_G, t + \tfrac{1}{2}\Delta t) \simeq \frac{3}{2}\underset{\sim}{V}(\underset{\sim}{x}_G, t) - \frac{1}{2}\underset{\sim}{V}(\underset{\sim}{x}_G, t - \Delta t)$$

    and then

    $$\underset{\sim}{\alpha} = \Delta t\, \underset{\sim}{V}(\underset{\sim}{x}_G + \tfrac{1}{2}\underset{\sim}{\alpha}, t + \tfrac{1}{2}\Delta t)$$

    trajectory second order accurate in Δt
  - Extrapolate using 3 time levels

    $$\underset{\sim}{V}(\underset{\sim}{x}_G, t + \tfrac{1}{2}\Delta t) \simeq \frac{15}{8}\underset{\sim}{V}(\underset{\sim}{x}, t) - \frac{10}{8}V(\underset{\sim}{x}, t - \Delta t) + \frac{3}{8}\underset{\sim}{V}(\underset{\sim}{x}, t - 2\Delta t)$$

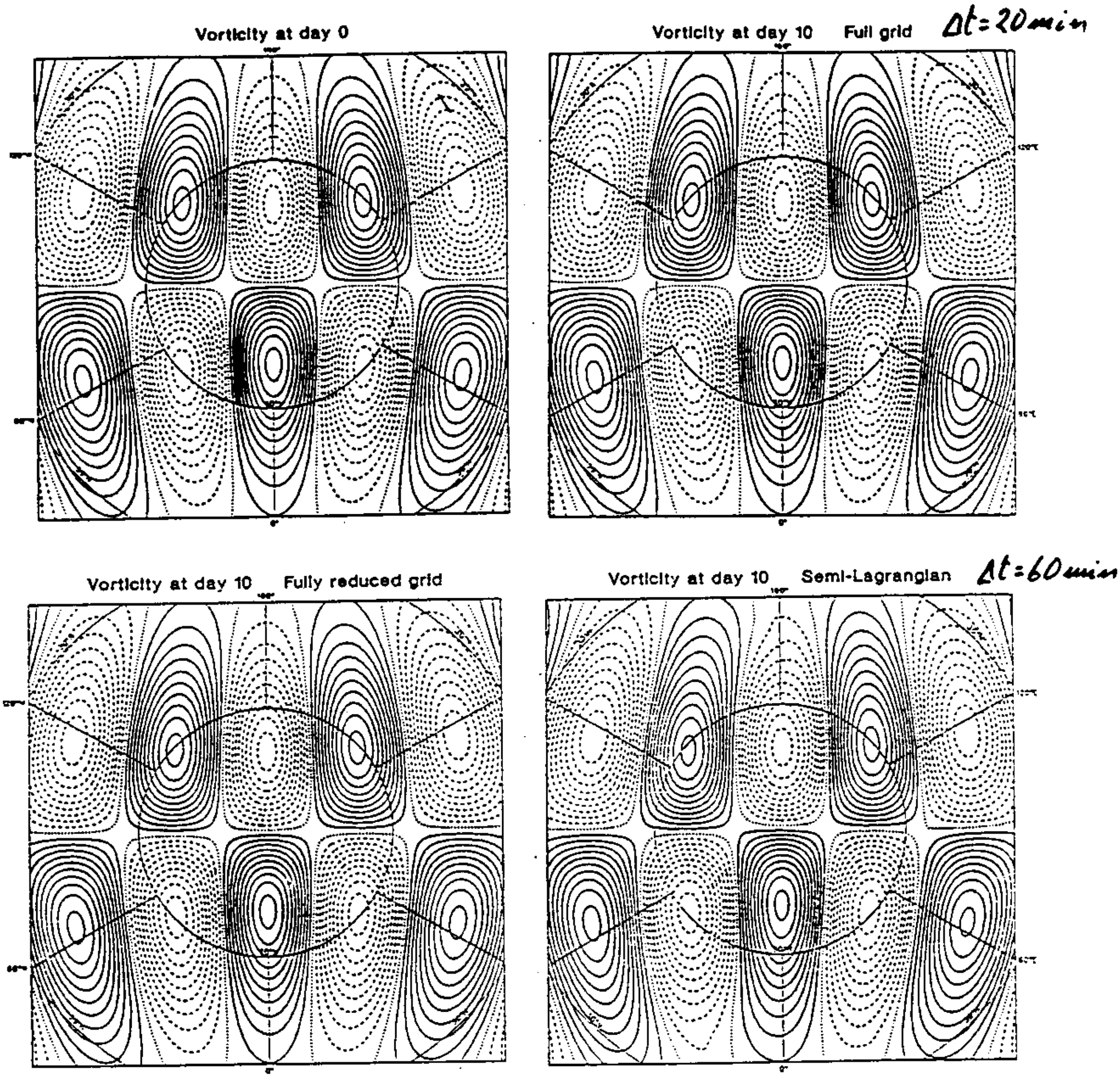
    then

    $$\underset{\sim}{\alpha} = \Delta t\, \underset{\sim}{V}(\underset{\sim}{x}_G + \tfrac{1}{2}\underset{\sim}{\alpha}, t + \tfrac{1}{2}\Delta t)$$

- Discretization then is

$$\frac{dF}{dt} = \frac{F_G^+ - F_0}{\Delta t}$$

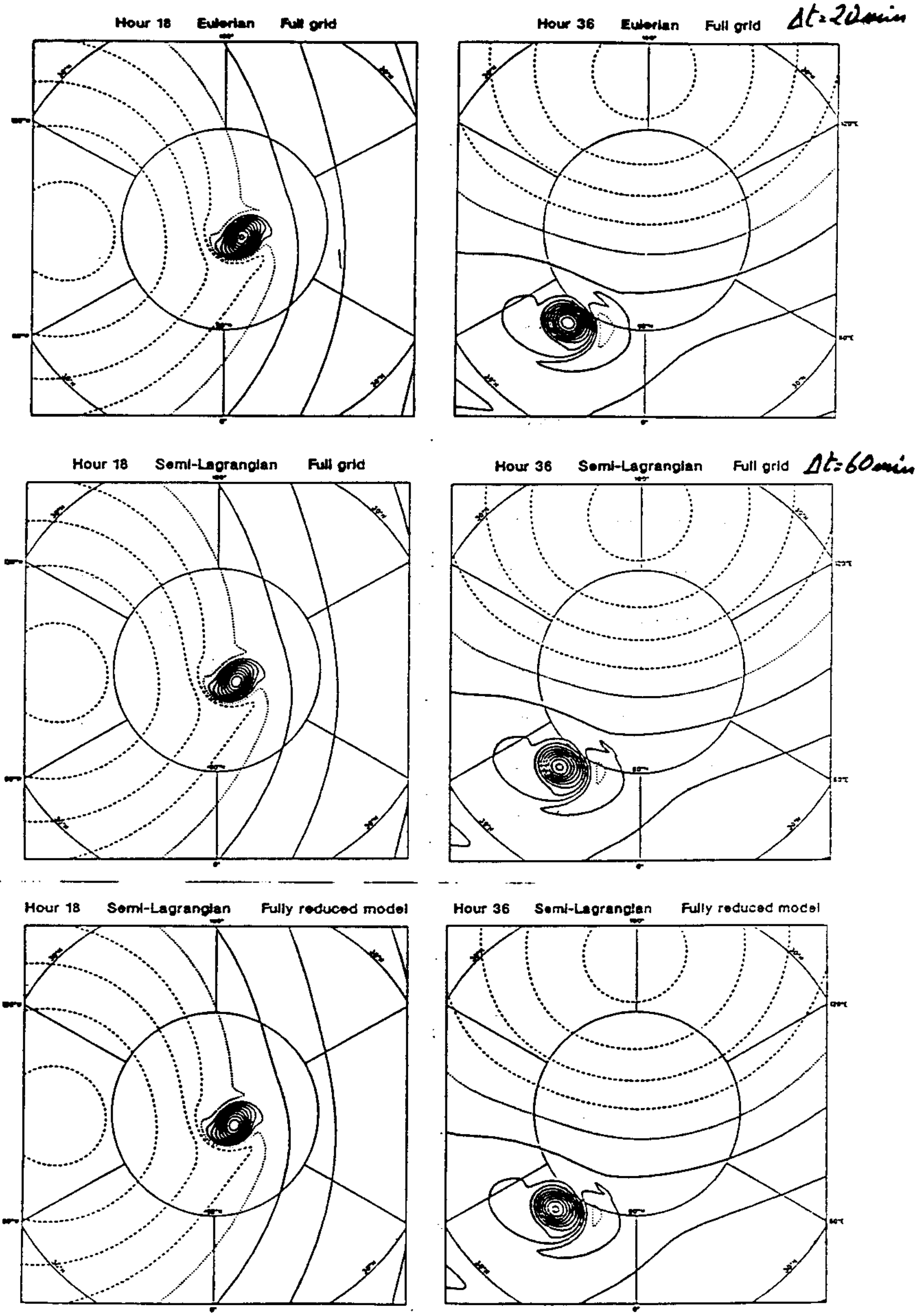
and could use Crank-Nicholson for the other terms



Barotropic vorticity model test
T106

Rossby-Haurwitz wave $\begin{cases} m = 8 \\ n = 9 \end{cases}$

Exact solution stationary in space



Barotropic model T106
Fig 7

SHALLOW WATER EQUATIONS

$$\frac{\partial \underset{\sim}{v}_H}{\partial t} + (\underset{\sim}{v}_H \cdot \nabla_H)\underset{\sim}{v}_H + \overline{\nabla_H \Phi}^{t} = - f \hat{k} \times \underset{\sim}{v}_H$$

(momentum)

$$\frac{\partial \Phi}{\partial t} + (\underset{\sim}{v}_H \cdot \nabla_H)\,\Phi + \Phi^{*}\overline{D}^{t} = - \Phi D$$

(continuity)

Compare EUL, ISL, NISL treatments of

$$\frac{d}{dt}(\ ) = \frac{\partial}{\partial t}(\ ) + \underset{\sim}{v}_H \cdot \nabla_H(\ \ )$$

with

$$\overline{(\ \ )}^{t} \doteq \frac{1}{2}\left[(\ )(\underset{\sim}{g}, t+\Delta t) + (\ )(\underset{\sim}{r}^{*}(t-\Delta t), t-\Delta t)\right]$$

$\underset{\sim}{g}$ for EUL

$\underset{\sim}{r}(t-\Delta t)$ for ISL

$\underset{\sim}{r}^{*}(t-\Delta t)$ for NISL

Right hand side terms ($f\hat{k}\times\underset{\sim}{v}_H, \Phi D, \underset{\sim}{v}_H\cdot\nabla_H(\ )$) are evaluated at mid-point of trajectory at time t. ($\Rightarrow$ nonlinear part is handled explicitly)

Handling the implicit (linear) part

Let $(\ )^{+} \sim (\ )(\underset{\sim}{g}, t+\Delta t)$

$(\ )^{-} \sim (\ )(\underset{\sim}{r}^{*}(t-\Delta t), t-\Delta t)$

$(\ )^{o} \sim$ evaluation at midpoint of trajectory

From time-discretized equations evaluate

$$\zeta^{+} = \hat{k} \cdot \nabla_H \times (\underset{\sim}{V}_H)^{+} \quad \text{and} \quad D^{+} = \nabla_H \cdot (\underset{\sim}{V}_H)^{+}$$

$$\Rightarrow \quad \zeta^{+} = L \qquad \text{(vorticity)} \qquad (1)$$

$$D^{+} + \Delta t \nabla_H^2 \Phi^{+} = M \qquad \text{(divergence)} \qquad (2)$$

$$\Phi^{+} + \Delta t\, \Phi^{*} D^{+} = Q \qquad \text{(continuity)} \qquad (3)$$

where L, M, Q are combinations of $(\ )^{o}$ and $(\ )^{-}$ terms.

(1) is an explicit equation for ζ^{+}.
Eliminating D^{+} from (2) and (3) ⇒

$$\Phi^{+} - \Delta t^2\, \Phi^{*} \nabla_H^2 \Phi^{+} = Q - \Delta t\, \Phi^{*} M$$

A Helmholtz equation that can be solved directly in spectral space since

$$\nabla_H^2 (\ ) \sim -\frac{n(n+1)}{a^2} (\ ).$$

ie.
The linear part of the problem (L.H.S. terms) can be solved directly.

EXPLICIT CORIOLIS TERM $\Rightarrow 2\Omega\Delta t \lesssim 1$

$$\Rightarrow \quad \Delta t \lesssim 1.9\ \mathrm{h}$$

SEMI-IMPLICIT CORIOLIS TERM

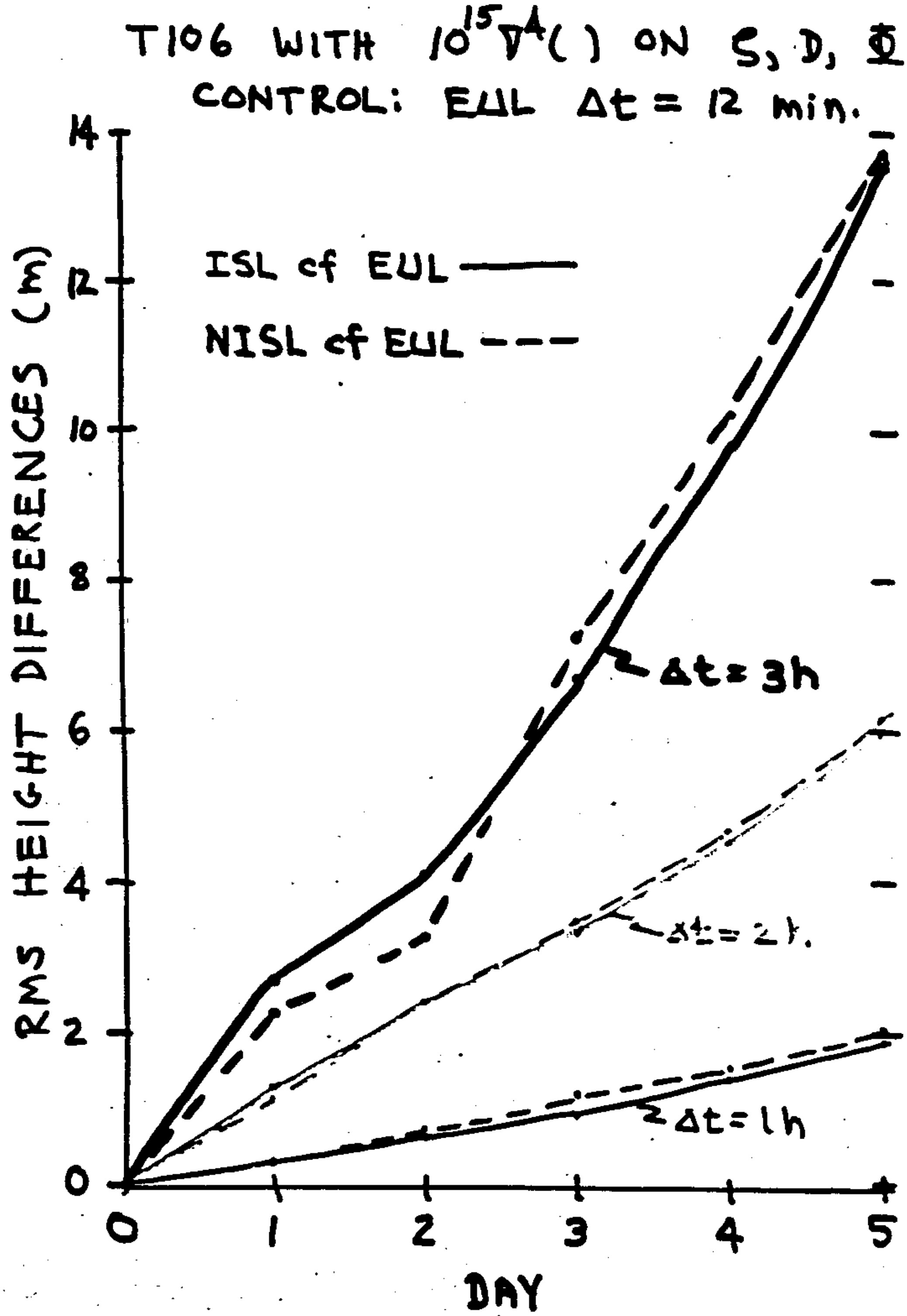
$$\frac{\partial \underset{\sim}{v}_H}{\partial t} + (\underset{\sim}{v}_H \cdot \nabla_H)\underset{\sim}{v}_H + \overline{f\,\hat{k}\times \underset{\sim}{v}_H}^{t} + \overline{\nabla_H \Phi}^{t} = \underset{\sim}{0}$$

(momentum)

$$\frac{\partial \Phi}{\partial t} + (\underset{\sim}{v}_H \cdot \nabla_H)\Phi + \Phi^{*}\,\overline{D}^{t} = -\,\Phi D$$

(continuity)

$\Rightarrow$ ISL, NISL unconditionally stable


T106 WITH $10^{15}\nabla^4(\ )$ ON ζ, D, Φ
CONTROL: EUL $\Delta t = 12$ min.
ISL of EUL
NISL of EUL
$\Delta t = 3h$
$\Delta t = 2h$
$\Delta t = 1h$
RMS HEIGHT DIFFERENCES (m)
DAY
0
2
4
6
8
10
12
14
1
3
5

Semi-Lagrangian model: current status at ECMWF

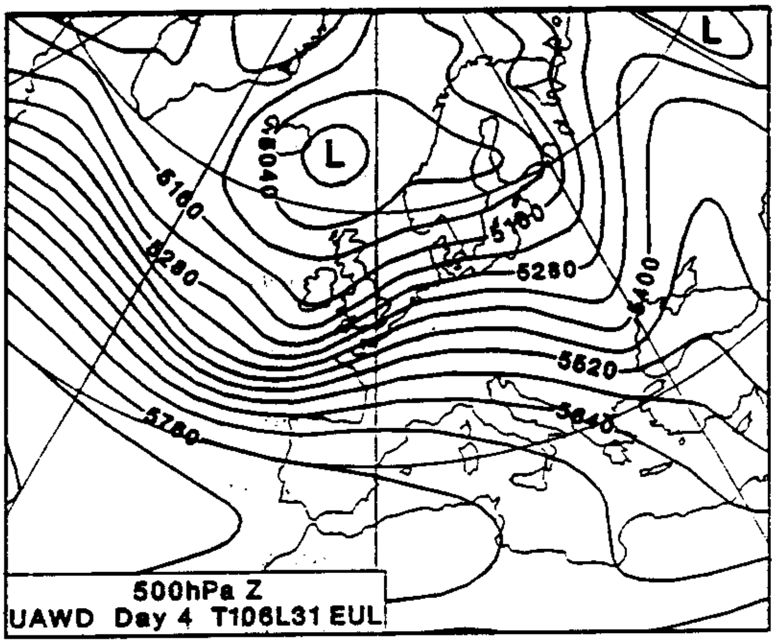
"Semi-implicit" terms are averaged along the trajectory, i. e. terms of the form $\Delta_{xx} X$ are evaluated as

$$(X(t-\Delta t)-X(t))|_{dep\ point} + (X(t+\Delta t)-X(t))|_{arr\ point}$$

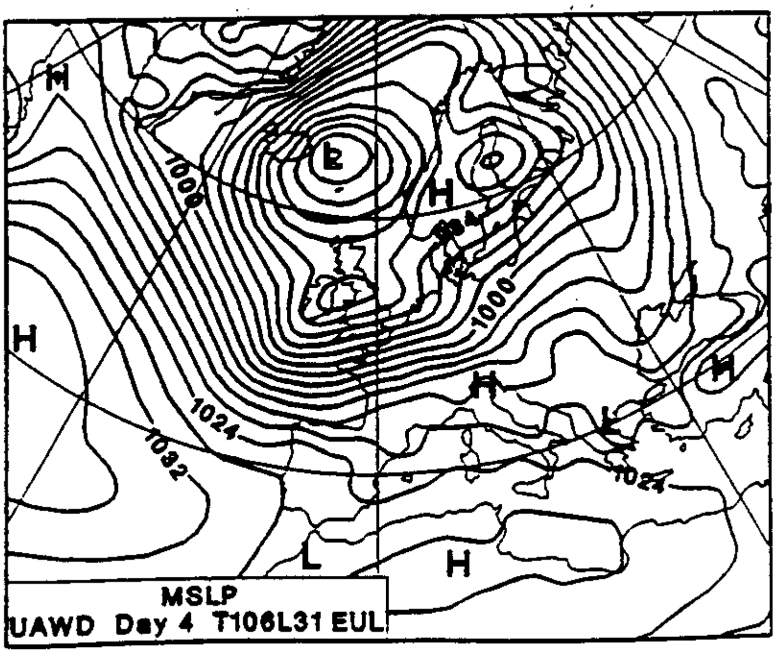
The right-hand sides of the momentum equations (Coriolis term + press gradient) are averaged along the trajectory as

$$\frac{1}{2}(X(t)|_{dep\ point} + X(t)|_{arr\ point})$$

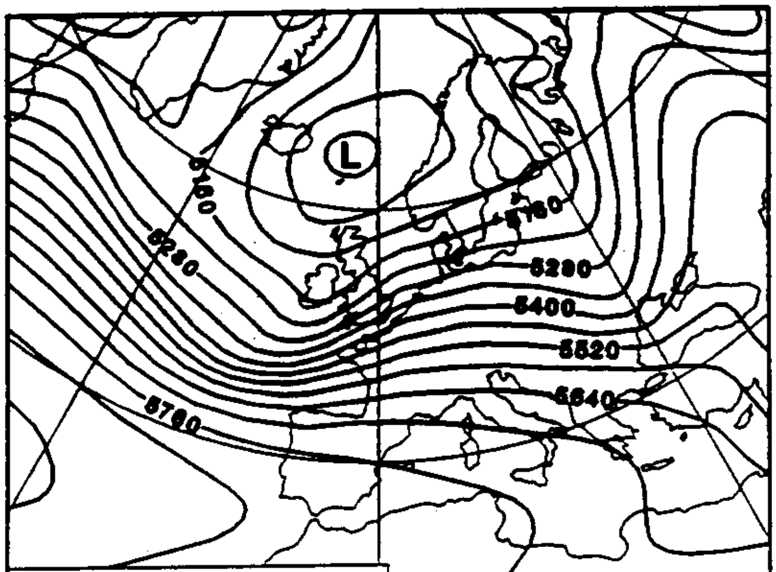
T106 L31 runs with Δt - 30 min, compared with Δt - 7 $\frac{1}{2}$ min for the Eulerian version



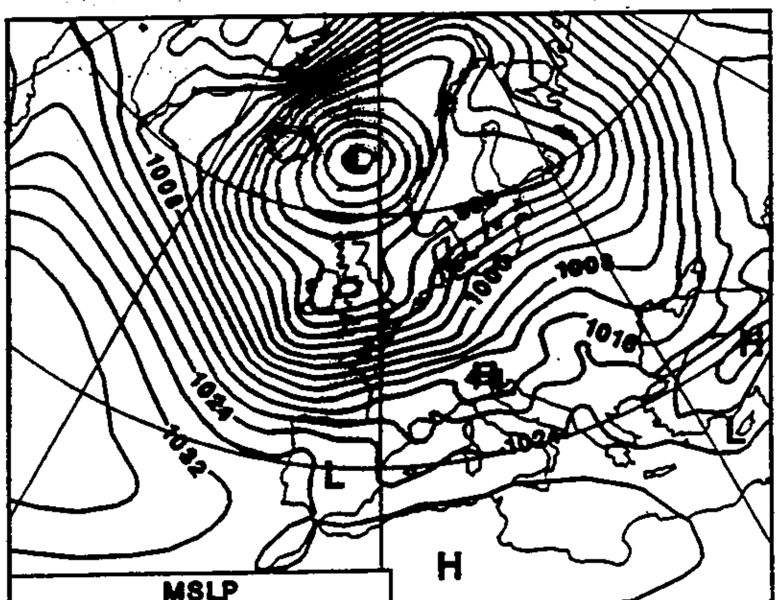
500hPa Z
UAWD Day 4 T106L31 EUL



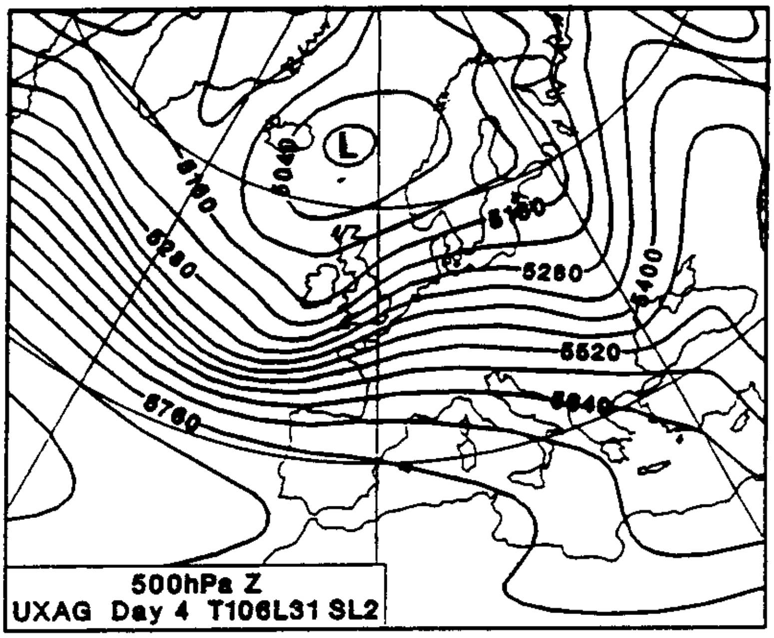
MSLP
UAWD Day 4 T106L31 EUL



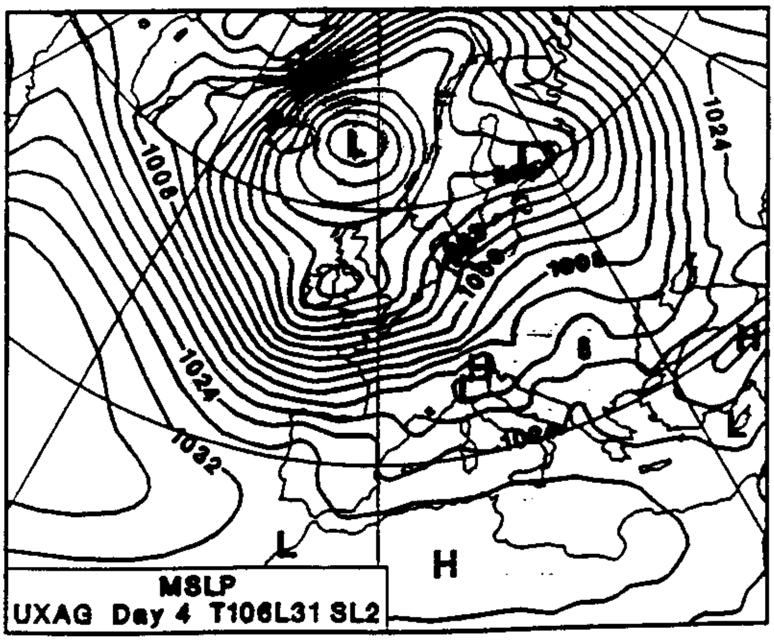
500hPa Z
UXAR Day 4 T106L31 SL1



MSLP
UXAR Day 4 T106L31 SL1



500hPa Z
UXAG Day 4 T106L31 SL2



MSLP
UXAG Day 4 T106L31 SL2

CONCLUSIONS

ISL, NISL are stable and accurate at high resolution with

$\Delta t_{\text{accuracy}} >> \Delta t_{\text{C.F.L. stability}}$

NISL has better conservation properties in extended integrations at lower resolution

Semi-Lagrangian operations vectorize fully on CRAY X-MP.

Efficiency gains comparable to those for grid point models (T126 $\Delta t = 60$ min cf EUL $\Delta t = 10$ min $\Rightarrow$ 4X, not fully optimized)

Momentum equation in vector $\underset{\sim}{v}_H$ formulation to avoid metric term instability near pole.