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"Semi-Lagrangian Integration Schemes for Atmospheric Models: A Review"

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SEMI-LAGRANGIAN INTEGRATION SCHEMES FOR ATMOSPHERIC MODELS - A REVIEW

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ABSTRACT

Recent developments in applying semi-Lagrangian methods to 2-d and 3-d atmospheric flows in both Cartesian and spherical geometries are reviewed. The models described are generally found to be one to two orders of magnitude more efficient than corresponding explicit leapfrog-based models run at the same resolution. The efficiency gains are presently more spectacular for 2-d models than 3-d ones, since they have been further developed, but considerable gains have already been realised for 3-d models with the hope of more to come. Operational applications to date include several 3-d weather forecast models and a 3-d emergency-response model. The challenge now is to develop $O(\Delta t^2)$-accurate two-time-level 3-d models, and obtain the further doubling of efficiency (demonstrated in 2-d) that accrues from using a *two*-time-level rather than a three-time-level scheme.

1. INTRODUCTION

Accurate and *timely* forecasts of weather elements are of great importance to both the economy and to public safety. Weather forecasters rely on guidance provided by Numerical Weather Prediction (NWP), a computer-intensive chain of operations beginning with the collection of data from around the world and culminating in the production of weather charts and computer-worded messages. At the heart of the system are the numerical models used to assimilate the data and to forecast future states of the atmosphere. The accuracy of the forecasts depends among other things on model resolution. Increased resolution, given the *real-time* constraints, can only be achieved by judiciously combining the most efficient numerical methods on the most powerful computers with the most appropriate programming techniques.

A longstanding problem in the integration of NWP models is that the maximum permissible timestep has been governed by considerations of stability rather than accuracy. For the integration to be stable, the timestep has to be so small that the time truncation error is very much smaller than the spatial truncation error, and it is therefore necessary to perform many more timesteps than would otherwise be the case. The choice of time integration scheme is therefore of crucial importance when designing an efficient weather forecast model, and this is also true when designing Environmental Emergency Response models. Early NWP models used an explicit leapfrog scheme, whose timestep is limited by the propagation speed of gravitational oscillations. By treating the linear terms responsible for these oscillations in an implicit manner, it is possible to lengthen the timestep by about a factor of six, at little additional cost and without degrading the accuracy of the solution [e.g. Kwizak and Robert (1971), Robert et al (1972)]. Such a scheme is termed *semi-implicit*. Nevertheless, the maximum stable timestep still remains much smaller than seems necessary from considerations of accuracy alone [Robert (1981)].

Discretization schemes based on a semi-Lagrangian treatment of advection have elicited considerable interest in the past decade for the efficient integration of weather-forecast models, since they offer the promise of allowing larger timesteps (with no loss of accuracy) than Eulerian-based advection schemes (whose timestep length is overly limited by considerations of stability). To achieve this end it is essential to associate a semi-Lagrangian treatment of advection with a sufficiently-stable treatment of the terms responsible for the propagation of gravitational oscillations. By associating a *semi-Lagrangian* treatment of advection with a *semi-implicit* treatment of gravitational oscillations, Robert (1981,1982) demonstrated a further increase of a factor of six in the maximum stable timestep, at some additional cost. This idea was demonstrated in the context of a *three*-time-level shallow-water finite-difference model in Cartesian geometry, and resulted in the time truncation errors finally being of the same order as the spatial ones.

Since Robert's seminal papers, the semi-Lagrangian methodology for advection-dominated fluid flow problems has been extended in several important ways. The purpose of this paper is to summarize the fundamentals of semi-Lagrangian advection (Section 2), to describe its application to coupled sets of equations (Section 3), to review recent extensions of the method (Section 4) not covered in the discussions of the previous sections, and to draw some conclusions (Section 5).

2. SEMI-LAGRANGIAN ADVECTION

In an *Eulerian* advection scheme an observer watches the world evolve around him at a fixed geographical point. Such schemes work well on regular Cartesian meshes (facilitating vectorisation and parallelisation of the resulting code), but often lead to overly-restrictive timesteps due to considerations of computational stability. In a *Lagrangian* advection scheme an observer watches the world evolve around him as he travels with a fluid particle. Such schemes can often use much larger timesteps than Eulerian ones, but have the disadvantage that an initially regularly-spaced set of particles will generally evolve to a highly-irregularly-spaced set at later times [Welander (1955)], and important features of the flow may consequently not be well represented. The idea behind *semi-Lagrangian* advection schemes is to try to get the best of both worlds: the regular resolution of Eulerian schemes and the enhanced stability of Lagrangian ones. This is achieved by using a different set of particles at each timestep, the set of particles being chosen such that they arrive exactly at the points of a regular Cartesian mesh at the end of the timestep. This idea gradually evolved from the pioneering work of Fjørtoft (1952,1955), Wiin-Nielsen (1959), Krishnamurti (1962), Sawyer (1963), Leith (1965) and Purnell (1976). Of the formulations introduced prior to that of Purnell (1976), those of Krishnamurti (1962) and Leith (1965) are perhaps the most similar to those used in present-day semi-Lagrangian advection schemes: however as formulated they are only valid for Courant numbers less than unity.

2 (a) Passive advection in 1-d

To present the basic idea behind the semi-Lagrangian method in its simplest context, we apply it to the 1-d advection equation

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{dx}{dt}\frac{\partial F}{\partial x} , \tag{1}$$

where

$$\frac{dx}{dt} = U(x,t) , \tag{2}$$

and $U(x,t)$ is a given function. Eq.(1) states that the scalar F is constant along a fluid path (or trajectory or characteristic). In Fig. 1, the *exact* trajectory in the (x-t) plane of the fluid particle that arrives at meshpoint x_m at time $t_n+\Delta t$ is denoted by the solid curve AC, and an *approximate* straight-line trajectory by the dashed line A'C. Let us assume that we know $F(x,t)$ at all meshpoints x_m at times $t_n-\Delta t$ and t_n, and that we wish to obtain values at the same meshpoints at time $t_n+\Delta t$. The essence of semi-Lagrangian advection is to approximately integrate (1) along the approximated fluid trajectory A'C. Thus

$$\frac{F(x_m,t_n+\Delta t) - F(x_m-2\alpha_m,t_n-\Delta t)}{2\Delta t} = 0 , \tag{3}$$

where α_m is the distance BD the particle travels in x in time Δt, when following the approximated space-time trajectory A'C. Thus if we know α_m, then the value of F at the arrival point x_m at time $t_n+\Delta t$ is just its value at the upstream point $x_m-2\alpha_m$ at time $t_n-\Delta t$. However we have not as yet determined α_m: even if we had, we only know F at meshpoints, and generally it still remains to evaluate F somewhere between meshpoints.

To determine α_m, note that U evaluated at the point B of Fig. 1 is just the inverse of the slope of the straight line A'C, and this gives the following $O(\Delta t^2)$ approximation to (2) [Robert (1981)]

$$\alpha_m = \Delta t\ U(x_m-\alpha_m,t_n) . \tag{4}$$

Eq.(4) may be iteratively solved for the displacement α_m, for example by

$$\alpha_m^{(k+1)} = \Delta t\ U(x_m-\alpha_m^{(k)},t_n) , \tag{5}$$

with some initial guess for $\alpha_m^{(0)}$, provided U can be evaluated between meshpoints. To evaluate F and U between meshpoints, spatial interpolation is used. The semi-Lagrangian algorithm for passive advection in 1-d in summary is thus:

(i) Solve (5) iteratively for the displacements α_m for all meshpoints x_m, using some initial guess (usually its value at the previous timestep), and an interpolation formula.

(ii) Evaluate F at upstream points $x_m-2\alpha_m$ at time $t_n-\Delta t$ using an interpolation formula.

(iii) Evaluate F at arrival points x_m at time $t_n+\Delta t$ using (3).

We defer the discussion of interpolation details to Section 2(d), and first generalize the above three-time-level algorithm to forced advection in several space dimensions [Section 2(b)], and to *two* time levels [Section 2(c)].

2 (b) Forced advection in multi-dimensions

Consider the forced advection problem

$$\frac{dF}{dt} + G(\mathbf{x},t) = R(\mathbf{x},t)\,, \tag{6}$$

where

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \mathbf{V}(\mathbf{x},t)\cdot\nabla F\,, \tag{7}$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{V}(\mathbf{x},t)\,, \tag{8}$$

$\mathbf{x}$ is the position vector (in 1-, 2- or 3-d), ∇ is the gradient operator, and G and R are forcing terms. A semi-Lagrangian approximation to (6) and (8) is then:

$$\frac{F^+ - F^-}{2\Delta t} + \frac{1}{2}\left[G^+ + G^-\right] = R^0\,, \tag{9}$$

$$\alpha = \Delta t\, \mathbf{V}(\mathbf{x}-\alpha,t)\,, \tag{10}$$

where the superscripts "+", "0" and "-" respectively denote evaluation at the arrival point $(\mathbf{x},t+\Delta t)$, the midpoint of the trajectory $(\mathbf{x}-\alpha,t)$ and the departure point $(\mathbf{x}-2\alpha,t-\Delta t)$. Here, $\mathbf{x}$ is now an arbitrary point of a regular (1-, 2- or 3-d) mesh.

The above is a centered $O(\Delta t^2)$ approximation to (6) and (8), where G is evaluated as the time average of its values at the endpoints of the trajectory, and R is evaluated at the midpoint of the trajectory. The trajectories are calculated by iteratively solving (10) for the *vector* displacements α in an analogous manner to the 1-d case for passive advection [eq.(5)]. If G is known (we assume that R is known since it involves evaluation at time t), then the algorithm proceeds in an analogous manner to the 1-d passive advection one and is thus:

(i) Solve (10) iteratively for the *vector* displacements α for all meshpoints $\mathbf{x}$, using some initial guess (usually its value at the previous timestep), and an interpolation formula.

(ii) Evaluate $F-\Delta tG$ at upstream points $\mathbf{x}-2\alpha$ at time $t-\Delta t$ using an interpolation formula. Evaluate $2\Delta tR$ at the midpoints $\mathbf{x}-\alpha$ of the trajectories at time t using an interpolation formula.

(iii) Evaluate F at arrival points $\mathbf{x}$ at time $t+\Delta t$ using

$$\begin{aligned} F(\mathbf{x},t+\Delta t) &= (F-\Delta tG)|_{(\mathbf{x}-2\alpha,t-\Delta t)} + 2\Delta tR|_{(\mathbf{x}-\alpha,t)} - \Delta tG|_{(\mathbf{x},t+\Delta t)} \\ &= (F-\Delta tG)^- + 2\Delta tR^0 - \Delta tG^+ \end{aligned} \tag{9'}$$

If G is not known at time $t+\Delta t$ (for instance if it involves another dependent variable in a set of coupled equations), then this leads to a coupling to other equations (more on this in Section 3).

2(c) Two-time-level advection schemes (and a pollutant-transport application)

Present semi-Lagrangian schemes are based on discretization over either *two* or *three* time levels, and thus far we have restricted our attention to three-time-level schemes. The principal advantage of ***two-time-level*** schemes over three-time-level ones is that they are potentially *twice* as fast. This is because three-time-level schemes require timesteps half the size of two-time-level ones for the same level of time truncation error [Temperton & Staniforth (1987)]. It is however important to maintain second-order accuracy in time in order to reap the full benefits of a two-time-level scheme (since enhanced stability with large timesteps is of no benefit if it is achieved at the expense of diminished accuracy). Early two-time-level schemes for NWP models unfortunately suffered from this deficiency [e.g. Bates & McDonald (1982), Bates (1984), McDonald (1986)]. The crucial issue is how to efficiently determine the trajectories to at least second-order accuracy in time [Staniforth & Pudykiewicz (1985), McDonald (1987)].

This problem arises in the context of self-advection of momentum. To see this we reexamine the algorithm of Section 2(a) for 1-d advection. Provided U is known at time t_n, *independently of F at the same time*, then it is possible to evaluate the trajectory, and then leapfrog the value of F from time $t_n-\Delta t$ to $t_n+\Delta t$, without knowing any value of F at time t_n.

Proceeding in this way, $F(t_n+3\Delta t)$ is then obtained using values of $F(t_n+\Delta t)$ and $U(t_n+2\Delta t)$. Thus we have two decoupled independent integrations, one using values of F at *even* timesteps and U at *odd* timesteps, the other using values of F at *odd* timesteps and U at *even* timesteps. Either of these two independent solutions is sufficient, thus halving the computational cost, and we obtain a two-time-level scheme (for the advected quantity F) by merely relabelling time levels t_n-Δt, t_n and $t_n+\Delta t$ respectively as t_n, $t_n+\Delta t/2$ and $t_n+\Delta t$ (see Fig. 2). Note that values of U (assumed known) only appear at time level $t_n+\Delta t/2$, and they are *solely* used to estimate the trajectories.

This is the essence of the 2-d advection-diffusion algorithm described and analyzed in Pudykiewicz & Staniforth (1984). It led to the development of a three-dimensional pollutant transport model [Pudykiewicz et al (1985)], where a family of chemical species are advected and diffused in the atmosphere using winds and diffusivities: these are either provided by a NWP model (for real-time prediction) or from analyzed data (for post-event simulations). This model is designed to provide real-time guidance in the event of an environmental accident and has been used to successfully simulate the dispersion of nuclear debris from the Chernobyl reactor accident [Pudykiewicz (1989)]. It has evolved into Canada's Environmental Emergency Response Model [Pudykiewicz (1990)].

Returning to the problem of self-advection of momentum, the above argument breaks down in the special case where F = U in (1) or F = V in (6), i.e. when the transported quantity U or V is advected by itself, as is the case for the momentum equations of fluid-dynamic problems in general, and NWP models in particular. This problem was addressed simultaneously and independently by Temperton & Staniforth (1987) and McDonald & Bates (1987), opening the way towards stable *and* accurate two-time-level schemes. The key idea here is to time-extrapolate the winds [with an $O(\Delta t^2)$-accurate extrapolator] to time-level $t+\Delta t/2$ using the known winds at time levels t and t-Δt: these winds are then used to obtain sufficiently-accurate [$O(\Delta t^2)$] estimates of the trajectories, which in turn are used to advance the

dependent variables from time level t to $t+\Delta t$. Thus the two-time-level algorithm to solve (6)-(8), analogous to the three-time-level one given by (9)-(10), is (see Fig.2)

$$\frac{F^+ - F^0}{\Delta t} + \frac{1}{2}\left[G^+ + G^0 \right] = R^{1/2} , \tag{11}$$

where

$$\alpha = \Delta t\, V^*(x-\alpha/2, t+\Delta t/2), \tag{12}$$

$$V^*(x,t+\Delta t/2) = (3/2)V(x,t+\Delta t) - (1/2)\, V(x,t-\Delta t) + O(\Delta t^2) , \tag{13}$$

the superscripts "+", "1/2" and "0" now respectively denote evaluation at the arrival point $(x,t+\Delta t)$, the midpoint of the trajectory $(x-\alpha/2, t+\Delta t/2)$ and the departure point $(x-\alpha,t)$, and α is still the distance the fluid particle is displaced in time Δt.

In the above formulation the evaluation of $R^{(1/2)}$ involves extrapolated quantities and therefore could potentially lead to instability. Temperton & Staniforth (1987) didn't find this to be a problem when some weak nonlinear metric effects were evaluated in this way in a shallow-water model integrated on a polar-stereographic projection, but it seems preferable to evaluate all non-advective terms (i.e. G in the above) as time averages along the trajectory whenever possible. [Subsequently Côté (1988) showed how to avoid evaluating the above-mentioned metric terms in terms of extrapolated quantities.] However Higgins and Bates (1990) report that evaluating the product term (of the geopotential perturbation and divergence) in the continuity equation of a global shallow-water model using time-extrapolated quantities [as in Bates et al (1990)] leads to the growth of computational noise, and the necessity to include divergence damping with a coefficient of appreciable magnitude. They found that by evaluating this product term using quantities at time t, rather than at time $t+\Delta t/2$, but still evaluating it at the trajectory midpoint $x-\alpha/2$, it is possible to integrate without the need for divergence damping. This mildly decenters the scheme and reduces the formal accuracy of this term to $O(\Delta t)$, but it is not a very large term in a shallow-water model. An alternative solution is to discretize the continuity equation in logarithmic form as in Côté & Staniforth (1990), at the price of making the elliptic-boundary-value problem of the semi-implicitly-treated terms mildly nonlinear: this has the advantage that it retains $O(\Delta t^2)$ accuracy because it is still a centered approximation. A further posible alternative is discussed in Section 4(d). Note that when *all* non-advective terms are evaluated implicitly as time averages along trajectories, then extrapolated quantities are used *solely* for the purpose of obtaining a sufficiently-accurate estimate of the trajectories.

Temperton & Staniforth (1987) examined several alternative ways of extrapolating quantities for the purpose of estimating trajectories. They found that those methods which keep a particle on its exact trajectory for solid-body rotation seem to give better results for the more general problem than those that do not. In particular, they found it advantageous to use a three-term extrapolator (using winds at times t, $t-\Delta t$, and $t-2\Delta t$ to obtain an extrapolated wind at $t+\Delta t/2$) instead of the two-term extrapolator (13). They also found that time extrapolating winds *along* the trajectory (their method 4) is less accurate than time extrapolating winds at meshpoints as in (13).

2 (d) Interpolation

A priori any interpolation could be used to evaluate F and U (or V) between meshpoints in the above algorithm. In practice the choice of interpolation formula has an important impact on the accuracy and efficiency of the method. Various polynomial interpolations have been tried including: linear; quadratic Lagrange; cubic Lagrange; cubic spline; and quintic Lagrange.

For step (ii) of the algorithm, it is found [see e.g. Purnell (1976), Bates & McDonald (1982), McDonald (1984), and Pudykiewicz & Staniforth (1984), for analysis] that cubic interpolation is a good compromise between accuracy and computational cost. While quadratic Lagrange interpolation is viable and was used in most of the early studies [e.g. Krishnamurti (1962,1969), Leith (1965), Mathur (1970,1974), Bates & McDonald (1982)], cubic interpolation has been widely adopted in recent studies [e.g. Robert et al (1985), McDonald

(1986), Bates & McDonald (1987), Ritchie (1988), Côté & Staniforth (1988), Bates et al (1990)]. Cubic interpolation gives 4th-order spatial accuracy with very little damping (it is very scale selective, affecting primarily the smallest scales), whereas linear interpolation [see McDonald (1984) for discussion] has unacceptably-large damping (it is also scale selective, but has a much less sharp response). Cubic spline interpolation has the useful property that it conserves mass [Bermejo (1990)]. Purser & Leslie (1988) recommend using at least 4th-order (i.e. cubic) interpolation, and have used quintic interpolation in their recent work [Leslie & Purser (1990)]. Improving the order of the interpolation increases the accuracy, but at additional cost, and the law of diminishing returns ultimately applies.

For step (i), the order of the interpolation is much less important. Theoretically McDonald (1987) has shown that one should use an interpolation of order one less than for step (ii), e.g. quadratic interpolation of U when using cubic interpolation of F. In practice however, in the context of both passive advection and coupled systems of equations in several spatial dimensions, it is found [Staniforth & Pudykiewicz (1985), Temperton & Staniforth (1987), Bates et al (1990)] that it is sufficient to use linear interpolation for the computation of the displacements,when using cubic interpolation for F, which is very economical. It is also found that there is no advantage in using more than two iterations for solving the displacement equation [step (i)]. McDonald (1987) has shown theoretically that it is not necessary to use the same order of interpolation for each iteration. For example, it is more economical and no less accurate to perform the first iteration using linear interpolation and the second using quadratic, than to use quadratic interpolation for both.

Pudykiewicz et al (1985) have shown that a sufficient condition for convergence of the iterative solution of step (i) is that Δt be smaller than the reciprocal of the maximum absolute value of the wind shear in any cordinate direction. Thus $\Delta t < [\max(|u_x|,|u_y|,|v_x|,|v_y|)]^{-1}$ for 2-d flow, where u and v are the two wind components. They estimated for atmospheric values that convergence is assured provided Δt is less than 3h, which is an order of magnitude larger than the maximum timestep permitted by Eulerian advection schemes in analogous circumstances.

2 (e) Stability and accuracy (and connection with other advection methods)

Analyses of the stability properties of the semi-Lagrangian advection scheme [e.g. Purnell (1976), Bates & McDonald (1982), McDonald (1984), Pudykiewicz & Staniforth (1984), Ritchie (1986,1987)] show that the maximum timestep isn't limited by the maximum wind speed, as is the case for Eulerian advection schemes, and consequently it is possible to stably integrate with Courant numbers ($C = |U|\Delta t/\Delta x$) that far exceed unity. To illustrate this point we reproduce (with permission) the results of Bermejo (1990) for the slotted cylinder test of Zalesak (1979). In Fig. 3a we show the slotted cylinder at initial time, and in Fig. 3b the corresponding result after *six* revolutions of solid-body rotation at uniform angular velocity about the domain center. The experiment was conducted using a cubic-spline interpolator at a Courant number of 4.2, which is considerably larger than that of an Eulerian advection scheme. This is recognized as being a challenging test, and the result is remarkably good. In particular the results illustrate the scheme's ability to handle sharp discontinuities without disastrous consequences (even though it was not designed specifically to do so) and the absence of noticeable dispersion problems (which are typically present for Eulerian advection schemes).

This behaviour in the presence of discontinuities or near discontinuities was also observed in the study of Kuo and Williams (1990) for a scale collapse problem. They concluded that semi-Lagrangian schemes are to be preferred to Eulerian schemes for this kind of problem since they have much smaller dispersion errors (which are localised around the shock) and can be integrated with significantly longer timesteps. Ritchie (1985) argued that the localisation of errors to the regions where the gradients are strongest when using semi-Lagrangian advection is a desirable property that may be advantageously exploited for the

treatment of moisture transport in NWP models, since large local gradients frequently occur in moisture fields (e.g. at fronts). He reported that semi-Lagrangian advection led to better results than Eulerian advection in the context of a 48h forecast.

In general it is found that semi-Lagrangian advection compares favorably with Eulerian advection with respect to accuracy, but it has the added advantage that this accuracy can be achieved at less computational cost, since models can be integrated stably with timesteps that far exceed the maximum-possible timesteps of Eulerian schemes. The aforementioned stability analyses show that semi-Lagrangian advection schemes have very good phase speeds with little numerical dispersion, but contrary to some Eulerian schemes (e.g. leapfrog-based schemes) there is some damping due to interpolation as discussed in Section 2(d). This damping is fortunately very scale selective (at least when using high-order interpolators). McCalpin (1988) has theoretically compared this damping with more traditional forms such as Laplacian and biharmonic dissipation, and derived some criteria to ensure that the damping due to semi-Lagrangian advection is less than that due to the more traditional forms. In practice Ritchie (1988) and Côté & Staniforth (1988) have found that semi-Lagrangian integration schemes have three times less damping than a typical Eulerian global medium-range forecast model run at typical resolution with a typical biharmonic dissipation.

Semi-Lagrangian advection is intimately connected with several other advection methods that have appeared in the literature over the years, including particle-in-cell [e.g. Raviart (1985)] and characteristic Galerkin [e.g. Morton (1985), Karpic & Peltier (1990)] methods. Indeed for uniform advection in 1-d, the simplest semi-Lagrangian advection scheme (using linear interpolation, and not recommended) is equivalent to both classical upwinding and to the simplest characteristic Galerkin method: and semi-Lagrangian advection using cubic-spline interpolation is equivalent to the higher-order characteristic Galerkin methods of Morton (1985) and Karpic & Peltier (1990), and also to a particle-in-cell method described in Eastwood (1987). Further, under more general conditions (including non-uniform advection in 2- and 3-d), Bermejo (1990) has shown that semi-Lagrangian advection using cubic-spline interpolation can be viewed as being a particle-in-cell finite-element method.

Several well-known Eulerian methods can also be interpreted as being special cases of semi-Lagrangian ones. Thus the Lax-Wendroff, Takacs (1985) 3rd-order, and Tremback et al (1987) schemes are respectively equivalent for 1-d uniform advection to semi-Lagrangian schemes with quadratic-Lagrange, cubic-Lagrange and n-th-order-Lagrange interpolation. Note however that these Eulerian methods are restricted to Courant numbers less than unity and are consequently less general than their semi-Lagrangian counterparts.

Although the semi-Lagrangian method is equivalent for uniform 1-d advection to several other methods, what distinguishes it from other methods is that it generalizes differently to non-uniform advection in multi-dimensions. The principal difference is the use of (10), introduced in Robert (1981), for the trajectory calculations. Of particular importance is that *the approximation of the trajectory equation (8) is $O(\Delta t^2)$ accurate*. It is possible to use a simpler, and cheaper, $O(\Delta t)$ accurate method to approximate the displacement equation (8) [as in e.g. Mathur (1970) and Bates & McDonald (1982)] but this can dramatically deteriorate the accuracy of the scheme, as shown by Staniforth & Pudykiewicz (1985) and Temperton & Staniforth (1987), and analysed by McDonald (1987). Consequently most, if not all, recent semi-Lagrangian schemes use an $O(\Delta t^2)$ method for discretizing the trajectory equation.

3. APPLICATION TO COUPLED SETS OF EQUATIONS

To illustrate how semi-Lagrangian advection can be advantagously used to solve coupled systems of equations, we describe its application to the discretization of the shallow-water equations

$$\frac{dU}{dt} + \phi_x - fV = 0, \qquad (14)$$

$$\frac{dV}{dt} + \phi_y + fU = 0, \qquad (15)$$

$$\frac{d \ln \phi}{dt} + U_x + V_y = 0, \qquad (16)$$

where U and V are the wind components, ϕ (=gz) is the geopotential height (i.e height multiplied by g) of the free surface of the fluid above a flat bottom, and f is the Coriolis parameter.

These equations are often used in NWP to test new numerical methods, since they are a 2-d prototype of the 3-d equations that govern atmospheric motions (they can be derived from them under certain simplifying assumptions). They share several important properties with their progenitor. A linearization of the equations reveals that there are two basic kinds of associated motion, slow-moving Rossby modes (which most affect the large-scale weather motions, and which move to leading order at the local wind speed) and small-amplitude fast-moving gravitational oscillations (which are inadequately represented at initial time due to the paucity of the observational network). From a numerical standpoint this has the important implication that the timestep of an explicit Eulerian scheme (e.g. leapfrog) is limited by the speed of the fastest-moving gravity mode. Since for atmospheric motions this speed is six times faster than those associated with the Rossby modes that govern the weather, this leads to timesteps that are six times shorter than those associated with an explicit treatment of advection. A time-implicit treatment of the pressure-gradient term of the vector momentum equation [2nd terms of (14) and (15)] and horizontal divergence of the continuity equation [2nd and 3rd terms of (16)], introduced in Kwizak and Robert (1971) and termed the *semi-implicit* scheme, allows stable integrations with no loss of accuracy using timesteps that are six times longer than that of the leapfrog scheme. The price to be paid for this increase in timestep length is the need to solve an elliptic-boundary-value problem once per timestep: nevertheless this improves efficiency by approximately a factor of five. Analysis shows that the maximum-possible timestep length is then limited by the Eulerian treatment of advection.

Early applications of semi-Lagrangian advection to coupled sets of equations [e.g. Krishnamurti (1962,1969), Leith (1965), Mathur (1970,1974), Mahrer & Pielke (1978)] didn't take advantage of the enhanced stability properties of the method, since the models were formulated in such a way that they were not, in the terminology of Bates and McDonald (1982), "multiply upstream" and so the Courant number (associated with the treatment of advection) was always less than unity. Nevertheless these studies did demonstrate that semi-Lagrangian advection is an acceptably-accurate method for advection. Robert (1981) reasoned that since semi-Lagrangian advection is stable for Courant numbers significantly larger than unity, it should be possible to associate a semi-Lagrangian treatment of vorticity advection with a semi-implicit treatment of the terms responsible for gravitational oscillations, and thereby obtain stable integrations with timesteps four to six times longer than that of a corresponding semi-implicit model employing an Eulerian treatment of advection. Using such a strategy he was able to obtain a computationally-stable solution with a 2 h timestep (approximately four times longer than that of a corresponding semi-implicit Eulerian model), although there was some evidence of a small noise problem at the western inflow boundary. It was also noted that there was an inconsistency in the formulation inasmuch as the advection terms in the divergence and continuity equations were not evaluated using the semi-Lagrangian technique, and the question of accuracy (as opposed to stability) was deferred to a later study.

It turned out [Robert (1982)] that the Robert (1981) integrations included a divergence diffusion term and a time filter, and that when these were removed an instability was observed. This was attributed to two factors: the explicit treatment of the Coriolis terms, and the application of the semi-Lagrangian technique to only the vorticity equation. To remedy these two deficiencies, Robert (1982) introduced a revised formulation using the primitive (instead of the differentiated vorticity/divergence) form of the equations together with a semi-Lagrangian treatment of all advected quantities and an implicit treatment of the Coriolis terms. This was done in the context of a three-time-level scheme where the metric terms of the momentum equation were treated explicitly at the midpoint of the trajectories [c.f. R in (9)] and all other non-advective terms as time averages of values at the endpoints of the trajectories [c.f. G in

(9)]. A stability analysis was given to demonstrate that this scheme should be stable with timesteps that exceed those of the gravitational, advective and inertial limits, and this was verified in sample integrations.

To illustrate the application of the semi-Lagrangian method we discretize (14)-(16) using the *two*-time-level semi-implicit semi-Lagrangian scheme of Temperton & Staniforth (1987), which permits a further doubling of efficiency with respect to the Robert (1982) algorithm at no extra cost. Thus

$$\frac{U^+ - U^0}{\Delta t} + \frac{\phi_x^+ + \phi_x^0}{2} - \frac{1}{2}\left[(f\,V)^+ + (f\,V)^0 \right] = 0 \,, \tag{17}$$

$$\frac{V^+ - V^0}{\Delta t} + \frac{\phi_y^+ + \phi_y^0}{2} + \frac{1}{2}\left[(f\,U)^+ + (f\,U)^0 \right] = 0 \,, \tag{18}$$

$$\frac{\ln\phi^+ - \ln\phi^0}{\Delta t} + \frac{1}{2}\left[(U_x + V_y)^+ + (U_x + V_y)^0 \right] = 0 \,, \tag{19}$$

where (14)-(16) have been discretized using (11) with R set to zero. Here advection terms are treated as time-differences along the trajectories and all other terms are treated as time-averages along the trajectories, leading to an $O(\Delta t^2)$-accurate scheme. Where traditional (three-time-level) semi-implicit time discretizations have an explicit time-treatment of the Coriolis terms, the above discretization employs a time-implicit treatment [as in Robert (1982)] in order to achieve an $O(\Delta t^2)$-accurate scheme: note that explicitly evaluating these terms at time t would not only reduce the accuracy to $O(\Delta t)$ but would also lead to instability. The trajectories are computed using the discretized equations (12)-(13) introduced by Temperton & Staniforth (1987) and McDonald & Bates (1987).

For the 1-d shallow-water equations it can be shown that there are three characteristic velocites in the coupled set, one being the local wind speed and associated with the slow Rossby modes that govern weather motions, the other two being associated with the propagation of gravitational oscillations. Thus the coupling of a semi-Lagrangian treatment of advection with a semi-implicit treatment of gravitational oscillations corresponds to integrating along the most important characteristic direction of the problem (i.e. that associated with the local windspeed): this is somewhat similar in spirit to a suggestion given on p. 860 of Morton (1985).

Eqs. (17)-(18) can be manipulated to give

$$U^{+} = -\frac{\Delta t}{2}\left[a\,\phi_x^{+} + b\,\phi_y^{+} \right] + \text{known}\,, \tag{20}$$

$$V^{+} = -\frac{\Delta t}{2}\left[a\,\phi_y^{+} - b\,\phi_x^{+} \right] + \text{known}\,, \tag{21}$$

where $a = [1+(f\Delta t/2)^2]^{-1}$ and $b = (f\Delta t/2)\,a$. Taking the divergence of (20)-(21) and eliminating this in (19) then leads to the elliptic-boundary-value problem

$$\left[(a\,\phi_x)_x + (a\,\phi_y)_y + (b\,\phi_y)_x - (b\,\phi_x)_y - 4\frac{\ln\phi}{\Delta t^2} \right]\Bigg|_{(x,t+\Delta t)} = \text{known}\,. \tag{22}$$

We now summarize the above as the following algorithm:

(i) Extrapolate V using (13) and solve (12) iteratively for the displacements α_m for all meshpoints x_m, using values at the previous timestep as initial guess , and an interpolation formula. Note that it is only necessary to perform this computation once per timestep, since the same trajectory is used for all three advected quantities.

(ii) Compute upstream (superscript 0) quantities in (17)-(19) by first computing derivative terms (e.g. U_x) and *then* evaluating quantities upstream (these two operations are *not* commutative!). Here it is more efficient to collect together all terms to be evaluated upstream in a given equation before interpolating (the distributive law applies).

(iii) Solve the elliptic-boundary-value problem (22) for $\phi(x,t+\Delta t)$.

(iv) Back substitute $\phi(x,t+\Delta t)$ into (20)-(21) to obtain $U(x,t+\Delta t)$ and $V(x,t+\Delta t)$.

The above elliptic-boundary-value problem is weakly non-linear and is solved iteratively using ϕ at the previous timestep as a first guess. It is only marginally more expensive to solve than the Helmholtz problem associated with traditional three-time-level semi-implicit Eulerian discretizations. The multi-grid method is particularly attractive for solving such elliptic-boundary-value problems because of its relatively-low arithmetic operation count. Such a solver is described in Barros et al (1990), and was successfully employed in the global model of Bates et al (1990) using a discretization scheme very similar to that described above.

For simplicity the above algorithm has been described for plane geometry. For spherical geometry, metric terms appear in the momentum equations (14)-(15). These can be trivially absorbed into the above formulation using the approach of Ritchie (1988) or Côté (1988).

Semi-Lagrangian advection has also been successfully coupled with the split-explicit method [Bates & McDonald (1982)] and the alternating-direction-implicit method [Bates (1984), Bates & McDonald (1987)]. Both of these approaches have the virtue of being simpler than the semi-implicit semi-Lagrangian one (there is no elliptic-boundary-value problem), but unfortunately they do not perform as well. The split-explicit-based model is less efficient [Bates (1984)] than the alternating-direction-implicit-based one, which in turn performs less well [Bates & McDonald (1987)] than the semi-implicit semi-Lagrangian model of McDonald (1986). This latter scheme was adopted in the study of McDonald & Bates (1989), and it was subsequently found [Bates et al (1990)] that its performance with large timesteps was not as good as had been hoped. This was attributed [McDonald (1989), Bates et al (1990)] to a time-splitting error introduced in the momentum equation associated with the Coriolis terms. To date it appears that the best schemes arise from associating semi-Lagrangian advection with a semi-implicit scheme, and that timesplitting is best avoided.

4. FURTHER ADVANCES

When Robert (1981) proposed associating a semi-Lagrangian treatment of advection with a semi-implicit treatment of gravitational oscillations, it was thought that this approach was restricted to three-time-level schemes in Cartesian geometry using a finite-difference discretization. This has happily proved not to be the case, and in this section we discuss some important extensions of the approach. Although important, the extension to *two-time-level schemes* has already been discussed in some detail, and will therefore only be briefly discussed in this section in the context of other extensions.

4 (a) Finite-element discretizations and variable-resolution

Pudykiewicz & Staniforth (1984) coupled semi-Lagrangian advection with a uniform-resolution finite-element discretization of the diffusion terms in the solution of the 2-d advection-diffusion equation, and this was extended to the 3-d case in Pudykiewicz et al (1985). Staniforth & Temperton (1986) extended the methodology in the context of a coupled system of equations (the shallow-water equations) in two ways. Firstly they showed that in this context the semi-Lagrangian method can be coupled to a spatial discretization scheme other than a finite-difference one, viz. a *finite-element* discretization, and secondly that it can also be applied on a *variable-resolution* Cartesian mesh. A set of comparative tests demonstrated that with a six-times-longer timestep it is as accurate as its analogous semi-implicit Eulerian version [Staniforth & Mitchell (1978)] when run with its maximium-possible timestep (which in turn uses a six-times-longer timestep than an Eulerian leapfrog scheme).

A further doubling of efficiency was then demonstrated in Temperton & Staniforth (1987) by replacing the three-time-level scheme of the Staniforth & Temperton (1986) model with a two-time-level one. Both these models use a differentiated (vorticity-divergence) form of the governing equations. This has the advantage of easily allowing variable resolution, but has the disadvantage of incurring additional interpolations and the need to solve two Poisson

problems, resulting in an approximately 20% overhead when compared to the ideal. This overhead can be eliminated by the use of the *pseudo-staggered* scheme proposed in Côté et al (1990) with no loss of accuracy.

4 (b) Non-interpolating schemes

The interpolation in a semi-Lagrangian scheme, as mentioned previously, leads to some damping of the smallest scales. While this damping is very scale selective, it may be argued that it would unacceptably degrade accuracy for very long simulations (e.g. many decades in the context of a climate model). To address this problem Ritchie (1986) proposed a non-interpolating version of semi-Lagrangian advection. The basic idea here is to decompose the trajectory vector into the sum of two vectors, one of which goes to the nearest meshpoint, the other being the residual. Advection along the first trajectory is done via a semi-Lagrangian technique that displaces a field from one meshpoint to another (and therefore requires no interpolation), while the advection along the second vector is done via an undamped three-time-level Eulerian approach such that the residual Courant number is always less than one. Thus the attractive stability properties of interpolating semi-Lagrangian advection are maintained but without the consequent damping.The non-interpolating scheme is also more efficient than a *three*-time level interpolating one, since there are only half the number of interpolations per timestep (i.e. there are no longer any interpolations associated with the middle time level). Ritchie (1986) demonstrated the non-interpolating scheme for a gridpoint shallow-water model on a polar-stereographic projection, and found it to be more efficient and slightly more accurate than an interpolating scheme run at the same resolution.

The non-interpolating methodology is not restricted to gridpoint discretizations and has also been successfully applied to spectral discretizations [Ritchie (1988,1990)]. This offers the possibility of retrofitting a non-interpolating semi-Lagrangian scheme into existing spectral models: there is however a minor technical complication inasmuch as present spectral models generally use the differentiated vorticity-divergence form of the equations whereas non-interpolating(and interpolating for that matter) semi-Lagrangian spectral models employ the primitive form, and this necessitates some changes to the spectral part of the formulation.

There are however a couple of disadvantages of the non-interpolating approach for problems where the small damping of the interpolating scheme is acceptable: this is generally the case for NWP applications, but probably not so for climate models (since they are generally run at much lower resolution and the damping is consequently more severe). Firstly, the non-interpolating method has the dispersive properties of its Eulerian component, which are not generally as good as those of interpolating semi-Lagrangian advection schemes. Secondly, being based on a three-time-level scheme it is potentially twice as expensive as a two-time-level interpolating scheme.

The scheme proposed in Rančić and Sindjić (1989) is advertised as being a non-interpolating one, but this is not in fact the case. Two schemes are derived for uniform advection in 1-d based on the Lax-Wendroff and Takacs (1985) schemes. A close examination of these schemes reveals that the Lax-Wendroff-based scheme is identical to a semi-Lagrangian one with quadratic Lagrange interpolation, whereas the Takacs-based scheme is identical to a semi-Lagrangian one using cubic Lagrange interpolation. A simple and interesting idea, somewhat buried in the detail of the Rančić and Sindjić (1989) paper, is to show how to make a two-time-level Eulerian advection scheme stable for Courant numbers greater than one. The idea however unfortunately seems to be limited to the 1-d case, since it is predicated on the assumption that a particle passes over a meshpoint at some time during the time interval of the timestep, which assumption does not hold for multiply-upstream particles in 2-d. It is of course possible to split the 2-d advection problem into two passes of the 1-d algorithm, but this then has the disadvantage that it usually introduces significant splitting errors for large timesteps [see e.g. Williamson & Rasch (1989)].

An alternative way of viewing the non-interpolating formalism of Ritchie (1986) is presented in Smolarkiewicz and Rasch (1990). They showed that it is possible to convert any advection algorithm into a semi-Lagrangian framework, thus permitting the use of much larger timesteps with the scheme for little additional cost. This interesting realisation is of potential benefit for models whose maximum timestep is limited by an Eulerian treatment of advection. To demonstrate this idea they successfully extended the stability limit of the Tremback et al (1987) family of algorithms. In so doing they obtained a family of schemes which is equivalent to using a time-split semi-Lagrangian scheme with Lagrange interpolation. They also successfully extended the stability limit of a family of positive-definite monotone advection algorithms. However after comparing results with those of semi-Lagrangian algorithms, they concluded that for problems where small undershoots and slight lack of conservation are acceptable, this family of positive-definite monotone algorithms cannot compete.

4 (c) Shape-preserving and monotonic interpolation

Although most authors have adopted polynomial schemes for the interpolatory steps of semi-Lagrangian schemes, other interpolators are also possible. Williamson & Rasch (1989) and Rasch & Williamson (1990a) have examined several different possible interpolators, designed to better preserve the shape of advected fields and to maintain monotinicity. They performed experiments in both Cartesian and spherical geometry, and concluded that the approach is viable.

Rasch & Williamson (1990b) pursued this work in the context of climate simulations by using shape-preserving interpolation for moisture transport. They indicated a preference for semi-Lagrangian advection, on the basis that although the global errors of semi-Lagrangian and Eulerian advection are comparable, the semi-Lagrangian method can be made to have smaller local errors by using shape-preserving interpolation. However they caution the reader that the two methods produce very different climatologies and that the numerical problems of moisture transport are still far from being resolved. In particular they note that the problem of moisture overshoot is just as important as that of undershoot, and that this issue still remains to be addressed. These problems are also of importance in NWP applications, but are probably less severe due to the generally-higher resolution of NWP models.

The principal difficulty with the shape-preserving and monotonic approaches appears to be to decide how to precisely determine the required attributes of the interpolator, and how to tailor it to respect them, since there is no universal best choice.

4 (d) Spherical geometry

The convergence of the meridians at the poles of an Eulerian finite-difference model in spherical geometry leads to unacceptably-small timesteps being required in order to maintain computational stability. The usual approach to this problem is to somehow filter the dependent variables in the vicinity of the poles. While this procedure does relax the stability constraint, it unfortunately deteriorates accuracy [e.g. Purser (1988)]. Ritchie (1987) demonstrated that it is possible to passively advect a scalar over the pole using semi-Lagrangian advection with timesteps far exceeding the limiting timestep of Eulerian advection schemes. This paved the way to applications in global *spherical* geometry. The first such application was to couple semi-Lagrangian advection with a *spectral* representation (i.e. expansion in terms of spherical harmonics) of the dependent variables to solve the shallow-water equations over the sphere [Ritchie (1988)]. A new problem arose here associated with the stable advection of a *vector* quantity (momentum). The solution proposed in Ritchie (1988) is to introduce a tangent plane to avoid a weak instability due to a metric term. The diagnosis of this problem, which led to the tangent plane algorithm, is described in Desharnais & Robert (1990).

An alternative solution, proposed by Côté (1988), is to use a Lagrange multiplier method. In this approach the horizontal momentum equations of the shallow-water equations on the sphere are written in 3-d vector form using the undetermined Lagrange multiplier

method. These equations are time discretized directly, and the Lagrange multiplier is then determined from the *discretized* equations to ensure that motion is constrained to follow the surface of the sphere. This is in contrast with the usual approach where the Lagrange multiplier is first determined from the *continuous* equations, followed by a discretization of the resulting equations. The procedure is applicable to any coordinate system and can also be extended to multilevel models.

Both methods give good results which are almost indistinguishable in practice. More recently Bates et al (1990) have described an approach based on the discretization of the vector form of the momentum equation. Although they state that their vector discretization is somewhat different from the Lagrange multiplier method of Côté (1988), it can be shown that the resulting algorithms are identical. It also turns out that the tangent-plane algorithm of Ritchie (1988) is identical to the Lagrange-multiplier one in the context of a two-time-level scheme.

Ritchie (1988) successfully integrated his shallow-water model with a timestep six-times longer than that of the limiting timestep of the corresponding Eulerian semi-implicit spectral model (which in turn uses a six-times-longer timestep than that of an Eulerian leapfrog model). Côté & Staniforth (1988) then further doubled the efficiency of the Ritchie (1988) model, by replacing its three-time-level scheme by a two-time-level one analogous to that of Temperton & Staniforth (1987) for Cartesian geometry.

The spectral method (i.e. expansion of the dependent variables in terms of spherical harmonics) has been the method of choice during the past decade for the horizontal discretization of global NWP models. However the spectral method ultimately becomes very expensive at high-enough resolution, due to the $O(N^3)$ cost of computing the Legendre transforms, where N is the number of degrees of freedom around a latitude circle. Finite-difference and finite-element methods on the other hand have a potential $O(N^2)$ cost. This, and the success of the semi-Lagrangian method in addressing the pole problem, suggests that it would be highly advantageous to use a semi-Lagrangian treatment of advection in a finite-difference or finite-element global model for medium-range forecasting.

A first tentative step in this direction was taken by McDonald and Bates (1989), who introduced semi-Lagrangian advection into a two-time-level global semi-implicit shallow-water model using the time discretization of McDonald (1986). Although their scheme was stable with time steps that exceeded the limiting time step of an Eulerian treatment of advection, the enhanced stability was unfortunately achieved at the expense of accuracy. The degradation of accuracy is attributable to a time-splitting error introduced in the momentum equation associated with the Coriolis terms. The solution to this problem is to avoid time-splitting altogether and then the algorithm [Bates et al (1990)] is very similar to that employed in Ritchie (1988) and Côté & Staniforth (1988), and results in significant improvements in accuracy for large timesteps. Nevertheless, Bates et al (1990) found it necessary to use divergence damping (with what appears to be a rather large coefficient) in order to integrate to 5 days, suggesting that there still remain some accuracy and/or stability problems.

Côté & Staniforth (1990) replaced the spectral discretization in the Côté & Staniforth (1988) model by a pseudo-staggered finite-element one [analogous to that described in Côté et al (1990)], to obtain a two-time-level semi-implicit semi-Lagrangian global model of the shallow-water primitive equations. Its performance at comparable resolution matched that of their corresponding 1988 model based on a spectral discretization, and this performance was achieved without recourse to any divergence damping, in contradistinction to the result reported in Bates et al (1990).

By evaluating the product term (of the geopotential perturbation and divergence) in the continuity equation using quantities at time t rather than at time $t+\Delta t/2$, but still evaluating it at the trajectory midpoint, Higgins & Bates (1990) show that it is possible to integrate the Bates et al (1990) model with no divergence damping, although this formally reduces the accuracy of the treatment of this term to $O(\Delta t)$. This result strongly suggests that the source of weak instability observed in the Bates et al (1990) results (without divergence damping) is somehow due to this term, but it still remains to explain why. We believe the explanation may be found in a stability analysis given in Côté and Staniforth (1988) for a somewhat similar time discretization, which analysis is valid for the Bates et al (1990) formulation.

Côté and Staniforth (1988) showed that such a time discretization is only stable provided $\phi^* > \phi_{max}$, where ϕ^* is the reference geopotential of the semi-implicit scheme and ϕ_{max} is the maximum-possible value of the geopotential. Thus where this condition is violated, such a time discretization is likely to be unstable, and this is most likely to occur in the tropics where the geopotential is generally largest. We believe that the ϕ^* of the Bates et al (1990) integrations (without divergence damping) is probably an average value of the geopotential (rather than its maximum value) and thus violates this stability criterion. An examination of the divergence-damping-free result [given in Higgins & Bates (1990)] of the Bates et al (1990) formulation reveals that the forecast is unstable in the tropics, but stable in the extra-tropics, consistent with the above argument. We therefore speculate that the Bates et al (1990) formulation could be stabilised by merely increasing the value of the reference geopotential, and that this solution would be preferable to the one proposed by Higgins & Bates (1990) since it is $O(\Delta t^2)$ [rather than $O(\Delta t)$] accurate.

4 (e) 3-d NWP applications

Thus far we have mostly discussed the use of semi-Lagrangian advection for extending the limiting timestep of 2-d applications for NWP. To be useful the method must also be applicable in 3-d. A first step in this direction was taken in Bates & McDonald (1982), where a semi-Lagrangian treatment of *horizontal* advection in a 3-d (baroclinic primitive equations) model was coupled with a split-explicit time scheme in the Irish Meteorological Service's operational model of the time. This was the first scheme to demonstrate the enhanced stability of semi-Lagrangian advection in a 3-d model, and the first to be used operationally. However it is only $O(\Delta t)$ accurate and although stable with long timesteps, the increase in timestep is consequently very much limited by accuracy considerations.

The 3-d model formulated in McDonald (1986) and improved in McDonald & Bates (1987) [by modifying the trajectory calculations to make them $O(\Delta t^2)$ accurate, which improves accuracy and allows longer timesteps] is four times more efficient than the Bates & McDonald (1982) model for the same accuracy and replaced it operationally. Nevertheless the resulting scheme still has some $O(\Delta t)$ truncation errors and the timestep is therefore smaller than it would be for an $O(\Delta t^2)$ scheme [see discussion in the preceding sub-section of the McDonald (1986) scheme in the context of a global model]. Bates & McDonald (1987) have also coupled a semi-Lagrangian treatment of horizontal advection in a 3-d model with the alternating-direction method, but found in comparative experiments that it doesn't perform as well as the McDonald & Bates (1987) scheme.

Robert et al (1985) introduced a three-time-level *$O(\Delta t^2)$-accurate* 3-d limited-area gridpoint model with a semi-Lagrangian treatment of *horizontal* advection, and were able to successfully integrate with longer timesteps than had hitherto been possible: however the model had no mountains and a very simple parameterization of physical processes. This semi-Lagrangian semi-implicit model does however demonstrate the practical importance of achieving a truly $O(\Delta t^2)$-accurate scheme. Although it employs a three-time-level scheme, it is only marginally more costly per timestep than the *nominally* two-time-level scheme of McDonald & Bates (1987) [which has several sub-steps] but can be integrated with longer timesteps. In principle it should be possible to further double the efficiency of the Robert et al (1985) algorithm by using a two-time-level scheme. While such an improvement has been achieved in 2-d [e.g Côté & Staniforth (1988), Bates et al (1990)] the extension to 3-d applications remains to be demonstrated.

A somewhat similar model to the Robert et al (1985) one, but with mountains included, is described in Kaas (1987). It was reported that when strong winds blow over steep mountains, instabilities may appear if the linear part $[\nabla(\phi+RT_0 \ln p_s)]$ of the horizontal pressure gradient term in sigma coordinates is evaluated as the average of values at the endpoints $[(\mathbf{x},t+\Delta t), (\mathbf{x}-2\alpha,t-\Delta t)]$ of the trajectory, but the nonlinear part $[R(T-T_0)\nabla \ln p_s]$ is evaluated at the midpoint $(\mathbf{x}-\alpha,t)$. This behaviour was attributed to a lack of balance (in the discrete approximation) between two large terms of opposite sign, due to their being evaluated at different geographical points. The reported solution to this problem is to evaluate the non-linear part as the average of its values at the geographical points associated with arrival $(\mathbf{x})$ and departure $(\mathbf{x}-2\alpha)$, both values being taken at the intermediate time level t. This stratagem has also recently been incorporated in the models of Robert et al (1985), Tanguay et al (1989) and Ritchie (1990).

While this approach appreciably mitigates the problem, it is not at all clear that it resolves it completely. Coiffier et al (1987) have studied it in the context of a 2-d linearized baroclinic model, and show that the use of semi-Lagrangian advection with *large* timesteps leads to an incorrect steady-state solution when the model is orographically forced. Their analysis to explain this behaviour also applies to the formulation proposed by Kaas (1987). It suggests that the seriousness of the problem is a function of timestep, windspeed and detail (the larger the timestep and windspeed, and the more detailed the orography, the worse is the problem), and of whether the time scheme is a two- or three-time-level one (two-time-level schemes are better since the problem first occurs with timesteps twice as long as those of three-time-level schemes). Although this problem has not prevented semi-Lagrangian models from being integrated with larger timesteps than Eulerian ones while obtaining results of equivalent accuracy, it does warrant further investigation.

The timesteps of the 3-d above-mentioned models are limited by the stability of an explicit Eulerian treatment of *vertical* advection: or put another way, vertical resolution is limited when using a large timestep [see Ritchie (1990) for an example]. This is an important limitation. An ever-increasing emphasis in model development is being put on the parameterization of physical processes in general, and that of the moist turbulent planetary-boundary-layer in particular, and results in ever-increasing demands on vertical resolution . To remove this limitation, Tanguay et al (1989) proposed a three-time-level model that uses semi-Lagrangian advection in all three space dimensions: this finite-element regional model uses a timestep which is three-times longer than that of the corresponding Eulerian version [Staniforth & Daley (1979)]. It is currently used by the Canadian Meteorological Center to operationally produce weather forecasts to 48 h twice daily.

Ritchie (1990) has recently introduced semi-Lagrangian advection into a three-time-level 3-d global spectral model in two different ways. The first uses an interpolating semi-Lagrangian scheme in all three dimensions, as in Tanguay et al (1989), whereas the second uses an interpolating semi-Lagrangian scheme for horizontal (2-d) advection and a non-interpolating scheme for vertical advection. These schemes are currently being introduced into the European Centre for Medium-range Weather Forecasts' spectral model. He reports that the latter scheme is more accurate than the former for the experiments he conducted, due to the former unduly smoothing fields in the vertical around the tropopause. The seriousness of this smoothing is a function of the resolution employed and of the order of the interpolator. The trend to higher vertical resolution should diminish the importance of this source of error in the future. In the meantime it suggests that higher-order vertical (i.e. quintic instead of cubic) interpolation is possibly warranted as proposed in Leslie & Purser (1990).

4 (f) Higher resolution and non-hydrostatic systems

As computers become ever more powerful, it becomes possible to run models at higher and higher resolution. A time is approaching [Daley (1988)] when it will be possible to run current hydrostatic baroclinic primitive-equation weather forecast models at resolutions for

which the hydrostatic assumption can no longer be assumed to hold, thus motivating the integration of non-hydrostatic systems of equations. Such systems admit *acoustic* modes, which travel much faster than either Rossby or gravity modes. Consequently if care is not exercised, the limiting timestep will be even more restrictive than that associated with an explicit primitive equations model.

Since the acoustic modes carry very little energy, it is permissible to slow them down by the use of a time-implicit treatment of the terms responsible for their existence, by analogy with the retarding of the gravity modes by the semi-implicit scheme. This is the approach taken by Tanguay et al (1990), who generalize the semi-implicit semi-Lagrangian methodology for the hydrostatic primitive equations to the non-hydrostatic case. They show that it is possible to integrate the fully-compressible non-hydrostatic equations (which are presumably more correct) for little additional cost, opening the way to highly-efficient non-hydrostatic models. Note that this a proof-of-concept study, since the model employed has several important deficiencies with respect to operational hydrostatic forecast models: it has no mountains, an extremely simple physical parameterization of physical processes, and very low vertical resolution (particularly in the planetary boundary layer) such that the timestep is not unduly limited by the Eulerian treatment of vertical advection (it is only the horizontal advection that is treated in a semi-Lagrangian manner). Nevertheless it represents a very important first step towards highly-efficient non-hydrostatic forecast models.

Increasing the resolution not only has important implications for the appropriate choice of governing equations, but also for the relative order of the temporal and spatial truncation errors. To date it has been found advantageous to couple semi-Lagrangian advection with a semi-implicit time scheme. This allows integration with larger timesteps than would otherwise be possible, chosen such that the temporal and spatial truncation errors are of the same magnitude. We are thus presently in the position where the $O(\Delta t^2)$ temporal truncation errors are approximately equal in magnitude to the $O(\Delta x^4)$ spatial ones (assuming cubic interpolation in the semi-Lagrangian discretization of advection). For sake of argument, assume that this Δt is four times larger than the limiting timestep of a corresponding semi-implicit model with an $O(\Delta x^4)$-accurate Eulerian advection scheme. We now ask the important question, what will be the size of the timestep (chosen such that the temporal and spatial truncation errors are of the same magnitude) of the semi-implicit semi-Lagrangian model for successive doublings of the spatial resolution, and how will this timestep compare to that of the corresponding semi-implicit Eulerian model?

For the first doubling of resolution, the spatial truncation errors wil be decreased by a factor of 16 ($=2^4$). To ensure that the temporal truncation errors will be of the same magnitude as the spatial ones it is therefore necessary to reduce them also by a factor of 16 ($=4^2$), which implies *reducing Δt by a factor of 4*. Comparing this timestep now with that of the corresponding Eulerian model for the same doubling of resolution (where the timestep is halved to respect the CFL stability criterion), we see that it is only twice as large (whereas before the doubling of resolution it was four times larger) and the relative advantage of the semi-Lagrangian timestep is thus halved.

Repeating the argument for a second doubling we see that the timestep of the semi-Lagrangian model now equals that of the Eulerian one, and there is no longer any advantage of timestep length for the semi-Lagrangian model. This is because the timestep of the semi-Lagrangian model is limited by accuracy considerations (it cannot be any larger otherwise the temporal truncation errors would dominate), and it so happens that the timestep of the Eulerian model is now limited by both stability *and* accuracy considerations. For any further increase in resolution beyond this critical resolution, the timesteps of the two models will be identical since they will be determined solely by accuracy considerations. So we conclude that for this example there is no timestep advantage for the semi-Lagrangian model at a quadrupling or more of resolution.

Since the semi-Lagrangian model is somewhat more expensive per timestep it is therefore debatable as to whether it would be advantageous in the above example to use a semi-Lagrangian treatment of advection at such resolutions rather than an Eulerian one [although one might argue that semi-Lagrangian advection might still be advantageous (particularly for moisture transport) since there are fewer dispersion problems]. The important implication of the above argument is that when the resolution of the Eulerian model is sufficiently high that the timestep is governed by its $O(\Delta t^2)$ temporal truncation error (rather than by its CFL stability criterion), it will become important to increase the order of the time discretization of the corresponding semi-Lagrangian model. How this might be done is discussed in McDonald (1987).

That said, there is perhaps a weakness in the above argument. We have assumed that the dominant source of spatial truncation error is $O(\Delta x^4)$, and for a realistic NWP model this means that we are implicitly assuming that the spatial truncation errors associated with the physical parameterization also behave as $O(\Delta x^4)$. In reality it is highly unlikely that a doubling of resolution reduces these errors by a factor of sixteen, and it is far more likely that they behave as $O(\Delta x^2)$, in which case this would be the leading source of horizontal truncation error and the efficiency advantage of the semi-Lagrangian model would be maintained for all resolutions.

A further important consideration is that the *temporal* discretization associated with the incorporation of physical processes in the models also be of higher order than the present $O(\Delta t)$ ones in order to benefit from the enhanced stability of semi-Lagrangian schemes. Present physical parameterizations appear to be rather sensitive to timestep length, and this problem requires further work.

5. CONCLUSIONS

During the past decade much progress has been achieved in using semi-Lagrangian advection to improve the efficiency of numerical models of the atmosphere. In this paper we have reviewed the semi-Lagrangian literature for atmospheric models, and drawn the following conclusions:

1) The semi-Lagrangian methodology has been extended from finite-difference applications in Cartesian geometry to finite-difference, finite-element and spectral applications in both Cartesian and spherical geometry.

2) The extension to finite-difference and finite-element discretizations for global applications is particularly noteworthy, since such discretizations are asymptotically much cheaper at high resolution than the present method of choice, the spectral method.

3) At the present state of development the efficiency gains are more spectacular in 2-d than in 3-d.

4) Two-time-level schemes are inherently twice as efficient as three-time-level schemes. This has been clearly shown in 2-d, but the full benefits of two-time-level schemes in 3-d remain to be demonstrated.

5) Best results are obtained when coupling semi-Lagrangian advection to a semi-implicit treatment of gravitational oscillations, rather than to splitting methods such as split - explicit and alternating-direction-implicit.

6) It is important to use the semi-Lagrangian method for vertical as well as for horizontal advection, in order to avoid unduly limiting vertical resolution.

7) Non-interpolating semi-Lagrangian schemes are attractive for climate applications, due to their lack of damping, a particularly important property for low-resolution simulations.

8) The semi-Lagrangian framework facilitates the incorporation of shape-preserving and monotonic schemes for moisture advection.

9) The semi-Lagrangian methodology is applicable not only to hydrostatic systems of equations, but also to non-hydrostatic ones.

10) Further research on the incorporation of mountains in semi-Lagrangian models is warranted.

11) Research on higher-order time discretizations is desirable.

12) Further research on the incorporation of physical parameterizations into models is needed, to reduce sensitivity to timestep length.

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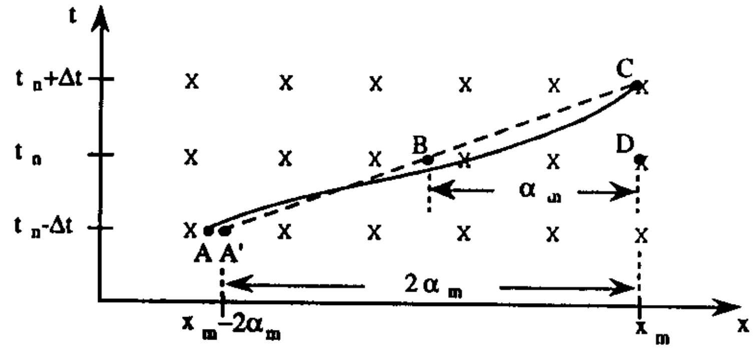
FIGURE LEGENDS

Fig. 1: Schematic for 3-time-level advection. Actual (solid curve) and approximated (dashed line) trajectories that arrive at meshpoint x_m at time $t_n+\Delta t$. Here α_m is the distance the particle is displaced in x in time Δt.

Fig. 2: Schematic for 2-time-level advection. Actual (solid curve) and approximated (dashed line) trajectories that arrive at meshpoint x_m at time $t_n+\Delta t$. Here α_m is the distance the particle is displaced in x in time Δt.

Fig. 3: The 'slotted' cylinder, (a)- at initial time, and (b)- after 6 revolutions.



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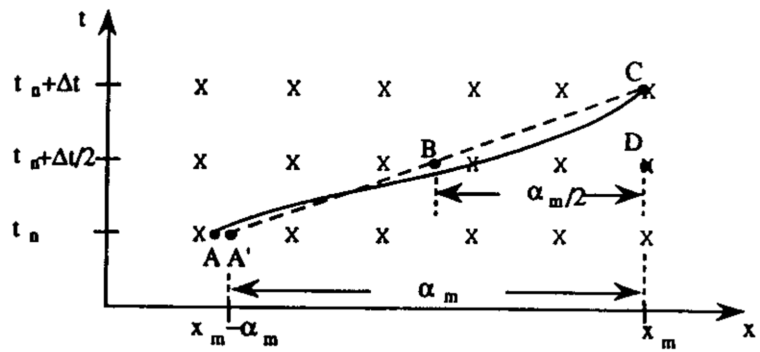

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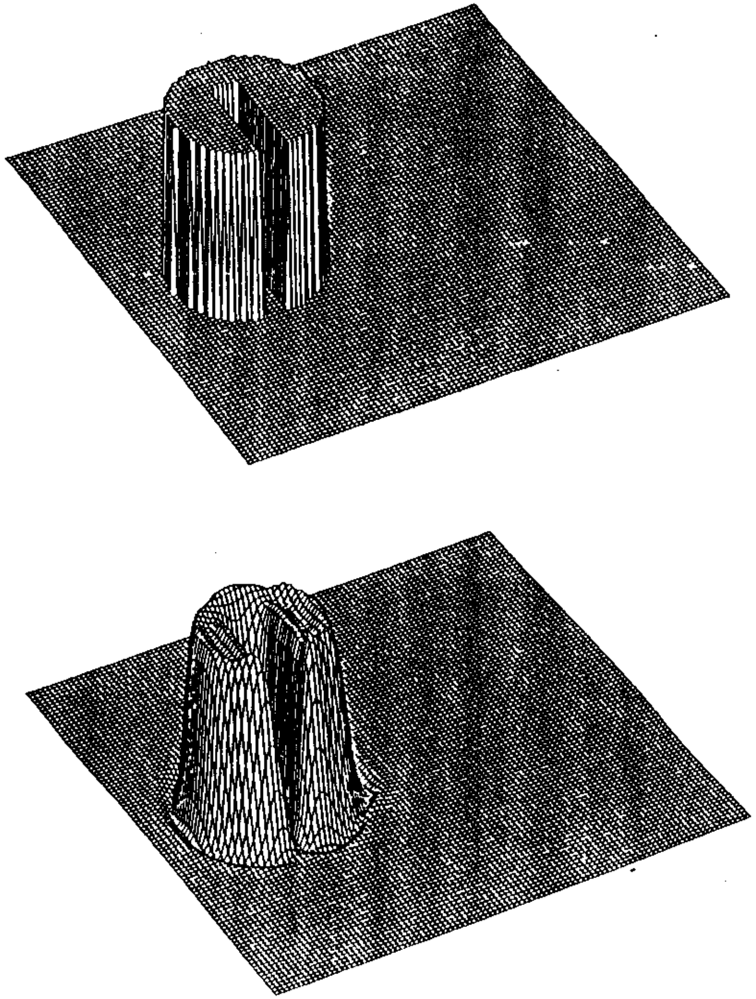

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