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ICTP/WMO WORKSHOP ON EXTRA-TROPICAL AND TROPICAL
LIMITED AREA MODELLING
22 October - 3 November 1990

"The Finite-Element Method for the Horizontal
& Vertical Discretization of Atmospheric Models"

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Please note: These are preliminary notes intended for internal

THE FINITE-ELEMENT METHOD FOR THE HORIZONTAL AND VERTICAL DISCRETIZATION OF ATMOSPHERIC MODELS

Andrew Staniforth

Recherche en prévision numérique
Environment Canada

OUTLINE

1) Introduction to FE method.
2) Formulation considerations.
3) Some recent results (hot off the press!). → deferred to Friday, Nov. 2.
4) Conclusions.

<u>FE REVIEW PAPERS</u>

A.N.Staniforth: The application of the finite-element method to meteorological simulations - a review.

Int. J. Num. Meth. Fluids, **4**, 1-12. 1984

A.N.Staniforth: Review: formulating efficient finite-element codes for flows in regular domains.

Int. J. Num. Meth. Fluids,, **7**, 1-16. 1987

FE

- Galerkin (like spectral)
- local (like FD's)

FD/FE issues

- staggered vs unstaggered grids
- primitive vs differentiated form of equations
- 2nd vs 4th order
- conservation properties
- uniform vs variable resolution

SPACE DISCRETIZATION

Choices

- FD
- FE } Galerkin
- spectral } Galerkin

Spectral (for limited area)

- similar to spherical application (half-wave Fourier series instead of spherical harmonics)
- aperiodicity requires special measures (e.g. add special basis functions)
- more efficient than spherical spectral (cheaper transforms)
- good phase speeds (but restrictive CFL)
- lbc's delicate

Galerkin FE method

(1) Expand dependent variables in terms of basis functions
- Fourier series or spherical harmonics for spectral discretization
- piecewise polynomials for FE discretization.

(2) Insert in governing equations.

(3) Orthogonalize error to basis
- multiply by arbitrary basis function
- integrate over domain.

GALERKIN METHOD

Governing equation: $Q\,[u(\underline{x}, t)] = 0$

1) Expand dependent variables

$$u(\underline{x}, t) = \sum_{n=1}^{N} u_n(t) f_n(\underline{x})$$

2) Insert into governing equations

$$Q\left(\sum_{n=1}^{N} u_n(t) f_n(\underline{x})\right) = E(\underline{x}, t)$$

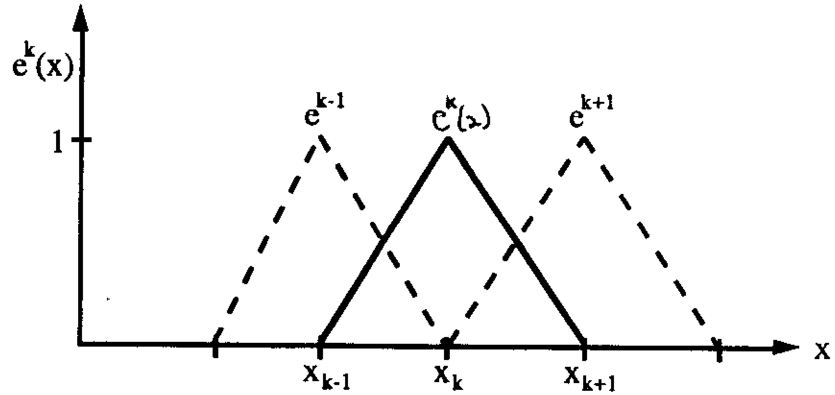
3) Orthogonalize error to basis

$$\int_D Q\left(\sum_{n=1}^{N} u_n(t) f_n(\underline{x})\right) f_k(\underline{x})\, d\underline{x} = 0$$

EXAMPLE: 1st derivative

Governing equation: $v = \dfrac{du}{dx}$

LINEAR ELEMENT IN 1-D



1) Expand dependent variables

$$u = \sum_{i=1}^{N} u_i e^i(x)$$

$$v = \sum_{i=1}^{N} v_i e^i(x)$$

2) Insert in governing equations

$$E(x) = \sum_{i=1}^{N} \left[v_i e^i(x) - u_i e^i_x(x)\right]$$

3) Orthogonalize error to basis

$$\int_{x_1}^{x_N} E(x)\, e^k(x)\, dx = 0$$

$$= \int_{x_1}^{x_N} \sum_{i=1}^{N} \left[v_i\, e^i(x) - u_i\, e^i_x(x)\right] e^k(x) dx$$

for $k = 1, 2, \cdots N$

FD vs FE for 1st derivatives

FD

$$v_k = \frac{u_{k+1} - u_{k-1}}{2h} + 0(h^2)$$

FE

$$\frac{1}{6}(v_{k-1} + 4v_k + v_{k+1}) = \frac{u_{k+1} - u_{k-1}}{2h} + 0(h^4)$$

"super-convergence at nodes" for FE result

EXAMPLE: product in 1-d

Governing equation: $w = u\,v$

1) Expand dependent variables

$$w = \sum_{i=1}^{N} w_i\, e^i(x) \qquad \ldots \text{ etc.}$$

2) Insert in governing equation

$$\sum_{i=1}^{N} w_i\, e^i(x) - \sum_{i=1}^{N} u_i\, e^i(x) \sum_{j=1}^{N} v_j\, e^j(x) = E(x)$$

3) Orthogonalize error to basis

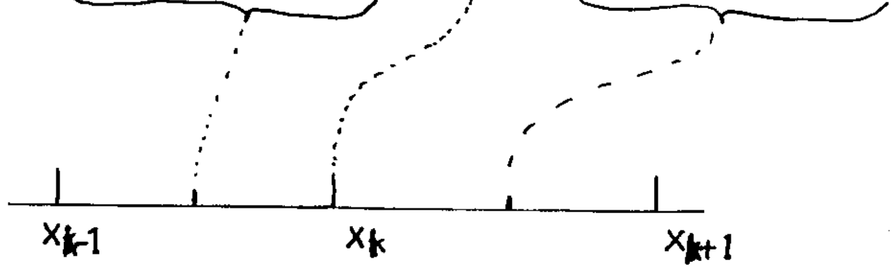
$$\int_{x_1}^{x_N} E(x)\, e^k(x)\,dx = 0 \quad \text{for } k = 1, 2, \ldots, N$$

Result:

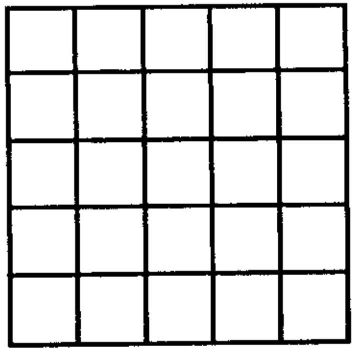
$$\frac{1}{6}\left(w_{k-1} + 4w_k + w_{k+1}\right)$$

$$= \frac{1}{12}\left[\left(u_{k-1}+u_k\right)\left(v_{k-1}+v_k\right) + 4u_k v_k + \left(u_k+u_{k+1}\right)\left(v_k+v_{k+1}\right)\right]$$

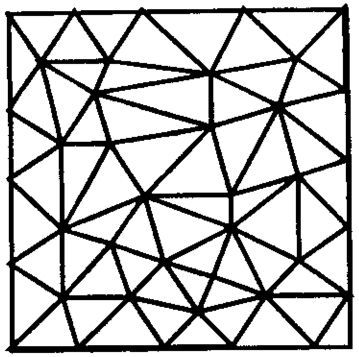
$$= \frac{1}{3}\left[\underbrace{\left(\frac{u_{k-1}+u_k}{2}\right)\left(\frac{v_{k-1}+v_k}{2}\right)} + \underbrace{u_k v_k} + \underbrace{\left(\frac{u_k+u_{k+1}}{2}\right)\left(\frac{v_k+v_{k+1}}{2}\right)}\right]$$



KEY ISSUE - ELEMENT CHOICE



RECTANGLES

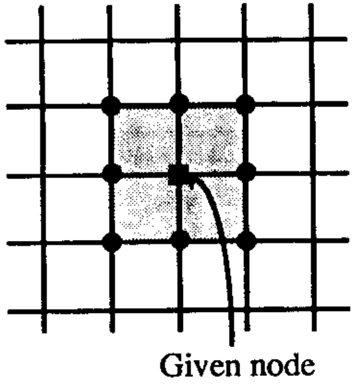


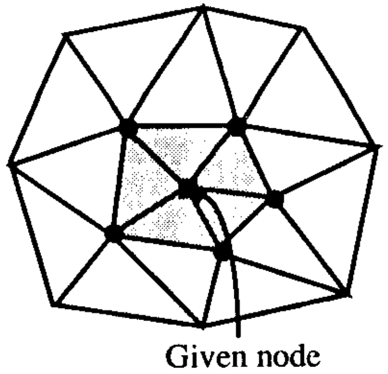
TRIANGLES

Q? Can we afford **generality** of triangles?

Q? Is it **computationally competitive** with FD's?

LINEAR ELEMENTS





Rectangles	**Triangles**
(a+bx)(c+dy)	(a+bx+cy)

Definition: $e^i(x,y) = 1$ at given node (■),

$= 0$ at neighbouring nodes (●),

$= 0$ outside shaded areas,

varies linearly over shaded areas.

MASS-MATRIX PROBLEM

Both $v = u_x$ and $v = uw$ lead to

$$P\underline{v} = \underline{r}$$

where $\underline{r}$ is known,

and P is **large** and **sparse**.

<u>COST</u> ?????

RECTANGULAR ELEMENTS

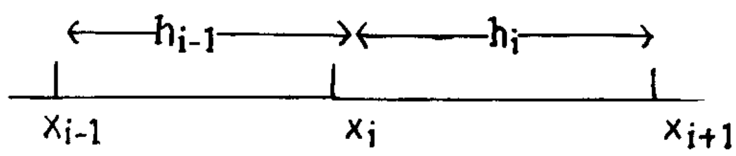
Can exploit structure of P.

Decompose P as

$$P = P^x P^y$$

where $P^x \sim \left[\frac{h_{k-1}}{6}, \frac{2(h_{k-1}+h_k)}{6}, \frac{h_k}{6},\right]$

Operational count:	O(1) per node
Memory:	O(1) per node
Generalizes to :	3d



TRIANGULAR ELEMENTS

Can exploit *bandwidth* of P,

but several orders of magnitude *more costly*.

Less accurate than rectangular elements

(no super-convergence at nodes)

TRICK FOR O(h^4) APPROXIMATION

(on uniform- resolution sub-domain)

TO:- $r = \nabla^2 f$

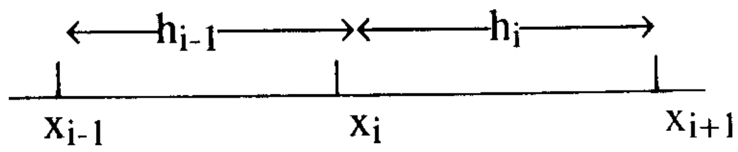
Example (in 1-d): $r = f_{xx}$

Usual: - O(h^2)

$$\frac{h_{i-1}}{6}(r_{i-1}+2r_{i-1}) + \frac{h_i}{6}(2r_i+r_{i+1}) = \frac{1}{h_i}(f_{i+1}-f_i) - \frac{1}{h_{i-1}}(f_i-f_{i-1})$$

Modified: - O(h^4)

$$\frac{h_{i-1}}{12}(r_{i-1}+5r_i) + \frac{h_i}{12}(5r_i+r_{i+1}) = \frac{1}{h_i}(f_{i+1}-f_i) - \frac{1}{h_{i-1}}(f_i-f_{i-1})$$



NUMERICAL QUADRATURE FOR PRODUCTS

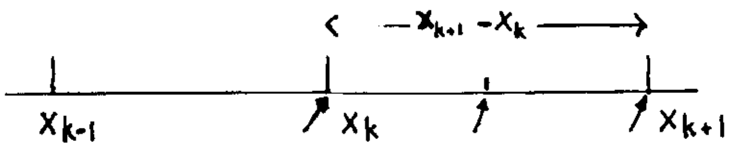
Example in 1-d:- $w = u\,v$

$$\int_{x_{k-1}}^{x_{k+1}} u\,v\,e^k(x)dx = \int_{x_{k-1}}^{x_k} u\,v\,e^k(x)dx + \int_{x_k}^{x_{k+1}} u\,v\,e^k(x)dx$$

Use *Simpson's rule* on each interval (exact for cubics):

$$\text{e.g. } I_k = \int_{x_k}^{x_{k+1}} u\,v\,e^k(x)dx = \int_{x_k}^{x_{k+1}} F\,dx$$

$$= \left(\frac{x_{k+1}-x_k}{6}\right)\left(F_k + 4F_{k+\frac{1}{2}} + F_{k+1}\right) + h^5\,f^{(iv)}(\xi)$$



where:

$$F_k = \left[u\,v\,e^k(x)\right]\big|_{x_k} = u_k \cdot v_k \cdot (1) = u_k v_k$$

$$F_{k+\frac{1}{2}} = \left[u\,v\,e^k(x)\right]\big|_{x_{k+\frac{1}{2}}} = u_{k+\frac{1}{2}} \cdot v_{k+\frac{1}{2}} \cdot \left(\frac{1}{2}\right)$$

$$= \frac{1}{8}(u_k+u_{k+1})(v_k+v_{k+1})$$

$$F_{k+1} = u_{k+1} \cdot v_{k+1} \cdot (0) = 0$$

Thus

$$I_k = \frac{(x_{k+1}-x_k)}{3}\left[2u_k v_k + \frac{(u_k+u_{k+1})}{2}\,\frac{(v_k+v_{k+1})}{2}\right]$$

Note: - method generalizes to 2- and 3-d

- yields algorithm that is more efficient than evaluating integrals analytically.

SPATIAL EVOLUTIONARY ERROR

Cullen & Morton considered two ways of approximating $w = u\frac{\partial v}{\partial x}$ (using linear elements) in context of time-dependent problem.

1) One-shot $w = u\frac{\partial v}{\partial x}$

2) Two shots $r = \frac{\partial v}{\partial x}$

$$w = u \cdot r$$

Both methods give $\mathbf{O(h^4)}$ estimate for spatial evolutionary error, **but** "two shots" *more accurate* and permits *code modularity*.

ALIASING

e.g. 2-d incompressible flow using linear elements (Jespersen)

$$\Rightarrow \quad \frac{\partial}{\partial t}\left(P\underline{\zeta}\right) = J(\psi, \zeta) = \psi_x\zeta_y - \psi_y\zeta_x$$

where $P \sim \frac{1}{36}\begin{bmatrix} 1 & 4 & 1 \\ 4 & 16 & 4 \\ 1 & 4 & 1 \end{bmatrix}$

and $J(\psi, \zeta)$ = **Arakawa's Jacobian**.

Thus if $P \rightarrow I$ ("*mass lumping*") we have Arakawa's scheme.

Note: - $J(\psi, \zeta)$ controls *aliasing*.

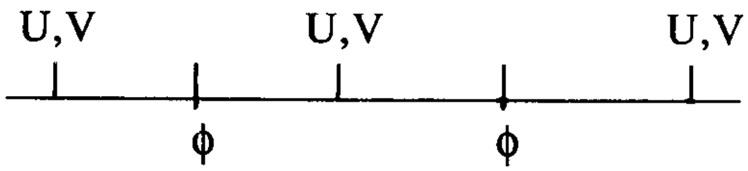
- P 'sharpens response', improves accuracy.

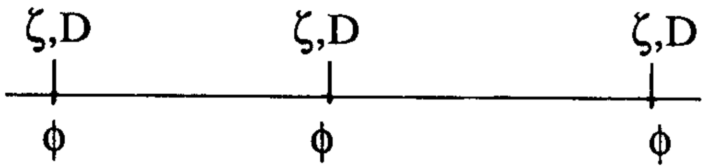
HORIZONTAL ISSUES

1) Staggered vs unstaggered meshes
2) Primitive vs differentiated form of equations

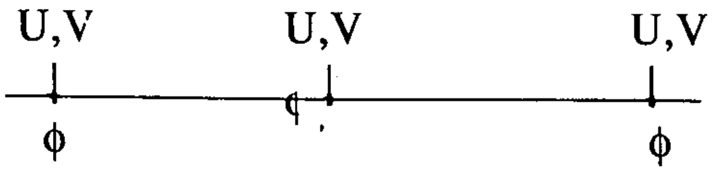
Analysis of linearised shallow-water equations shows (Williams) that FE considerations similar to FD ones.

Good:





Bad:



But:

Pseudo staggering **good** in SL applications

Example: $v = \dfrac{du}{dx}$

$$\Rightarrow \quad \frac{1}{2}(v_k + v_{k+1}) = \frac{u_{k+1} - u_k}{h_k}$$

Note: This is a **one**-interval approximation to a 1st derivative, rather than the usual **two**-interval one.
It is obtained by orthogonalizing the error to a different set of basis functions (piecewise-constant instead of piecewise linear).
It has certain advantages for some variable-res applications.

VERTICAL DISCRETIZATIONS

Several schemes exist using piecewise-*linear* andpiecewise-*constant* elements.

Some have interesting **conservation** properties- e.g Yakimiw & Girard and Steppeler & Burridge.

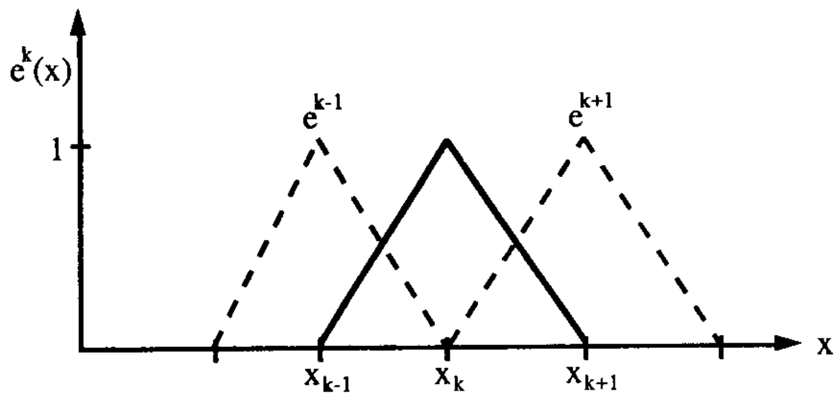
Yakimiw &Girard show how to obtain several well-known *FD* schemes using *FE*'s!

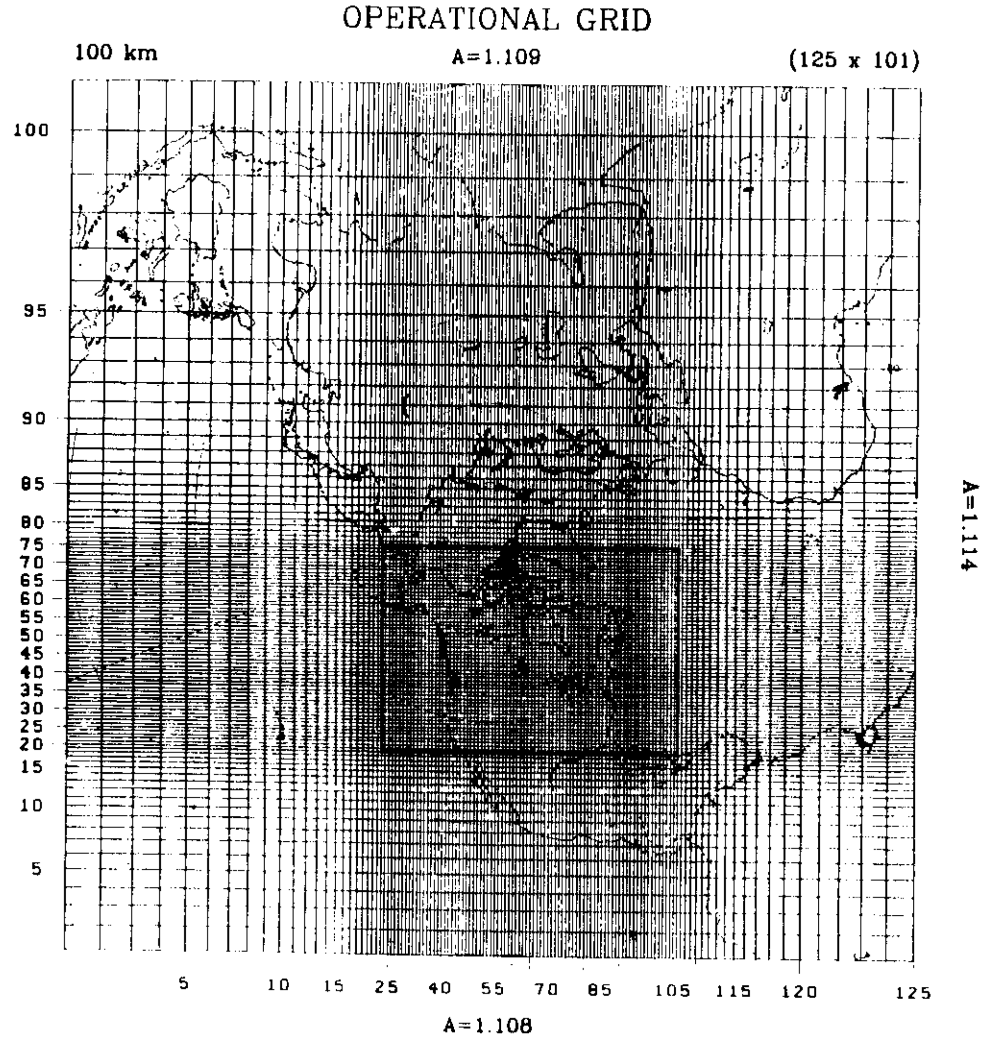
Beaudoin & Staniforth and Steppeler & Burridge found sensitivity to application of **top b.c.**

ELEMENT CHOICE FOR OPERATIONAL CANADIAN REGIONAL FE MODEL

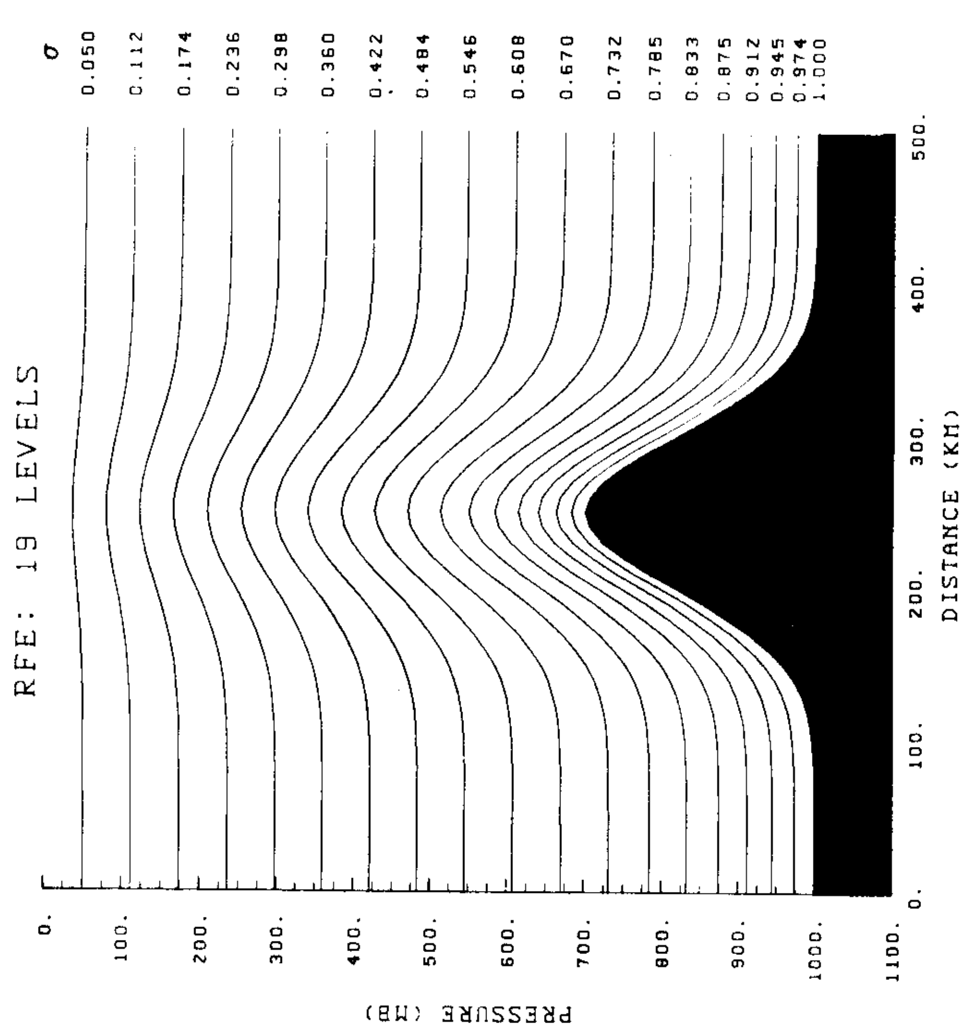
- tri-linear elements on a variable-res Cartesian mesh

$$f^{ijk}(x,y) = e^i(x)\, e^j(y)\, e^k(\sigma)$$





OPERATIONAL GRID
100 km
A=1.109
(125 x 101)
A=1.114
A=1.108



RFE: 19 LEVELS
σ
0.050
0.112
0.174
0.236
0.298
0.360
0.422
0.484
0.546
0.608
0.670
0.732
0.785
0.833
0.875
0.912
0.945
0.974
1.000
DISTANCE (KM)
PRESSURE (MB)

REASONS

- Simplicity
- Flexibility of variable resolution
- Accuracy (particularly when coupled with semi-Lagrangian advection)
- Separable basis gives efficiency
    - vectorized algorithms
    - code modularity
- Low-order elements reduce # of computational modes
- Ease of incorporating b.c.'s

SURPRISING FACT

Semi-Lagrangian advection with cubic-spline interpolation is a finite-element method (using linear basis functions)

FINITE-ELEMENT REGIONAL MODEL

DYNAMICS

Tanguay,Simard & Staniforth, MWR,89

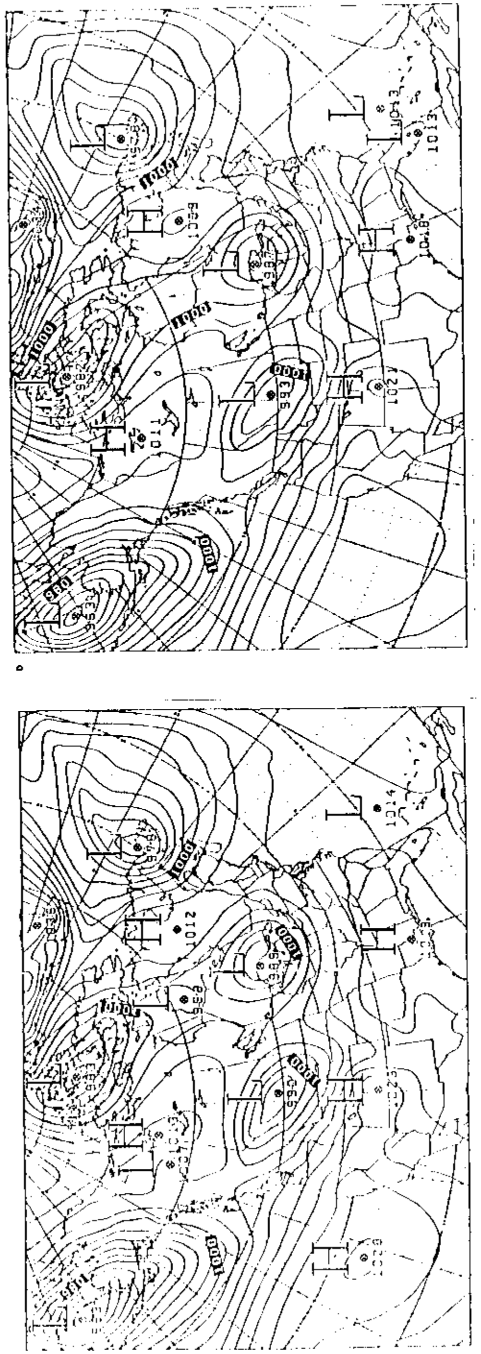
- hydrostatic primitive equations
- variable-resolution 125x101 horizontal grid
- uniform 100 km res over N/A
- 19 levels (top at sigma = 0.05)
- trilinear finite elements
- semi-implicit semi-Lagrangian time scheme
- orography

PARAMETERIZATIONS

Benoit , Côté & Mailhot , MWR,89

- turbulent fluxes of momentum, heat & moisture based on turbulent KE
- prediction of temperature & moisture over land
- diurnal cycle based on surface energy budget
- solar & infrared radiation interactive with clouds
- moist convection and precipitation

3-d regional finite-element model
48-hour mean-sea-level forecasts



Semi-Lagrangian dt = 1200 secs

Eulerian dt = 400 secs

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WHAT DOES IT ALL MEAN?

1) Triangular elements not viable.

2) Higher-order elements ⇒ law of diminishing returns.

3) Analyse linear versions of models for best choice of elements/ form of equations.

4) Advantageous to stray outside FE space for 2nd derivatives.

5) FE's viable for vertical discretization of both regional and global models.

6) Linear FE's competitive with FD's for regional modelling.

7) Semi-Lagrangian advection is a FE method.

8) Particularly advantageous to associate variable-res FE's with semi-Lagrangian advection (in both regional and global models).

9) FE's with semi-Lagrangian advection now offers promise of being more efficient than spectral method for global modelling.