INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE CENTRATOM TRIESTE

SMR/534-10

ICTP/WMO WORKSHOP ON EXTRA-TROPICAL AND TROPICAL LIMITED AREA MODELLING
22 October - 3 November 1990

"Spectral Method in LAMs"

A. SEGAMI
Japan Meteorological Agency
Tokyo
Japan

Please note: These are preliminary notes intended for internal distribution only.

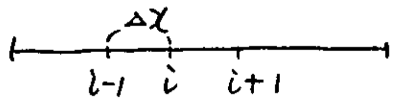
Spectral Method in LAMs

A. Segami

1. Grid point method
2. Spectral method
   advantage and disadvantage
3. Regional spectral method
4. Intercomparison between
   spectral and grid point LAMs

1. Grid point method

$$u = u(x)$$



$\downarrow$ introduction of a set of grid points

$$u_i = u(i\Delta x) \quad (i = 0, 1 \cdots\cdots N)$$

$$x_i = i\Delta x$$

— finite difference method

$$\frac{du}{dx}$$

$$\left(\frac{du}{dx}\right)_i \doteqdot \frac{u_{i+1} - u_i}{\Delta x} \quad \cdots\cdots \text{forward difference}$$

$$\doteqdot \frac{u_i - u_{i-1}}{\Delta x} \quad \cdots\cdots \text{backward difference}$$

$$\doteqdot \frac{u_{i+1} - u_{i-1}}{2\Delta x} \quad \cdots\cdots \text{centerd difference}$$

Truncation error of the finite difference

- forward difference

$$\frac{u_{i+1}-u_i}{\Delta x} = \left(\frac{du}{dx}\right)_i + \varepsilon_f$$

$$\varepsilon_f = \frac{1}{2}\left(\frac{d^2u}{dx^2}\right)_i (\Delta x) + \frac{1}{3!}\left(\frac{d^3u}{dx^3}\right)_i (\Delta x)^2 + \cdots$$

$$\varepsilon_f = O(\Delta x)$$ the first order of accuracy

- Centered difference

$$\frac{u_{i+1}-u_{i-1}}{2\Delta x} = \left(\frac{du}{dx}\right)_i + \varepsilon_c$$

$$\varepsilon_c = \frac{1}{3!}\left(\frac{d^3u}{dx^3}\right)_i (\Delta x)^2 + \frac{1}{5!}\left(\frac{d^5u}{dx^5}\right)(\Delta x)^4 + \cdots$$

$$\varepsilon_c = O((\Delta x)^2)$$ the second order of accuracy

2. Spectral method (Galerkin method)

$$\frac{\partial \psi(x,t)}{\partial t} = F(\psi(x,t), x, t) \qquad (1)$$

$$\psi(x,t) \doteqdot \sum_{n=1}^{N} C_n(t)\, a_n(x) \qquad (2)$$

a_n : orthogonal functions (orthonormal)

normalized

$$\underbrace{(a_n, a_m)}_{\text{inner product}} = \delta_{nm} \begin{cases} = 1 & (n=m) \\ = 0 & (n \neq m) \end{cases}$$

ex. Fourier series, Spherical Harmonics

(2) $\longrightarrow$ (1) and inner product

$$\frac{\partial C_n(t)}{\partial t} = \left(a_n(x), F\left(\sum_m C_m a_m(x), x, t\right)\right)$$

$$C_n(0) = (a_n, \psi(x, 0))$$

Advantages of spectral method

(compared with a grid point method)

(1) Accuracy

Space derivative can be evaluated accurately

— elimination of computational dispersion
retardation of phase speed

— elimination of aliasing error and
nonlinear instability

(2) Simplicity

— the ease of modeling (conservation law)

— the ease of incorporation of
NNMI and semi-implicit scheme

(3) Economy

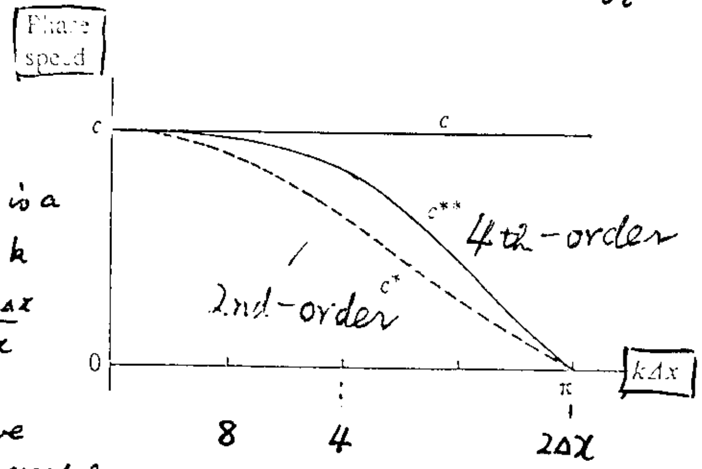
almost same as grid point model with the use of
FFT and transform method

∘ Computational dispersion

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}$$

• The phase speed is a function of k

$$C^* = C \cdot \frac{\sin k\Delta x}{k\Delta x}$$



• All waves have smaller phase speed than the true one

$$C^* < C$$

∘ Aliasing error

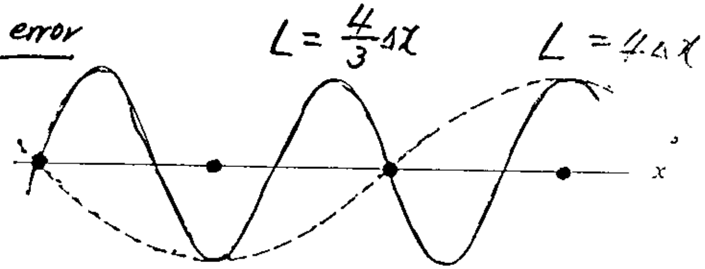
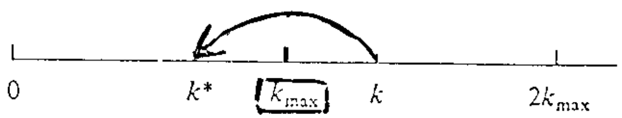


Figure 6.1 A wave of wave length $4\Delta x/3$, misrepresented by the finite difference grid as a wave of wave length $4\Delta x$.



$$k^* = 2k_{max} - k$$

Disadvantage

Difficulty in treating a time dependent lateral boundary condition

$\downarrow$

Solved by Tatsumi (1985, 1986)

- Fourier sine series

$$f(x) = \sum_{k=1}^{K} S_k \sin \frac{\pi k x}{L} \qquad \left(\frac{d^{2n} f}{dx^{2n}}\right)_{x=0,L} = 0$$

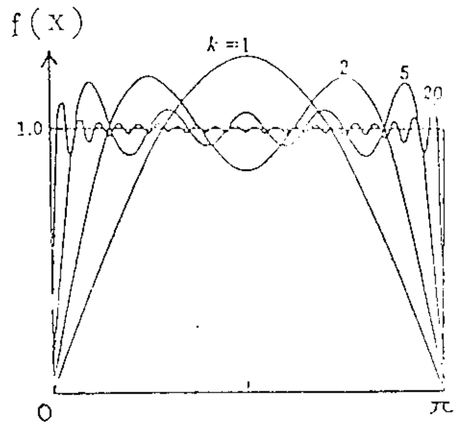
$$f = 0$$

- Fourier cosine series

$$f(x) = \sum_{k=0}^{K} C_k \cos \frac{\pi k x}{L} \qquad \left(\frac{d^{2n+1} f}{dx^{2n+1}}\right)_{x=0,L} = 0$$

$$\frac{df}{dx} = 0$$

- Gibbs phenomena



$f(x) = 1$

expansion with Fourier sine series

$$f(x) = \sum_{k=1} \sin kx$$

3. Regional spectral method

$$f(x) = f_N(x) + f_o(x)$$

f_N : non-orthogonal term (additional term)

large scale field which is derived from a time dependant lateral boundary condition.

f_o : orthogonal term

$\Sigma S_k \sin kx$ or $\Sigma C_k \cos kx$

deviation from $f_N(x)$

- Necessary condition for $f_N(x)$ at the L.B.

$$\left.\begin{array}{l} f_N(x) \neq 0 \\ \dfrac{df_N(x)}{dx} \neq 0 \end{array}\right\} \text{ if } \left\{\begin{array}{l} \underbrace{f_o(x) = 0}_{\text{sine series}} \\ \underbrace{\dfrac{df_o}{dx} = 0}_{\text{cosine series}} \end{array}\right\} \text{ at L.B.}$$

Additional bases

$$f_N(x) \begin{cases} : \underline{1}, \ \underline{\cos x} & \text{for } f_o(x) \text{ --- sine series} \\ \quad f_N \neq 0 \text{ at L.B.} & \\ : \underline{\sin x}, \ \underline{\sin 2x} & \text{for } f_o(x) \text{ --- cosine series} \\ \quad \dfrac{df_N}{dx} \neq 0 \text{ at L.B.} & \end{cases}$$

Modified Fourier expansion

- Modified sine series

$$f(x) = \overbrace{\underbrace{S_0 \cos 0x + S_{-1} \cos x}_{\text{prescribed to satisfy L.B.C.}}}^{\text{non-orthogonal term}} + \overbrace{\sum_{k=1}^{K} \underbrace{\boxed{S_k}}_{\text{prognostic variable}} \sin kx}^{\text{orthogonal term}}$$

$$= \sum_{k=-1}^{K} S_k \sin^* kx$$

- Modified cosine series

$$f(x) = C_{-1} \sin x + C_{-2} \sin 2x + \sum_{k=0}^{K} C_k \cos kx$$

$$= \sum_{k=-2}^{K} C_k \cos^* kx$$

— Modified sine

$$f(0) = S_0 + S_{-1}, \quad f(\pi) = S_0 - S_{-1}$$

$$\Downarrow$$

$$S_0 = \frac{f(0) + f(\pi)}{2}, \qquad S_{-1} = \frac{f(0) - f(\pi)}{2}$$

values are given from L.B.C.

— Modified cosine

$$f'(0) = C_{-1} + 2C_{-2}, \quad f'(\pi) = -C_{-1} + 2C_{-2}$$

$$\Downarrow$$

$$C_{-1} = \frac{f'(0) - f'(\pi)}{2}$$

$$C_{-2} = \frac{f'(0) + f'(\pi)}{4}$$

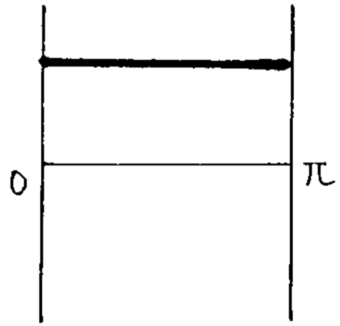
$$\left(f' \equiv \frac{df}{dx} \right)$$

derivatives are given from L.B.C.

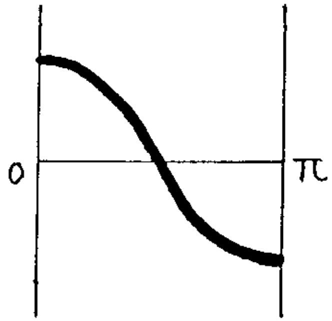
Additional bases $F_N(x)$

non-orthogonal term

1. For modified $\boxed{\text{sine}}$ series

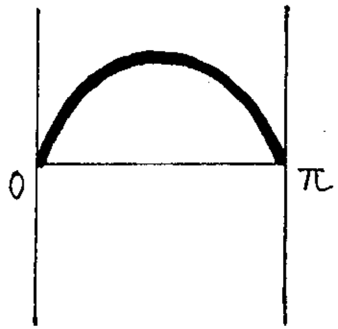


cos0x

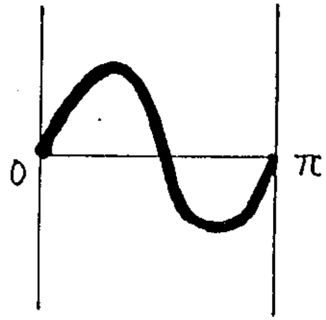


cos1x

2. For modified $\boxed{\text{cosine}}$ series

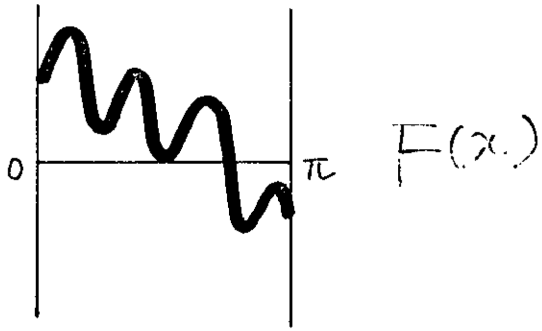


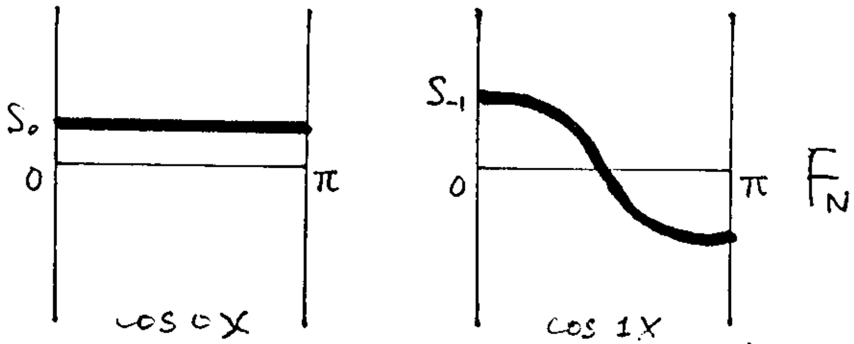
sin1x



sin2x

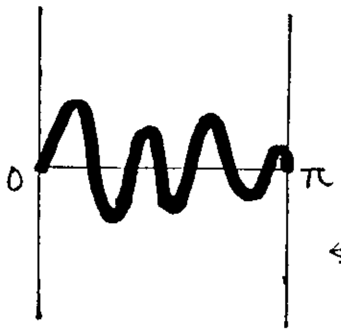
- Modified sine expansion





cos 0x cos 1X

non-orthogonal term
prescribed from the boundary value



$F_O = \sum f_k \sin kx$

← Orthogonal term
prognostic

Prognostic eq. of S_k

$$\frac{\partial u}{\partial t} = F\left(u, \frac{\partial u}{\partial x}, x, t\right)$$

$$u = \sum S_k \sin^* kx = S_0 + S_{-1}\cos x + \sum_1^K S_k \sin kx$$

$$\frac{\partial u}{\partial t} = \frac{\partial S_0}{\partial t} + \frac{\partial S_{-1}}{\partial t}\cos x + \sum \frac{\partial S_k}{\partial t}\sin kx$$

Fourier transform

$$\therefore \frac{\partial S_k}{\partial t} = \varepsilon_k \int_0^\pi \sin kx \left[\frac{\partial u}{\partial t} - \frac{\partial S_0}{\partial t} - \frac{\partial S_{-1}}{\partial t}\cos x\right] dx$$

(ε_k : normalizing factor)

- Interaction coefficient method

$$F = -u \cdot \frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial t} = F = -\sum_{l,m}^{K} S_l \, m S_m \sin^* lx \, \frac{d \sin^* mx}{dx}$$

summation $O(K^2)$ operations

— convolution is calculated analytically

$$\left(\int_0^\pi \sin kx \sin^* lx \frac{d \sin^* mx}{dx} dx \right)$$

- Transform method

$\frac{\partial u}{\partial t}$ is calculated over grid points

$$\left(\frac{\partial u}{\partial t}\right)_i = F_i = -u_i \cdot \left(\frac{\partial u}{\partial x}\right)_i$$

analytical

$$\int_0^{\pi} \sin kx \, \frac{\partial u}{\partial t} \, dx \Longrightarrow \boxed{\sum \sin \frac{k\pi i}{N} \left(\frac{\partial u}{\partial t}\right)_i \Delta x}$$

FFT

integral ═══ trapezoidal formula

$$\left(\begin{array}{l} \text{if} \quad N > \frac{3K}{2} \\ \text{or} \quad IM > \frac{3K}{2} + 1 \end{array} \right)$$

elimination of aliasing error

$$\underset{0}{\overset{I}{\vdash}}\!\!\text{———————}\!\!\underset{N}{\overset{IM}{\dashv}}$$

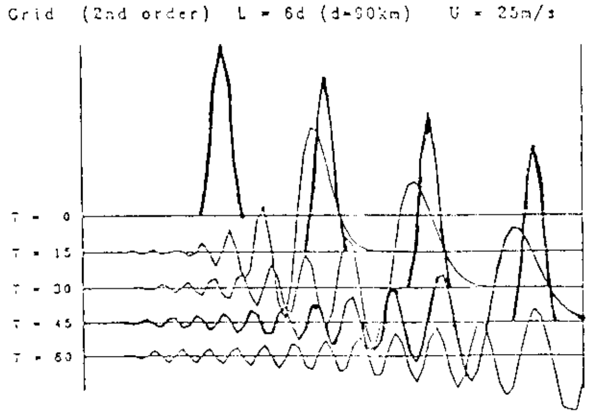
Transform method + FFT

$O(K \log K)$ operations

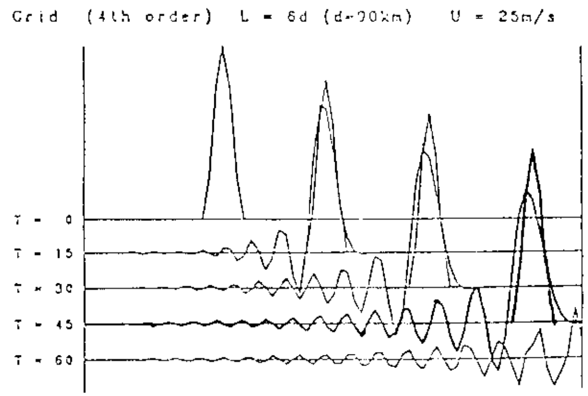
Linear advection

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = 0 \qquad L = 6d$$

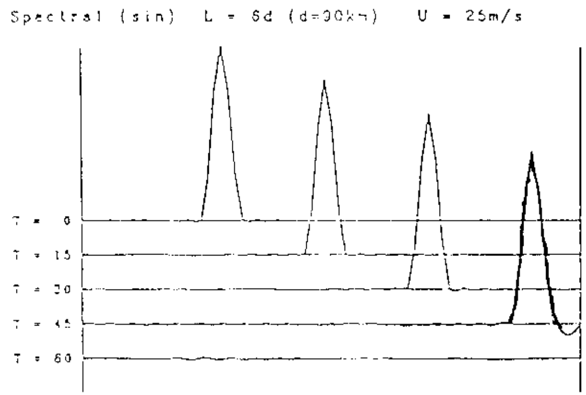
Grid (2nd order)



Grid (4th order)



Spectral (sin)



3-dim. model

Vertical ···· finite difference method

horizontal ···· double Fourier series

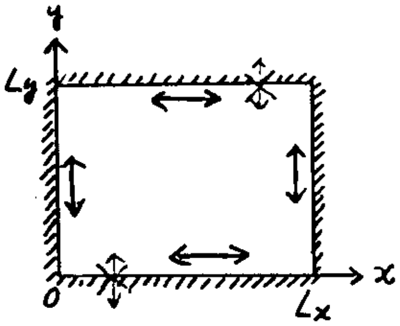
u^* : $\sin^* k\hat{x} \cdot \cos^* l\hat{y}$

v^* : $\cos^* k\hat{x} \cdot \sin^* l\hat{y}$

T, π^*, m^2 : $\cos^* k\hat{x} \cdot \cos^* l\hat{y}$

q, f : $\sin^* k\hat{x}\ \sin^* l\hat{y}$

$\left(\hat{x} = \dfrac{\pi x}{L_x},\ \hat{y} = \dfrac{\pi y}{L_y} \right)$



- Orthogonal bases satisfy

  free-slip wall boundary condition. (excpt. q)

  (mirror boundary condition)

- Fourier functions produced from r.h.s. of eqs agree with the ones of l.h.s. of eq. (orthogonal basis)

  ↓

  elimination of errors when you calculate convolutions

$$u^* = \frac{u}{m},\ v^* = \frac{v}{m},\ T,\ q,\ \pi^* = \ln p_s$$

Equations of motion

$$\begin{aligned}
\frac{\partial u^*}{\partial t} &= -m^2\left\{u^*\frac{\partial u^*}{\partial x} + v^*\frac{\partial u^*}{\partial y}\right\} \\
&\quad -\frac{1}{2}u^*\left\{u^*\frac{\partial m^2}{\partial x} + v^*\frac{\partial m^2}{\partial y}\right\} \\
&\quad -\dot{\sigma}\frac{\partial u^*}{\partial \sigma} - \left\{\frac{\partial \phi}{\partial x} + RT_v\frac{\partial \pi^*}{\partial x}\right\} \\
&\quad +v^*\left\{f + \frac{1}{2}\left(u^*\frac{\partial m^2}{\partial y} - v^*\frac{\partial m^2}{\partial x}\right)\right\} \\
&\quad +Du - \frac{g}{mp_s}\frac{\partial \tau_x}{\partial \sigma} \qquad (2.1)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial v^*}{\partial t} &= -m^2\left\{u^*\frac{\partial v^*}{\partial x} + v^*\frac{\partial v^*}{\partial y}\right\} \\
&\quad -\frac{1}{2}v^*\left\{u^*\frac{\partial m^2}{\partial x} + v^*\frac{\partial m^2}{\partial y}\right\} \\
&\quad -\dot{\sigma}\frac{\partial v^*}{\partial \sigma} - \left\{\frac{\partial \phi}{\partial y} + RT_v\frac{\partial \pi^*}{\partial y}\right\} \\
&\quad -u^*\left\{f + \frac{1}{2}\left(u^*\frac{\partial m^2}{\partial y} - v^*\frac{\partial m^2}{\partial x}\right)\right\} \\
&\quad +Dv - \frac{g}{mp_s}\frac{\partial \tau_y}{\partial \sigma} \qquad (2.2)
\end{aligned}$$

Thermodynamic equation

$$\begin{aligned}
\frac{\partial T_v}{\partial t} &= -m^2\left\{u^*\frac{\partial T_v}{\partial x} + v^*\frac{\partial T_v}{\partial y}\right\} \\
&\quad -\sigma^{\kappa}\dot{\sigma}\frac{\partial}{\partial \sigma}(T_v\sigma^{-\kappa}) \\
&\quad +\kappa T_v\left\{\frac{\partial \pi^*}{\partial t} + m^2\left(u^*\frac{\partial \pi^*}{\partial x} + v^*\frac{\partial \pi^*}{\partial y}\right)\right\} \\
&\quad +\frac{Q^*}{C_p} + \frac{g}{C_p p_s}\frac{\partial F_H}{\partial \sigma} + D_H \qquad (2.3)
\end{aligned}$$

Equation of water vapor

$$\begin{aligned}
\frac{\partial q}{\partial t} &= -m^2\left\{u^*\frac{\partial q}{\partial x} + v^*\frac{\partial q}{\partial y}\right\} \\
&\quad -\dot{\sigma}\frac{\partial q}{\partial \sigma} + M + \frac{g}{p_s}\frac{\partial F_q}{\partial \sigma} + Dq \qquad (2.4)
\end{aligned}$$

Tendency equation and continuity equation

$$\begin{aligned}
\frac{\partial \pi^*}{\partial t} &= -\int_0^1 m^2\left\{\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y}\right\}d\sigma \\
&\quad -\int_0^1 m^2\left\{u^*\frac{\partial \pi}{\partial x} + v^*\frac{\partial \pi}{\partial y}\right\}d\sigma \qquad (2.5)
\end{aligned}$$

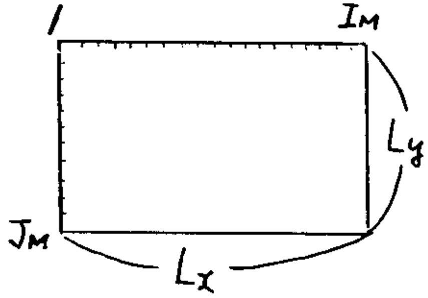
$$\begin{aligned}
\dot{\sigma}_{\sigma+\Delta\sigma} &= \dot{\sigma}_\sigma - m^2\int_\sigma^{\sigma+\Delta\sigma} \\
&\quad \left\{\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} + u^*\frac{\partial \pi^*}{\partial x}\right. \\
&\quad \left. + v^*\frac{\partial \pi^*}{\partial y}\right\}d\sigma - \Delta\sigma\frac{\partial \pi^*}{\partial t} \qquad (2.6)
\end{aligned}$$

Hydrostatic equation

$$\frac{\partial \phi}{\partial \sigma^\kappa} = -C_p T_v \sigma^{-\kappa} \quad : \quad \frac{\partial \phi}{\partial \sigma} = -\frac{RT_v}{\sigma} \qquad (2.7)$$

Maximum wavenumber

$$\frac{m^2}{K_m} \cdot \frac{u \cdot \frac{\partial u}{\partial x}}{2K}$$



elimination of aliasing error

$$2(I_M - 1) > 2K + K_m + K$$

$$\begin{cases} I_M > \dfrac{3K + K_m}{2} + 1 \\ J_M > \dfrac{3L + L_m}{2} + 1 \end{cases} \qquad (K_m = L_m = 5)$$

K, L : max. wavenumber for x, y-direction

K_m, L_m : max. wavenumber of m^2

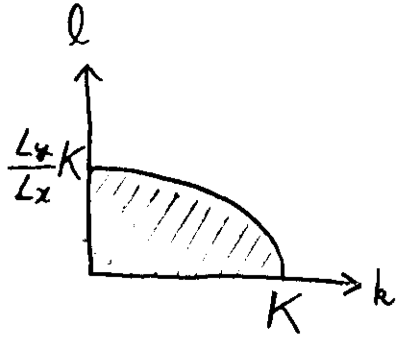
Wavenumber truncation

$$f = \frac{\sin}{\cos}\left(\frac{k\pi x}{L_x}\right) \cdot \frac{\sin}{\cos}\left(\frac{l\pi y}{L_y}\right)$$

$$\left(\frac{k\pi}{L_x}\right)^2 + \left(\frac{l\pi}{L_y}\right)^2 \leqq \left(\frac{K\pi}{L_x}\right)^2$$

$$k^2 + \left(\frac{L_x}{L_y}\right)^2 l^2 \leqq K^2$$

$$\downarrow \left(\frac{I_M - 1}{-}\right)^2$$



eliptic truncation

Boundary relaxation

to reduce the spurious solution near lateral boundary

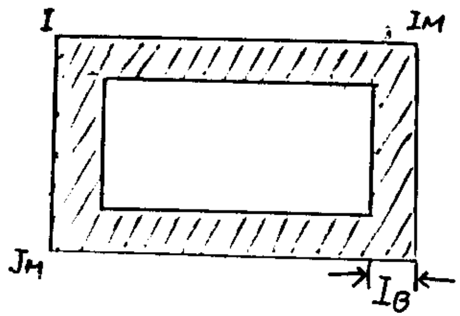
$$\frac{\partial f}{\partial t} = \cdots \quad - \beta\,(f - \hat{f}\,)$$

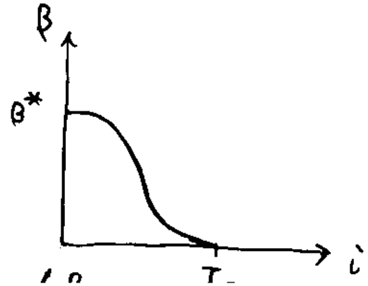
(in physical space)

f : interior field

$\hat{f}$: boundary field

linearly interpolated in time





Spectral LAM in ECMWF

Hoyer (1987)

$$f(x) = \underbrace{f_N(x)}_{\text{ECM Global forecasts}} + \underbrace{f_0(x)}_{\text{Double Fourier}}$$

f_0 : $\sin k\lambda \sin l\theta$ for ζ

: $\cos k\lambda \cos l\theta$ for $D,\ T,\ q$

Reference

Grid point method

Mesinger, F. & A. Arakawa, 1976: Numerical methods used in atmospheric models. GARP Publication Series, No.17.

Arakawa, A. & V.R. Lamb, 1977: Computational design of the basic dynamical processes of the UCLA general circulation model. Methods in Comp. Phys. 17, 174-265.

Spectral method

Bourke, W., B. McAvaney, K. Puri, and R. Thurling, 1977: Global modeling of atmospheric flow by spectral methods. Methods in Comp. Phys. 17, 268-324.

Spectral method in LAMs

Tatsumi, Y., 1985: A spectral limited-area model with time dependent lateral boundary condition and its application to a multi-level primitive equation model. JMA/NPD Tech. Rep. No3.

Tatsumi, Y., 1986: AS Tatsumi(1985) above. J. Met. Soc. Japan, 64, 637-664.

Ikawa, M., 1987: A consideration of spectral methods for limited area models. J. Met. Soc. Japan, 65, 439-453.

Takano, I., H. Nakamura and Y. Tatsumi, 1990: Non-linear normal mode initializaion for a spectral limited-area model. J. Met. Soc. Japan, 68, 265-280.

Hoyer, J.M., 1987: The ECMWF Spectral limited area model. ECMWF Workshop Proceedings, 2-4 November 1987.

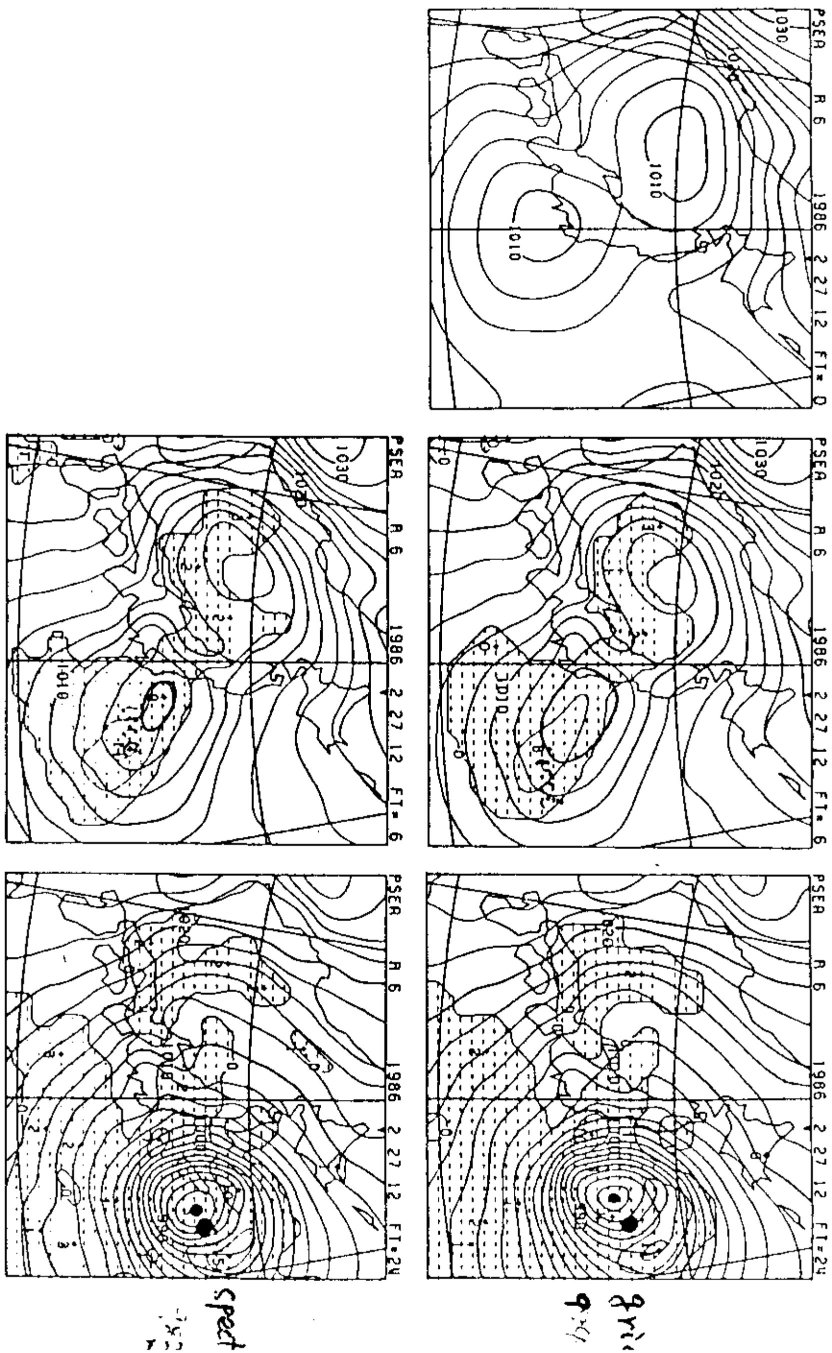


Fig.1 Sea level pressure fields and 6-hour accumulated precipitation amounts. The top panels represent the results using the grid point model, while the bottom panels represent the results using the spectral model. The analyzed central position is shown by black circles in the figures.

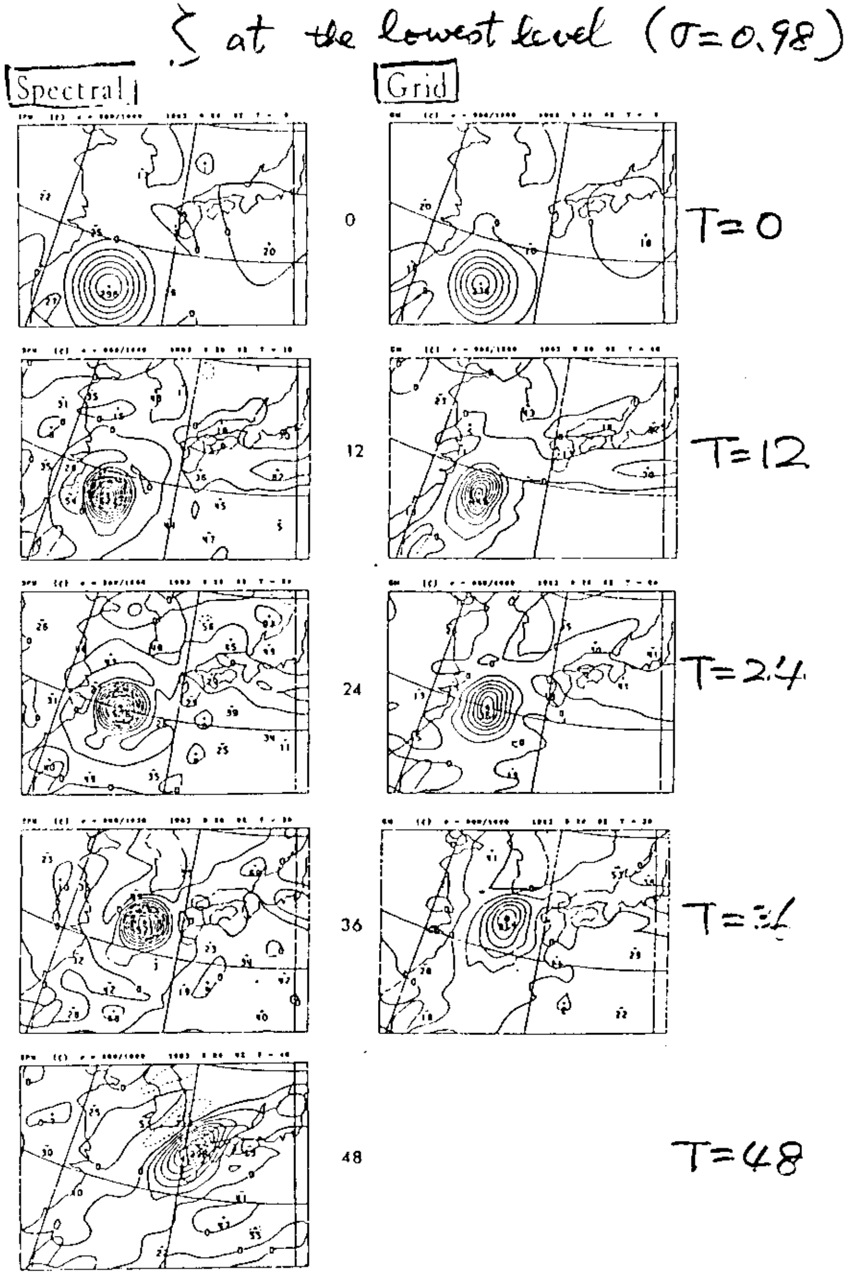


Fig. 15a The forecast comparison of the vorticity of typhoon Forrest (T8310) at the lowest sigma level (sigma=0.98) of the spectral 12L-FLM with E46 truncation (left column) and the grid point 12L-FLM (right column), from initial (top panels) to 48 hours (bottom panel). Contour line of the negative vorticity is shown by broken lines. The integer numbers in the middle column show the forecast time (hour).