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SECOND COLLEGE ON THEORETICAL AND EXPERIMENTAL RADIOPROPAGATION PHYSICS
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INTRODUCTION

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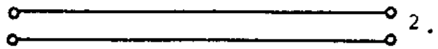
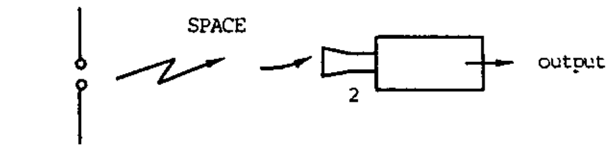
Introduction

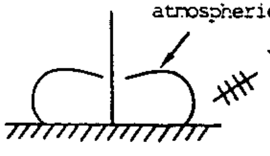
"Guided wave" communication channel

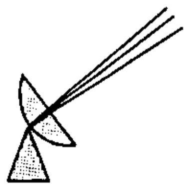


"Free space" communication channel

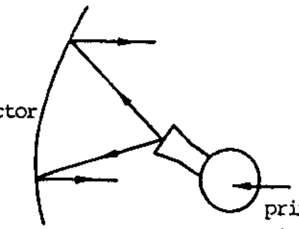


Distribution of radiated power





Radiating currents



1. Maxwell's equations

In the early part of the 19th century Faraday discovered the inducti[on] law, which asserts that a pulsating magnetic flux creates an electric field $\overline{e}$. In integral form (Fig. 1.1) :

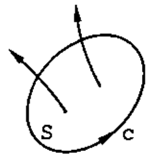


Fig. 1.1

$$\int_c e.\overline{d\ell} = -\frac{\partial \Phi}{\partial t} \qquad (1.1)$$

where Φ is the flux of the magnetic induction $\overline{b}$ through S. In differential form :

$$\boxed{\text{curl } \overline{e} = -\frac{\partial \overline{b}}{\partial t}} \qquad (1.2)$$

The integral form (1.1) can be obtained from (1.2) by application of the theorem of Stokes. The flux Φ which appears in (1.1) is conservative, i.e. the flux of $\overline{b}$ through a closed surface vanishes. We write (Fig. 1.2a)

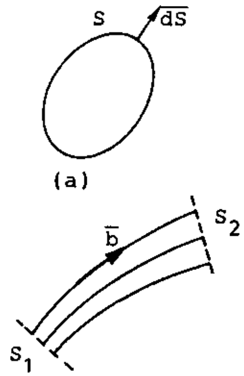


Fig. 1.2

$$\iint_S \overline{b}.\overline{dS} = 0 \qquad (1.3)$$

Applied to the tube of force shown in Fig. 1.2b this relationship implies that the flux Φ through S_2 is the same as through S_1. The differential form of integral relationship (1.3) is

$$\boxed{\text{div } \overline{b} = 0} \qquad (1.4)$$

The flux of the electric field is not conservative. For a closed surface :

$$\iint_S \overline{e}.\overline{dS} = \frac{1}{\varepsilon_o} Q_t \qquad (1.5)$$

where Q_t is the enclosed quantity of electricity. In differential form :

$$\boxed{\text{div } \bar{e} = \frac{1}{\varepsilon_o} \rho_t} \qquad (1.6)$$

The charge density ρ_t is the total charge density, given explicitly in (1.13). At about the time of Faraday's discovery Ampère showed that an electric current creates a magnetic field. In integral form (Fig. 1.1) :

$$\int \bar{b}.\bar{d\ell} = \frac{1}{\mu_o} I \qquad (1.7)$$

where I is the current through S. In differential form :

$$\text{curl } \bar{b} = \mu_o \bar{j}_t \qquad (1.8)$$

where $\bar{j}_t$ is the current density. In a region devoid of currents $\bar{b}$ is irrotational. Maxwell, clearly bothered by the lack of symmetry of (1.2) and (1.8), made a fundamentally deep remark in assuming that, because a pulsating $\bar{b}$ creates an $\bar{e}$, similarly a pulsating $\bar{e}$ must create a $\bar{b}$. This assumption led him to rewrite Ampère's law as

$$\boxed{\text{curl } \bar{b} = \mu_o \bar{j}_t + \varepsilon_o \mu_o \frac{\partial \bar{e}}{\partial t}} \qquad (1.9)$$

It is easy to see, by application of (1.2), (1.4) and (1.9), that (in vacuo)

$$\nabla^2 \bar{b} = \text{grad div } \bar{b} - \text{curl curl } \bar{b} = \varepsilon_o \mu_o \frac{\partial^2 \bar{b}}{\partial t^2} \qquad (1.10)$$

The $\bar{b}$ field therefore satisfies a wave equation, and electromagnetic waves are seen to propagate with a velocity

$$c = \frac{1}{\sqrt{\varepsilon_o \mu_o}} \qquad (1.11)$$

This is the velocity of light in vacuo. Light waves are therefore electromagnetic waves. This fundamental result could not be obtained if the displacement current

$$\bar{j}_{dis} = \varepsilon_o \frac{\partial \bar{e}}{\partial t} \qquad (1.12)$$

had not been introduced in (1.9).

Equations (1.2)(1.4)(1.6) and (1.9) hold in vacuo. In a material medium electric and magnetic polarizations of densities $\bar{m}_e$ and $\bar{m}_m$ may appear. They contribute to the total charge and current densities, which are

$$\begin{aligned} \rho_t &= \rho - \text{div } \bar{m}_e \\ \bar{j}_t &= \bar{j} + \bar{j}_a + \frac{\partial \bar{m}_e}{\partial t} + \text{curl } \bar{m}_m \end{aligned} \qquad (1.13)$$

The densities ρ, $\bar{j}$ and $\bar{j}_a$ represent the contributions of the free charges, which move under the influence of resp. electromagnetic effects (term $\bar{j}$) and non magnetic effects (term $\bar{j}_a$). In an electron gas, for example,

$$\bar{j}_t = \underbrace{- n q \mu \bar{e}}_{\bar{j} = \sigma \bar{e}} + \underbrace{q D \text{ grad } n}_{\bar{j}_a} \qquad (1.14)$$

where $(-q)$ is the charge of the electron, n the electron density, μ the mobility and D the diffusion coefficient. By introducing the notations

$$\begin{aligned} \bar{d} &= \varepsilon_o \bar{e} + \bar{m}_e \\ \bar{h} &= \frac{1}{\mu_o} \bar{b} - \bar{m}_m \end{aligned} \qquad (1.15)$$

Maxwell's equations may be written as

$$\text{curl } \bar{e} = - \frac{\partial \bar{b}}{\partial t} \qquad (1.16)$$

$$\text{curl } \bar{h} = \bar{j} + \bar{j}_a + \frac{\partial \bar{d}}{\partial t} \qquad (1.17)$$

$$\text{div } \bar{d} = \rho \qquad (1.18)$$

$$\text{div } \bar{b} = 0 \qquad (1.19)$$

These are eight scalar equations for the twelve components of $(\bar{e}, \bar{d}, \bar{b}, \bar{h})$. They must be complemented by the constitutive equations. The most frequent ones concern a linear isotropic medium . For such a medium

$$\bar{b} = \mu_r \mu_o \bar{h} = \mu \bar{h} \qquad (1.20)$$

$$\bar{d} = \varepsilon_r \varepsilon_o \bar{e} = \varepsilon \bar{e} \qquad (1.2\ )$$

$$\bar{j} = \sigma \bar{e} \qquad (1.22)$$

where $(\varepsilon, \mu, \sigma)$ are frequency independent. In many technical applications, however, the constitutive parameters are frequency dependent (e.g. in a biomaterial), or tensorial (in anisotropic media such as crystals). The medium can also be nonlinear, in which case the solution of Maxwell's equations becomes much more difficult to perform.

2. Potent.als. Boundary conditions

Maxwell's equation (1.4) can be satisfied if we set

$$\boxed{\bar{b} = \text{curl } \bar{a}} \qquad (2.1)$$

Inserting this relationship in (1.2) yields

$$\text{curl}(\bar{e} + \frac{\partial \bar{a}}{\partial t}) = 0 \qquad (2.2)$$

The vector between brackets is irrotational, hence can be derived from a scalar potential. We therefore write

$$\boxed{\bar{e} = -\text{grad } \phi - \frac{\partial \bar{a}}{\partial t}} \qquad (2.3)$$

Use should now be made of the remaining Maxwell's equations to derive differential equations for ϕ and $\bar{a}$. To simplify matters we shall limit ourselves to fields in vacuo. Inserting (2.1) and (2.3) in (1.6) and (1.9) gives, as $c^2 = (1/\varepsilon_o \mu_o)$,

$$\begin{aligned} -\text{curl curl } \bar{a} - \frac{1}{c^2} \frac{\partial^2 \bar{a}^2}{\partial t^2} - \frac{1}{c^2} \text{grad} \frac{\partial \phi}{\partial t} &= -\mu_o \bar{j} \\ \nabla^2 \phi + \text{div} \frac{\partial \bar{a}}{\partial t} &= -\frac{\rho}{\varepsilon_o} \end{aligned} \qquad (2.4)$$

These equations contain both $\bar{a}$ and ϕ. They may be uncoupled by noticing that the potentials are not unique, hence may be subjected to the additional "Lorenz" condition

$$\text{div } \bar{a} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0 \qquad (2.5)$$

For such c.oice (2.4) becomes

$$\boxed{\begin{aligned} \nabla^2 \bar{a} - \frac{1}{c^2} \frac{\partial^2 \bar{a}}{\partial t^2} &= -\mu_o \bar{j} \\ \nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} &= -\frac{\rho}{\varepsilon_o} \end{aligned}} \qquad (2.6)$$

The last equation is well-known in mathematical physics. It is a scalar wave equation with second member, which is also satisfied, for example, by the acoustic pressure. The solution of this equation is classical. We shall not derived it explicitly, but merely quote the end result, which is

$$\phi(\bar{r},t) = \frac{1}{4\pi\varepsilon_o} \iiint \frac{\rho(\bar{r}', t - \frac{|\bar{r}-\bar{r}'|}{c})}{|\bar{r}-\bar{r}'|} dV' \qquad (2.7)$$

The deep physical meaning of this expression is evident. The formula shows that the effect on an observer located in $\bar{r}$ at time $\underline{t}$ does not depend on the value of the elementary source $\rho\ dV'$ at time $\underline{t}$, but at a previous time $(t - |\bar{r}-\bar{r}'|/c)$. In other words, effects are not felt instantaneously , but the contribution of the source element propagates with a finite velocity $\underline{c}$. This retardation property is the basis of a system such as <u>sonar</u>. It is obviously due to the term in $(1/c^2)(\partial^2\phi/\partial t^2)$ in the equation, as the solution of the equation without second time derivative corresponds to $c = \infty$, hence to vanishing retardation times. It is to be noticed that this <u>static</u> solution yields a good approximation at low frequencies, i.e. when the dimensions of the system are small with respect to $\underline{\lambda}$.

The solution for $\bar{a}$ can be obtained by triple application of (2.7). Thus,

$$\bar{a}(\bar{r},t) = \frac{\mu_o}{4\pi} \iiint \frac{\bar{j}(\bar{r}', t - \frac{|\bar{r}-\bar{r}'|}{c})}{|\bar{r}-\bar{r}'|} dV' \qquad (2.8)$$

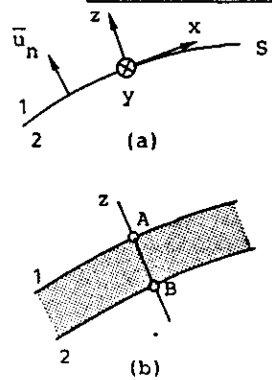


Fig. 2.1

At the separation between two media $\underline{1}$ and $\underline{2}$, the various components of the fields behave in a well defined manner. Let us imagine that a very thin transition layer exists between $\underline{1}$ and $\underline{2}$ (Fig. 2.1b). In this layer (1.4) gives

$$\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} + \frac{\partial b_z}{\partial z} = 0 \qquad (2.9)$$

Let us integrate this equation from B to A, along the normal (the z-axis). This gives

$$b_z(A) - b_z(B) = - \int_B^A \left(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y}\right) dz \qquad (2.10)$$

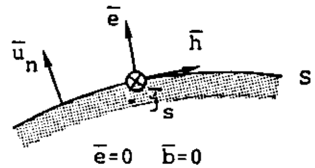
As the derivatives in the $\underline{x}$ and $\underline{y}$ directions are bounded, the integral approaches zero as B approaches A. We conclude that the normal components of $\bar{b}$ have the same value on both sides of the boundary. The other boundary conditions follow by analogous manipulations. Written in detail they are

$b_{n1} = b_{n2}$ (b_n continuous)	(2.11)
$d_{n1} - d_{n2} = \rho_s$	(2.12)
$\bar{u}_n \times \bar{e}_1 = \bar{u}_n \times \bar{e}_2$ ($\bar{e}_{tang}$ continuous)	(2.13)
$\bar{u}_n \times \bar{h}_1 - \bar{u}_n \times \bar{h}_2 = \bar{j}_s$	(2.14)

The notation $\bar{u}_a$ denotes a unit vector in the $\underline{a}$ direction.

The boundary conditions show that a charge density ρ_s on S produces a jump in the normal component of $\bar{d}$, while a surface current $\bar{j}_s$ produces a jump in the tangential component of $\bar{h}$. At the boundary between dielectrics there are no ρ_s and $\bar{j}_s$, unless they are artificially introduced in the form of e.g. current-carrying windings. Except for that case both d_n and $\bar{h}_{tang}$ are continuous.

The boundary conditions take a simple form at the surface of a perfect conductor (a useful model). In a perfect conductor $\bar{e}$ must vanish, otherwise $\bar{j}$ would be infinite, from (1.22), and the Joule losses too (see Sec. 3). It follows, from (2.13), that <u>$\bar{e}$ is perpendicular to</u>



the boundary. On the other hand (1.2) implies that $\bar{b}$ is time independent. The constant value of $\bar{b}$ must be zero because it was so at $t = -\infty$, i.e. before the sources were turned on. We conclude, from (2.11), that $\bar{b}$ (and $\bar{h}$) are tangential. In fact, $\bar{h}$ and $\bar{j}_s$ are connected by the relationships

$$\bar{h} = \bar{j}_s \times \bar{u}_n$$
$$\bar{j}_s = \bar{u}_n \times \bar{h} \qquad (2.15)$$

3. Transmission lines

Basic equations

The potential difference between points x and x + dx may be written as

$$v(x+dx,t)-v(x,t) = \frac{\partial v}{\partial x}\,dx = -r\,dx\,i - \ell\,\frac{\partial i}{\partial t}\,dx + v_g dx \qquad (3.1)$$

In this equation r is the linear resistance in $\Omega\,m^{-1}$, sum of the resistances of conductors 1 and 2, ℓ is the linear inductance (in $H\,m^{-1}$) and v_g the applied voltage (often zero

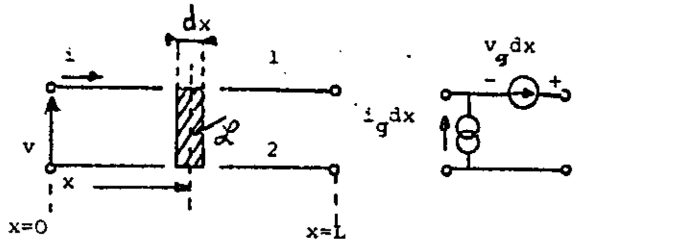


Fig. 3.1

A similar budget may be written for the current in terms of the linear capacitance C (in $F\,m^{-1}$), the linear conductance g (in $S\,m^{-1}$) and a possible current source i_g (in $A\,m^{-1}$). Taking the limit $dx \to 0$ yields the differential equations

$$\frac{\partial v}{\partial x} = -ri - \ell\,\frac{\partial i}{\partial t} + v_g \qquad (3.2)$$

$$\frac{\partial i}{\partial x} = -g\,v - c\,\frac{\partial v}{\partial t} + i_g \qquad (3.3)$$

We shall only consider the sourceless situation ($v_g = 0$, $i_g = 0$), and pay special attention to sinusoidal phenomena. For such case (3.2) and (3.3) become

$$\frac{dV}{dx} = -r\,I - j\omega\ell\,I = -(r+j\omega\ell)\,I = -ZI \qquad (3.4)$$

$$\frac{dI}{dx} = -g\,V - j\omega c\,V = -(g + j\omega c)V = -YV \qquad (3.5)$$

Incident and reflected waves

When the line is lossless the basic equations are

$$\boxed{\begin{aligned}\frac{\partial v}{\partial x} &= -\ell\,\frac{\partial i}{\partial t}\\ \frac{\partial i}{\partial x} &= -c\,\frac{\partial v}{\partial t}\end{aligned}} \qquad (3.6)$$

Eliminating i yields

$$\boxed{\frac{\partial^2 v}{\partial x^2} - \ell c\,\frac{\partial^2 v}{\partial t^2} = 0} \qquad (3.7)$$

The general solution of this equation is obtained by way of the change of variables

$$u = x - \frac{1}{\sqrt{lc}}\,t \qquad v = x + \frac{1}{\sqrt{lc}}\,t \qquad (3.8)$$

which converts (3.7) into

$$\frac{\partial^2 v}{\partial u\,\partial w} = 0 \qquad (3.9)$$

The general solution of this equation is

$$v = f(u) + g(w) = f(x - \frac{1}{\sqrt{lc}}\,t) + g(x + \frac{1}{\sqrt{lc}}\,t) \qquad (3.10)$$

The corresponding current has the form

$$i = \frac{1}{R_c}\left[f(x - v_{ph}t) - g(x + v_{ph}t)\right] \quad (R_c = \sqrt{\frac{\ell}{c}}) \qquad (3.11)$$

Here R_c is the characteristic resistance of the line (in Ω), and v_{ph} is the phase velocity $(1/\sqrt{lc})$. The functions f and g are arbitrary, and their actual form is determined by the boundary

conditions at x = 0 and x = L. The function f represents a wave to increasing x, as shown clearly in Fig. 3.2. The function g represents a wave to decreasing x. Normally a generator is connected in x = 0 and a load in x = L. The f wave then becomes an incident wave and the g wave a reflected one. It is easy to check that the incident wave is the only one to exist when the line is infinite or, equivalently, when it is loaded by R_c. For such a case there are no reflections, and the line is matched.

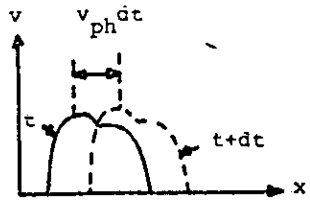


Fig. 3.2

Time harmonic signals

Voltage and current have the general form

$$v(x,t) = v_1 \cos(\omega t - kx + \phi_1) + v_2 \cos(\omega t + kx + \phi_2) \qquad (3.12)$$

$$i(x,t) = \frac{1}{R_c}\left[v_1 \cos(\omega t - kx + \phi_1) - v_2 \cos(\omega t + kx + \phi_2)\right]$$

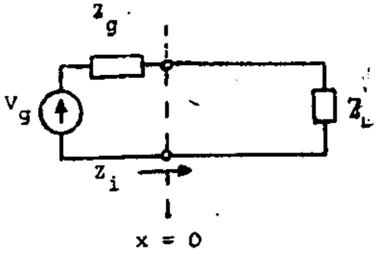
where

$$k = \frac{\omega}{v_{ph}} = \omega\sqrt{\ell c} = \frac{2\pi}{\lambda_g} \qquad (3.13)$$

The quantity λ_g is the wavelength along the line. In phasor form :

$$\begin{aligned} V(x) &= V_1\,e^{-jkx} + V_2\,e^{jkx}\\ I(x) &= \frac{1}{R_c}\left[V_1\,e^{-jkx} - V_2\,e^{jkx}\right]\end{aligned} \qquad (3.14)$$

where $V_1 = v_1\,e^{j\phi_1}$ and $V_2 = v_2\,e^{j\phi_2}$. Writing (3.14) at x = 0 and x = d leads to the value of the input impedance of the line



$$\frac{Z_i}{R_c} = Z_i' = \frac{Z_L' + j\ \mathrm{tg}\ kd}{1 + j\ Z_L'\ \mathrm{tg}\ kd} \qquad (3.15)$$

Fig. 3.3

When the line is lossy the formulas become

$$V = V_1\ e^{-\alpha x} e^{-j\beta x} + V_2\ e^{\alpha x} e^{j\beta x}$$

$$I = \frac{1}{Z_c}\ [V_1\ e^{-\alpha x} e^{-j\beta x} - V_2\ e^{\alpha x} e^{j\beta x}] \qquad (3.16)$$

Attenuation and propagation constant are given by

$$\gamma^2 = (\alpha + j\beta)^2 = Y.Z = \alpha^2 - \beta^2 + 2j\alpha\beta = (rg - \omega^2 \ell c) + j(\omega \ell g + \omega rc) \qquad (3.17)$$

The characteristic impedance becomes

$$Z_c^2 = \frac{Z}{Y} = (R_c + j\ X_c)^2 = \frac{rg + \omega^2 \ell c + j(\omega \ell g - \omega rc)}{g^2 + \omega^2 c^2} \qquad (3.18)$$

The input impedance is now

$$Z_i' = \frac{Z_i}{Z_c} = \frac{Z_L' + \mathrm{th}\gamma\ d}{1 + Z_L'\mathrm{th}\gamma\ d} \qquad (3.19)$$

If the load is matched ($Z_l' = 1$) the input impedance is Z_c, irrespective of the length of the line.

Reflection coefficient on a lossless line

From (3.14) the ratio between reflected and incident voltages (the reflection coefficient) is

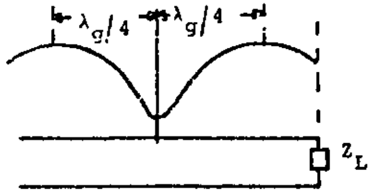


Fig. 3.4

$$K = K_V = \frac{V_2\ e^{jkx}}{V_1\ e^{-jkx}} = \frac{V_2}{V_1}\ e^{2jkx} \qquad (3.20)$$

At a distance d from the load, therefore,

$$K = K_L\ e^{-2j\ kd} \qquad (3.21)$$

Reflection coefficient and impedance are connected by

$$K(x) = \frac{Z_i'(x) - 1}{Z_i'(x) + 1} \qquad Z_i' = \frac{1+K}{1-K}) \qquad (3.22)$$

This formula shows that a measurement of Z_i may be obtained from a measurement of K (in amplitude and phase). The value of K may be deduced from an observation of the interference between reflected and incident waves down the line. The amplitude of the voltage is given by

$$|V| = \sqrt{V_1^2 + V_2^2 + 2V_1V_2 \cos\ (2kx + \phi_2 - \phi_1)}$$

$$= V_1\sqrt{1 + |K|^2 + 2\ |K|\ \cos\ (2kx + \phi_2 - \phi_1)} \qquad (3.23)$$

The general appearance of the curve $|V(x)|$ is as shown in Fig. 3.4. The maxima occur at points where the two waves are in phase, i.e. for

$$-kx + \phi_1 = kx + \phi_2 + n2\pi \qquad (3.24)$$

These points are separated by $(\lambda_g/2)$. The minima (destructive interference) correspond to

$$-kx + \phi_1 = kx + \phi_2 + \pi + 2n\pi \qquad (3.25)$$

They are located $(\lambda_g/4)$ from the maxima. An observation of the ratio of maximum to minimum (the standing wave ratio) gives $|K|$:

$$\text{VSWR} = \frac{V_1+V_2}{V_1-V_2} = \frac{1+\frac{V_2}{V_1}}{1-\frac{V_2}{V_1}} = \frac{1+|K|}{1-|K|} \qquad (3.26)$$

The location of the maxima and minima gives $(\phi_2 - \phi_1)$, i.e. the phase angle of K_L. One is then able to construct K_L, and from there to determine the unknown Z_L by use of (3.22). The method is the classical way of determining impedances at high frequencies.

Matching

In most applications it is desirable to match the load to the line. The reasons are :

(1) The power to the load. It is given by

$$\mathcal{P} = \frac{1}{2}\,\text{Re}\,(VI^*) = \frac{V_1^{\;2}}{2R_c} - \frac{V_2^{\;2}}{2R_c} = \frac{V_1^{\;2}}{2R_c}\,(1 - |K|^2) \qquad (3.27)$$

$\mathcal{P}$ is maximum when $|K| = 0$.

(2) Reflections create peaks of voltage. For a given absorbed power $\mathcal{P}$:

$$|V_{max}| = \sqrt{2\mathcal{P}R_c}\,\sqrt{\frac{1+K}{1-K}} \qquad (3.28)$$

The danger for breakdown increases with $|K|$.

(3) the input impedance is much more sensitive to small frequency excursions when the load is unmatched. This "long line effect" will be explained in the problem session, together with methods to match an (originally unmatched) load, and the use of the Smith chart.

These various considerations may be applied, in slightly modified form, to the lossy line. There,

$$K = K_L\, e^{-2\gamma d} = K_L\, e^{-2\alpha d}\, e^{-2j\beta d} \qquad (3.29)$$

$$K = \frac{Z'-1}{Z'+1}$$

4. Power budget. Poynting's vector

Scalar multiplication of Maxwell's equations (1.16) and (1.17)
with resp. $\bar{h}$ and $\bar{e}$ gives

$$\bar{h}.\text{curl}\ \bar{e} = -\bar{h}\ .\ \frac{\partial \bar{b}}{\partial t} \qquad (4.1)$$

$$\bar{e}.\text{curl}\ \bar{h} = \bar{e}.(\bar{j} + \bar{j}_a + \frac{\partial \bar{d}}{\partial t}) \qquad (4.2)$$

Subtraction of the second equation from the first yields

$$\text{div}(\bar{e} \times \bar{h}) = \bar{h}.\text{curl}\ \bar{e} - \bar{e}.\text{curl}\ \bar{h} = -\bar{e}.\bar{j}_f - \bar{h}.\ \frac{\partial \bar{b}}{\partial t} - \bar{e}.\ \frac{\partial \bar{d}}{\partial t}$$

$$= -\bar{j}_f\ .\ (\frac{\bar{j}_f}{\sigma} - \bar{e}_a) - (\bar{h}.\ \frac{\partial \bar{b}}{\partial t} + \bar{e}.\ \frac{\partial \bar{d}}{\partial t}) \qquad (4.3)$$

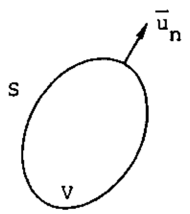


Fig. 4.1

where we have eliminated $\bar{e}$ by writing the free charge current as

$$\bar{j}_f = \bar{j} + \bar{j}_a = \sigma\bar{e} + \sigma\bar{e}_a \qquad (4.4)$$

Vector $\bar{e}_a$ is the applied electric field. Let us now integrate (4.3) over a volume V, to obtain (Fig. 4.1)

$$\iint_S \bar{u}_n.(\bar{e} \times \bar{h})dS = -\iiint_V \frac{1}{\sigma}|\bar{j}_f|^2 dV + \iiint_V \bar{e}_a.\bar{j}_f dV - \iiint_V (\bar{h}.\ \frac{\partial \bar{b}}{\partial t} + \bar{e}.\ \frac{\partial \bar{d}}{\partial t})dV \qquad (4.5)$$

The variation in the electromagnetic energy contained in V is

$$d\mathcal{E} = \iiint_V [\bar{h}.d\bar{b} + \bar{e}.d\bar{d}]\ dV \qquad (4.6)$$

It follows that (4.5) can be rewritten in the form of a power budget, to wit

$$\boxed{\iiint \bar{e}_a.\bar{j}_f dV = \iiint_V \frac{1}{\sigma}|\bar{j}_f|^2 dV + \frac{\partial \mathcal{E}}{\partial t} + \iint_S \bar{u}_n.(\bar{e} \times \bar{h})dS} \qquad (4.7)$$

Let us interpret the various terms of this relationship. The left-hand member is the power delivered by the applied electric field (in the form of, for example, chemical or thermoelectric power). The second member shows how this power is spent :

- the first term is Joule power (a generalization of the ri^2 formula of network theory)
- the second term is the rate of increase of the electromagnetic energy
- the third term is the electromagnetic power leaving the volume. It is the flux of the vector of Poynting

$$\boxed{\bar{s} = \bar{e} \times \bar{h}} \quad (W\ m^{-2}) \qquad (4.8)$$

An interesting application is sketched in Fig. 4.2, in which a radiating antenna is surrounded by a perfectly conducting screen. Boundary condition (2.15) implies that Poynting's vector $\bar{s}$ lies in the tangent plane of S_1. It follows that no power penetrates through S_1, hence that the outside world has been screened from the radiating system.

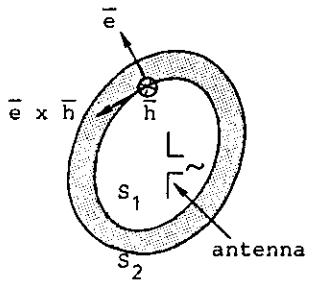


Fig. 4.2

5. Sinusoidal phenomena. Polarization

Signals with a sinusoidal time dependence play an important role in Telecommunications. Such signals are carried by fields with time-harmonic components of the form

$$a_x = a_{xm}\cos(\omega t + \phi_x)$$
$$a_y = a_{ym}\cos(\omega t + \phi_y) \qquad (5.1)$$
$$a_z = a_{zm}\cos(\omega t + \phi_z)$$

The vector $\bar{a}$ could be an electric field, a current density, a velocity or any other relevant vector quantity. The components of $\bar{a}$ can be written in phasor form as

$$A_x = a_{xm}e^{j\phi_x}$$
$$A_y = a_{ym}e^{j\phi_y} \qquad (5.2)$$
$$A_z = a_{zm}e^{j\phi_z}$$

where capital letters are used for the complex quantities. Complex form (5.2) contains the relevant phase and amplitude information. It allows restoring the actual time dependence by the simple operation

$$a_x(t) = \mathrm{Re}\,[A_x\, e^{j\omega t}] \qquad (5.3)$$

The full vector $\bar{a}$ can analogously be written in complex form as

$$\bar{A} = A_x\bar{u}_x + A_y\bar{u}_y + A_z\bar{u}_z$$
$$= (a_{xm}\cos\phi_x\bar{u}_x + a_{ym}\cos\phi_y\bar{u}_y + a_{zm}\cos\phi_z\bar{u}_z) \qquad (5.4)$$
$$+ j(a_{xm}\sin\phi_x\bar{u}_x + a_{ym}\sin\phi_y\bar{u}_y + a_{zm}\sin\phi_z\bar{u}_z)$$
$$= \bar{a}_r + j\,\bar{a}_i$$

We have separated the complex vector into its real and imaginary components $\bar{a}_r$ and $\bar{a}_i$. This allows restoring the actual time dependence by triple application of (5.3). Thus

$$\bar{a}(t) = \mathrm{Re}\,[\bar{A}\, e^{j\omega t}] = \bar{a}_r\cos\omega t - \bar{a}_i\sin\omega t \qquad (5.5)$$

This equation clearly shows that the tip of $\bar{a}$ describes a curve

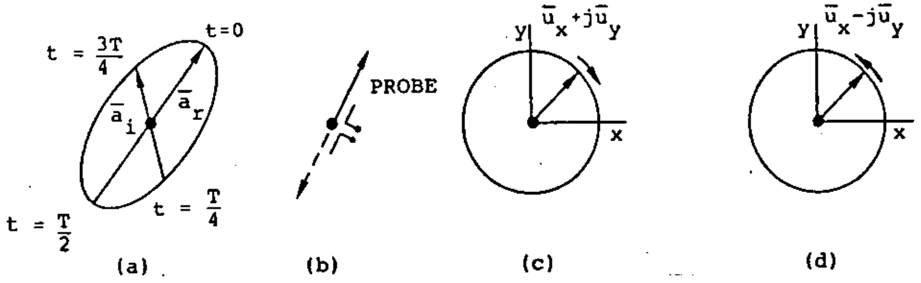


Fig. 5.1

in the $(\bar{a}_r,\bar{a}_i)$ plane. This curve is an ellipse, as can be shown by the following simple calculation. Let us take the $(\bar{a}_r,\bar{a}_i)$ plane as the (x,y) plane. The coordinates of the tip of the vector are

$$X = a_{rx}\cos\omega t - a_{ix}\sin\omega t$$
$$Y = a_{ry}\cos\omega t - a_{iy}\sin\omega t \qquad (5.6)$$

The time-coordinate <u>t</u> is a parameter which may be eliminated by making use of the relationship

$$\sin^2\omega t + \cos^2\omega t = 1 \qquad (5.7)$$

This gives

$$(Xa_{ry} - Ya_{rx})^2 + (Xa_{iy} - Ya_{ix})^2 = (a_{rx}a_{iy} - a_{ry}a_{ix})^2 \qquad (5.8)$$

which is the equation of an ellipse (Fig. 5.1a). A time harmonic signal is consequently <u>elliptically polarized</u>. An important particular case is the <u>linear polarization</u>, which occurs when $\bar{a}_r$ and $\bar{a}_i$ are parallel. For such case the oscillating vector remains parallel with a given direction during its whole period of variation T (Fig. 5.1b). An advantage of the linear polarization (examples of which are the vertical and horizontal polarizations) is that a small probe in the form of a short antenna will pick up a maximum voltage when parallel with the field, and no signal when perpendicular to the latter. This consideration leads to the use of polarization diversity, a system in which frequency use is increased by transmitting two signals of identical frequency on waves with mutually perpendicular polarizations.

Another important particular case is the circular polarization, in which the $\bar{a}_r$ and $\bar{a}_i$ vectors have the same magnitude, but are perpendicular. The tip of the vector now describes a circle, either in the clockwise direction(with respect to the z-axis : see Fig. 5.1c)or in the counterclockwise direction (Fig. 5.1d).

The basic vector equations discussed in the previous sections may be written in complex form by replacing $\frac{\partial}{\partial t}$ by $j\omega$ whenever needed. Maxwell's equations, for example, become

$$\begin{aligned} \text{curl } \bar{E} &= -j\omega\bar{B} \\ \text{curl } \bar{H} &= j\omega\bar{D} + \underbrace{\bar{J} + \bar{J}_a}_{\bar{J}_{tot}} \\ \text{div } \bar{B} &= 0 \\ \text{div } \bar{D} &= \rho \end{aligned} \tag{5.9}$$

It is to be noticed that a complex vector such as $\bar{E}$ is a function of the space coordinates (x,y,z).

The form of the power budget under time harmonic conditions may be obtained from (5.9). In a medium of parameters (ε,μ,σ) we write

$$\bar{H}^*.\text{curl } \bar{E} - \bar{E}.\text{curl } \bar{H}^* = -j\omega\bar{H}^*.\bar{B} - \bar{E}.(-j\omega\bar{D}^* + \bar{J}^*_{tot}) \tag{5.10}$$

or

$$\text{div}(\bar{E} \times \bar{H}^*) = -j\omega\mu|\bar{H}|^2 + j\omega\varepsilon|\bar{E}|^2 - \frac{|\bar{J}_{tot}|^2}{\sigma} + \bar{E}_a.\bar{J}^*_{tot} \tag{5.11}$$

In this expression the notation $|\bar{A}|^2$ stands for $\bar{A}.\bar{A}^* = |\bar{a}_r|^2 + |\bar{a}_i|^2$.

A classical result from circuit theory asserts that the time-average of the product of two sinusoidal quantities $\underline{a}$ and $\underline{b}$

is given by $\frac{1}{2}\text{Re}(AB^*)$, where A and B are the complex representations of $\underline{a}$ and $\underline{b}$. Applied to power budget (5.11) this formula gives

$$\boxed{\frac{1}{2}\text{Re}\iiint_V \bar{E}_a.\bar{J}^*_{tot}dV = \frac{1}{2}\iiint_V \frac{|\bar{J}_{tot}|^2}{\sigma}dV + \frac{1}{2}\text{Re}\iint_S (\bar{E} \times \bar{H}^*).\overline{dS}} \tag{5.12}$$

It is seen that the average power delivered by the applied field (left-hand member) is spent in the form of

- an average Joule effect in the conductors (first term, second member)
- an average radiated power (second term, second member).

The vector $\bar{E} \times \bar{H}^*$ is the complex vector of Poynting. Half its real part represents the average power flux per m^2.

6 . Modes and eigenfunctions

We shall discuss the basic ideas of the eigenfunction method on the very simple example of the flexible string, a basic component of several musical instruments (Fig. 6.1). The string is under tension T, and is acted upon by an external force of density p(x) (in N m^{-1}). The force on a small element of string dx is therefore p(x)dx. The equation satisfied by the small displacement y(x,t) of the string is a wave equation, viz.

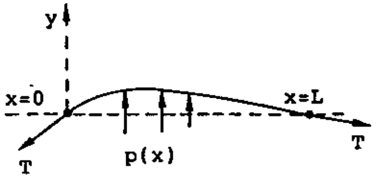


Fig. 6.1

$$\begin{cases} \dfrac{\partial^2 y}{\partial x^2} - \dfrac{\rho}{T}\dfrac{\partial^2 y}{\partial t^2} = -\dfrac{p(x,t)}{T} \\ y = 0 \text{ in } x = 0 \text{ and } x = L \end{cases} \qquad (6.1)$$

The symbol ρ denotes the mass density of the string (in kg m^{-1}). The velocity of propagation is $c = \sqrt{(T/\rho)}$. Under time-harmonic conditions :

$$\begin{cases} \dfrac{d^2 Y}{dx^2} + \dfrac{\omega^2\rho}{T} Y = -\dfrac{P(x)}{T} \\ Y = 0 \text{ at } x = 0 \text{ and } x = L \end{cases} \qquad (6.2)$$

A first question concerns the existence of free vibrations, i.e. of time-harmonic displacements which may be sustained in the absence of external forces. From (12.2) the problem reduces to the determination of functions $y_n(x)$ satisfying

$$\begin{cases} \dfrac{d^2 y_n}{dx^2} = \lambda_n y_n \\ y_n = 0 \text{ at } x = 0 \text{ and } x = L \end{cases} \qquad (6.3)$$

These functions are the eigenfunctions of the operator (d^2/dx^2). Their main property is that, acted upon by the operator, they

reproduce their own form, but with a coefficient λ_n termed the eigenvalue. It is easy to solve (6.3) explicitly. Thus,

$$y_n = \sin \frac{n\pi x}{L}$$
$$\lambda_n = -\left(\frac{n\pi}{L}\right)^2 \qquad (6.4)$$

The frequency of the free vibrations, obtained by equating $(\omega^2\rho/T)$ to $(-\lambda_n)$, is

$$\nu_n = n\,\frac{1}{2L}\sqrt{\frac{T}{\rho}} \qquad (6.5)$$

The eigenfunctions enjoy a crucial property, which is : they are orthogonal. By this we mean that

$$\int_0^L y_n\, y_m\, dx = 0 \qquad (\text{for } m \neq n) \qquad (6.6)$$

To solve a system such as (6.2) by the method of eigenfunctions, we expand Y(x) in a series

$$Y(x) = \sum_{n=1}^{\infty} A_n\, y_n(x) \qquad (6.7)$$

The problem is to determine the A_n. To do so, we write

$$\frac{d^2 Y}{dx^2} = \Sigma\, A_n \frac{d^2 y_n}{dx^2} = \Sigma\, \lambda_n A_n y_n \qquad (6.8)$$

$$P(x) = \Sigma\, B_m\, y_m(x) \qquad (6.9)$$

The validity of the term by term differentiation involved in (6.8) is not automatically guaranteed, but in the present case it can be shown to hold. Making use of (6.6) allows determination of B_n. Multiplying, indeed, (6.9) with $y_n(x)$, and integrating from 0 to L, yields

$$\int_0^L y_n(x)P(x)dx = B_n \int_0^L y_n^2\, dx = \frac{L}{2} B_n \qquad (6.10)$$

or

$$B_n = \frac{2}{L}\int_0^L y_n \, P \, dx \qquad (6.11)$$

Inserting (6.7),(6.8) and (6.9) in (6.2) gives, upon equating coefficients of y_n,

$$A_n = -\frac{1}{\rho}\frac{B_n}{\omega^2-\omega_n^2} \qquad (6.12)$$

The solution for Y(x) is therefore

$$Y(x) = -\frac{1}{2\pi^2\rho L}\sum \frac{\int_0^L P(x')\sin\frac{n\pi x'}{L}dx'}{\nu^2-\nu_n^2}\sin\frac{n\pi x}{L} \qquad (6.13)$$

This solution is obtained in the form of an infinite sum, the "building blocks" of which are the eigenfunctions. Such a sum may easily be programmed on a digital computer. Notice the infinite amplitudes which occur at the resonant frequencies ν_n. These infinities disappear when the frictional and radiative losses of the string are taken into account. Notice that expansion (6.7) can also be used to solve the general wave equation (6.1). Coefficient A_n is now a function of time $a_n(t)$, and steps similar to the previous ones show that a_n satisfies

$$\frac{d^2a_n(t)}{dt^2} + \frac{T}{\rho}\left(\frac{n\pi}{L}\right)^2 a_n(t) = \frac{2}{\rho L}\int_0^L p(x,t)\sin\frac{n\pi x}{L}dx \qquad (6.14)$$

This is the equation of an (L,C) circuit.

7. Closed electromagnetic waveguides

The modal expansion in a waveguide makes use of the eigenfunctions of the Dirichlet problem

$$\bar f_m = \text{grad}\,\phi_m$$
$$\nabla^2\phi_m + \mu_m^2\phi_m = 0 \quad \text{in } S \qquad (7.1)$$
$$\phi_m = 0 \quad \text{on } C$$
$$\iint_S |\bar f_m|^2\, dS = 1$$

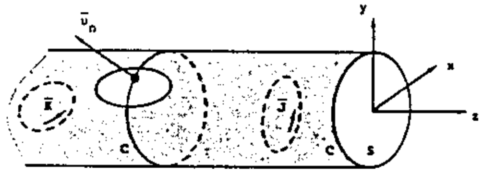


Fig. 7.1

The index m really stands for a double index (m,n). In a rectangle, for example (Fig. 7.3),

$$\phi_{mm} = \frac{2}{\sqrt{\frac{m^2b}{a}+\frac{n^2a}{b}}}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad (7.2)$$

We also need the eigenfunctions of the Neumann problems

$$\bar g_m = \text{grad}\,\psi$$
$$\nabla^2\psi_m + \nu_m^2\psi_m = 0 \quad \text{in}$$
$$\frac{\partial\psi_m}{\partial n} = 0 \quad \text{on} \qquad (7.3)$$
$$\iint_S |\bar g_m|^2\, dS = 1$$

The field expansions are

$$\bar E(r) = \sum_m V_m(z)\,\text{grad}\,\phi_m + \sum_n V_n(z)\,\text{grad}\,\psi_n \times \bar u_z + \sum A_m(z)\phi_m\,\bar u_z$$
$$\bar H(\bar r) = \sum_m I_m(z)\bar u_z \times \text{grad}\,\phi_m + \sum I_n(z)\,\text{grad}\,\psi_n + \sum B_n(z)\psi_n\,\bar u_z \qquad (7.4)$$

The problem is to find the expansion coefficients. The method proceeds by requiring (7.4) to satisfy Maxwell's equations. The source terms are volume electric and magnetic currents, $\vec J$ and $\vec K$, and sidewall

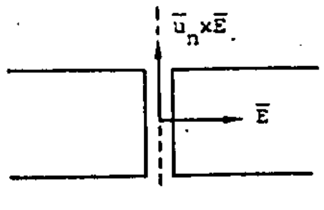


Fig. 7,2

aperture fields. Detailed calculations give, for the E (or TM) modes

$$\frac{dV_m}{dz} + j\omega\mu_o I_m - A_m = -\iint_S \bar{K}.(\bar{u}_z \times \text{grad } \phi_m)dS$$

$$-\int_c (\bar{u}_n \times \bar{E}).(\bar{u}_z \times \text{grad } \phi_m)dc \qquad (7.5)$$

$$\frac{dI_m}{dz} + j\omega\varepsilon_o V_m = - \iint_S \bar{J}.\text{grad } \phi_m \, dS$$

$$I_m + \frac{j\omega\varepsilon_o}{\mu_m^2} A_m = - \iint_S (\bar{J}.\bar{u}_z)\phi_m \, dS$$

Here, $\bar{u}_n$ is a unit vector perpendicular to the waveguide wall. The contour integrals containing $(\bar{u}_n \times \bar{E})$ represent the excitation through the aperture. (7.6)

Similar equations hold for the H (or TE) modes :

$$\frac{dV_n}{dz} + j\omega\mu_o I_n = -\iint_S \bar{K}.\text{grad } \psi_n \, dS - \int_c (\bar{u}_n \times \bar{E}).\text{grad } \psi_n \, dc$$

$$\frac{dI_n}{dz} + j\omega\varepsilon_o V_n - B_n = - \iint_S \bar{J}.(\text{grad } \psi_n \times \bar{u}_z)dS$$

$$V_n + \frac{j\omega\mu_o}{\nu_n^2} B_n = - \iint_S (\bar{K}.\bar{u}_z)\psi_n \, dS - \int_S (\bar{u}_n \times \bar{E}).\bar{u}_z \, \psi_n \, dc$$

For both modes two of the three unknowns may be eliminated, and an equation for the third obtained. For the TE modes, for example, and for perfectly conducting walls,

$$\frac{d^2V_n}{dz^2} + (k^2 - \nu_n^2)V_n = f(z) \qquad (7.7)$$

where f(z) is a source term which vanishes outsides the source region. Far away from the sources, therefore, V_n is a linear combination of exponentials. When $k > \nu_n$, i.e. when the frequency is above cut-off, V_n is of the form

$$A\,e^{-j\sqrt{k^2-\nu_n^2}\,z} + B\,e^{j\sqrt{k^2-\nu_n^2}\,z} \qquad (7.8)$$

outside the sources. The mode is propagated, and transmission line theory may be applied. The wavelength in the guide is

$$\lambda_g = \frac{\lambda}{\sqrt{1 - \frac{\nu_n^2}{k^2}}} \qquad (7.9)$$

and the phase velocity

$$v_{ph} = \frac{c}{\sqrt{1 - \frac{\nu_n^2}{k^2}}} \qquad (7.10)$$

A propagated mode is therefore dispersive. When $k < \nu_n$, i.e. below cut-off and outside the sources, V_n is of the form

$$A\,e^{-\sqrt{\nu_n^2 - k^2}\,z} + B\,e^{+\sqrt{\nu_n^2 - k^2}\,z} \qquad (7.11)$$

The mode is attenuated. The same considerations hold for the TM modes.

It is clear that the number of propagated modes is finite. By suitable choice of the frequency it is possible to launch only one mode (monomode operation). This lowest mode always belongs to the TE family. In a rectangular guide the relevant data are

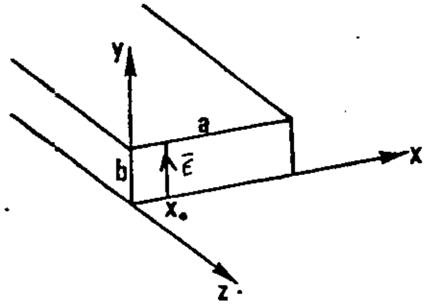


Fig. 7.3

$$\psi_{10} = \cos \frac{\pi x}{a}$$

$$\nu_{10} = \frac{\pi}{a} \qquad (7.12)$$

$$\text{cut off freq} = \frac{c}{2a}$$

8. Plane waves. Reflection and transmission at plane interfaces

By definition the fields in a plane wave depend on a single space coordinate, say z. Keeping therefore only derivatives with respect to z and t in Maxwell's equations (1.16) to (1.19) yields

$$\begin{cases} \dfrac{\partial e_x}{\partial z} = -\mu \dfrac{\partial h_y}{\partial t} \\ \dfrac{\partial h_y}{\partial z} = -\sigma e_x - \varepsilon \dfrac{\partial e_x}{\partial t} \end{cases} \qquad (8.1)$$

$$\begin{cases} \dfrac{\partial e_y}{\partial z} = -\mu \dfrac{\partial (-h_x)}{\partial t} \\ \dfrac{\partial (-h_x)}{\partial z} = -\sigma e_y - \varepsilon \dfrac{\partial e_y}{\partial t} \end{cases} \qquad (8.2)$$

A plane wave is therefore the superposition of two linearly polarized waves, one with $\bar{e}$ in the x direction, the other with $\bar{e}$ in the y direction. Equations (8.1) and (8.2) are of the transmission line type, and it is clear that (e_x, h_y) play the role of v and i. In a lossless medium :

$$e_x = f(z-ct)+g(z+ct)$$
$$h_y = \frac{1}{R_c}\,[f(z-ct)-g(z+ct)] \qquad (8.3)$$

where

$$R_c = \sqrt{\frac{\mu}{\varepsilon}} \qquad \text{(the characteristic resistance)}$$
$$c = \frac{1}{\sqrt{\varepsilon\mu}} \qquad \text{(the propagation velocity)} \qquad (8.4)$$

In a lossy medium, under time-harmonic conditions,

$$E_x = E_1\, e^{-\gamma z} + E_2\, e^{\gamma z} = E_1\, e^{-\alpha z} e^{-j\beta z} + E_2\, e^{\alpha z} e^{j\beta z}$$
$$H_y = \frac{1}{Z_c}\,(E_1\, e^{-\gamma z} - E_2\, e^{\gamma z}) \qquad (8.5)$$

where E_1 and E_2 are arbitrary complex coefficients, and

$$\gamma^2 = -\omega^2\varepsilon\mu + j\omega\mu\sigma = (\alpha + j\beta)^2$$
$$Z_c = \sqrt{\frac{j\omega\mu}{\sigma+j\omega\varepsilon}} \quad \text{(the characteristic impedance)} \qquad (8.6)$$

In a good conductor the displacement current $|j\omega\bar{d}|$ is, by definition, negligible with respect to the conduction current $|\sigma\bar{e}|$. For such case

$$\gamma = (1 + j)\sqrt{\frac{\omega\mu\sigma}{2}} = \frac{1+j}{\delta}$$
$$Z_c = \sqrt{\frac{\omega\mu}{2\sigma}}\,(1+j) \qquad (8.7)$$

The quantity δ is the penetration depth, defined as

$$\boxed{\delta = \sqrt{\frac{2}{\omega\mu\sigma}}} \qquad (8.8)$$

This depth, which is of great importance in practice, is of the order of $1\mu m$ at 10GHz for a good conductor such as Cu or Al. The practical meaning of δ is clearly illustrated by condidering the transmission of power in a conductor illuminated by a plane wave at normal incidence.

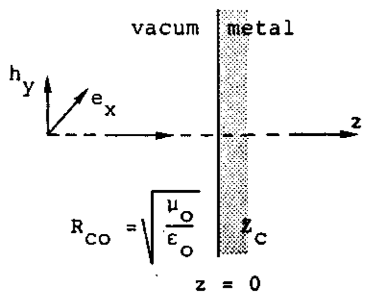


Fig. 8.1

The fields in vacuum can be written as

$$E_x = e^{-jkz} + K\, e^{jk_o z}$$
$$H_y = \frac{1}{R_{co}}\,(e^{-jk_o z} - K\, e^{jk_o z}) \qquad (8.9)$$

In this formulas $k_o = (2\pi/\lambda)$ and R_{co} refer to vacuum, while K is the reflection coefficient. In the good conductor :

$$E_x = T\, e^{-\frac{z}{\delta}}\, e^{-j\frac{z}{\delta}}$$
$$H_y = \frac{T}{Z_c}\, e^{-\frac{z}{\delta}}\, e^{-j\frac{z}{\delta}} \qquad (8.10)$$

The values of K and T are obtained by expressing continuity of E_x and H_y at $z = 0$. This gives

$$T \approx \frac{2Z_c}{R_{co}} = 2\sqrt{\frac{\omega\varepsilon_o}{\sigma}}\, e^{j\frac{\pi}{4}} = 2\sqrt{Q}\, e^{j\frac{\pi}{4}} \qquad (8.11)$$

It is seen, from (8.10), that fields and currents decrease exponentially, and that practically no fields are left at a depth of, say, 5δ. Such a property is obviously important for the design of shields. It is also to be noticed that the z-dependence embodied in (8.10) still holds under <u>oblique</u> incidence. The proof is omitted here.

When the conductor is perfect total reflection occurs, i.e. the amplitude of the reflected field is equal to that of the incident field. The reflection pattern

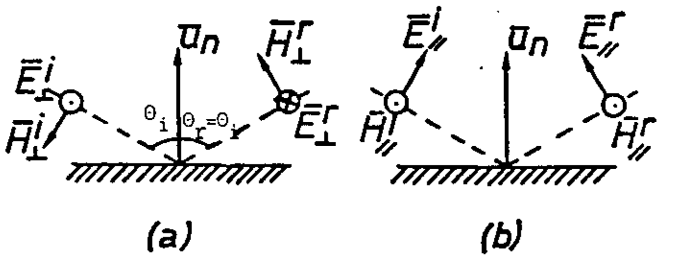


Fig. 8.2

is shown in Fig. 8.2 for both fundamental polarizations. This pattern is obtained by imposing the boundary conditions discussed in Sec. 2, which require the tangential component of $\bar{e}$ and the normal component of $\bar{h}$ to vanish on the metal. The surface charge and current densities which appear on the conductor are found (a posteriori) from (2.12) and (2.15). Thus, for both polarizations,

$$\boxed{\begin{aligned} P_s &= \varepsilon_o E_n = 2\varepsilon_o(\bar{u}_n \cdot \bar{E}_i) \\ \bar{J}_s &= \bar{u}_n \times \bar{H} = 2(\bar{u}_n \times \bar{H}_i) \end{aligned}} \qquad (8.12)$$

These formulas remain approximately valid when the conducting surface is slightly curved. This remark is of considerable importance for the solution of the scattering problem discussed

in Sec. 16.

As a last configuration we consider the boundary between two lossless, non magnetic dielectrica (Fig. 8.3). Assume first that $\bar{E}$ is polarized perpendicularly to the plane of incidence (i.e. the plane formed by $\bar{u}_i$ and $\bar{u}_n$). In this case the reflected and transmitted fields are given, at a typical point O, by

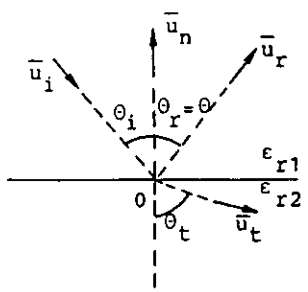


Fig. 8.3

$$\bar{E}_r = K_\perp \bar{E}_i = \frac{\cos\Theta_i - \sqrt{N - \sin^2\Theta_i}}{\cos\Theta_i + \sqrt{N - \sin^2\Theta_i}}\,\bar{E}_i$$

$$\bar{E}_t = T_\perp \bar{E}_i = \frac{2\cos\Theta_i}{\cos\Theta_i + \sqrt{N - \sin^2\Theta_i}}\,\bar{E}_i \qquad (8.13)$$

where $N = (\varepsilon_{r2}/\varepsilon_{r1})$. For the polarization in which $\bar{H}$ is perpendicular to the plane of incidence :

$$\bar{H}_r = K_{/\!/} \bar{H}_i = \frac{N\cos\Theta_i - \sqrt{N - \sin^2\Theta_i}}{N\cos\Theta_i + \sqrt{N - \sin^2\Theta_i}}\,\bar{H}_i$$

$$\bar{H}_t = T_{/\!/} \bar{H}_i = \frac{2N\cos\Theta_i}{N\cos\Theta_i + \sqrt{N - \sin^2\Theta_i}}\,\bar{H}_i \qquad (8.14)$$

In both cases the transmission angle Θ_t is given by

$$\sin\Theta_t = \frac{1}{\sqrt{N}}\sin\Theta_i \qquad (8.15)$$

These formulas find an application in, for example, the study of the influence of the ground on the radiation pattern of an antenna. The relationships given above must be reinterpreted when $\varepsilon_{r1} > \varepsilon_{r2}$, a situation which occurs, for example, when a wave propagates

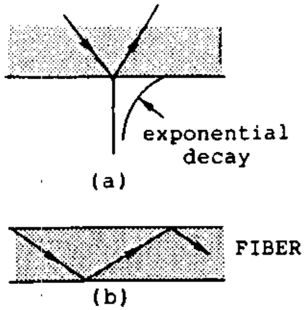


Fig. 8.4

from a dielectric into an air-filled region. For such case N is less than unity, and (8.15) cannot be satisfied if $\sin\Theta_i > \sqrt{N}$. Under these circumstances the wave in air is not a classical plane wave, but a field pattern which remains "glued" to the boundary on the air side, and the wave in the dielectric is totally reflected at the boundary (Fig. 8.4 a). This behaviour explains how an optical fiber is capable of guiding a wave while suffering low radiative losses (Fig. 8.3b).

9. Ray tracing

Ray tracing is a method used at very high frequencies, i.e. at short wavelengths. Wavelengths λ are said to be short when the characteristics of the medium supporting the wave vary little over a distance λ. For such case the wave behaves locally as a plane wave, and its direction of propagation is that of the ray. A typical time-harmonic component is written as

$$E_x = \mathcal{E}_x(\bar{r},k)e^{-jkS(\bar{r})} \qquad (9.1)$$

The surfaces $S(\bar{r})$ = const. are the phase fronts. At high frequencies the amplitude $\mathcal{E}_x$ is expanded in a series in the small parameter $(1/k)$. Thus,

$$\mathcal{E}_x(\bar{r},k) = \mathcal{E}_{xo}(\bar{r}) + \frac{1}{k}\mathcal{E}_{x1}(\bar{r}) + \dots \qquad (9.2)$$

Expansions of this kind are inserted in Maxwell's equations. Taking into account that

$$\mathrm{curl}\ \bar{E} = e^{-jkS}(\mathrm{curl}\ \bar{\mathcal{E}} - jk\ \mathrm{grad}\ S \times \bar{\mathcal{E}}) \qquad (9.3)$$

yields

$$\begin{aligned} \mathrm{curl}\ \bar{\mathcal{E}} - jk\ \mathrm{grad}\ S \times \bar{\mathcal{E}} &= -jk\, R_{co}\, \mu_r \bar{\mathcal{H}} \\ \mathrm{curl}\ \bar{\mathcal{H}} - jk\ \mathrm{grad}\ S \times \bar{\mathcal{H}} &= \frac{jk}{R_{co}}\, \epsilon_r \bar{\mathcal{E}} + \sigma\bar{\mathcal{E}} \end{aligned} \qquad (9.4)$$

Equating the dominant terms (the terms in k) leads to

$$\begin{aligned} \mathrm{grad}\ S \times \bar{\mathcal{E}}_o &= \mu_r R_{co} \bar{\mathcal{H}}_o \\ \mathrm{grad}\ S \times \bar{\mathcal{H}}_o &= -\frac{\epsilon_r}{R_{co}} \bar{\mathcal{E}}_o \end{aligned} \qquad (9.5)$$

Equations (9.5) show that grad S is perpendicular to both $\bar{\mathcal{E}}_o$ and $\bar{\mathcal{H}}_o$, and that $\bar{\mathcal{E}}_o$ is perpendicular to $\bar{\mathcal{H}}_o$. They also imply the "eikonal equation"

$$\boxed{|\mathrm{grad}\ S|^2 = \epsilon_r\mu_r = n^2} \qquad (9.6)$$

The index of refraction $\underline{n}$ is a function of x,y,z. The $\underline{\text{rays}}$ are the orthogonal trajectories of $S(x,y,z)$; they are therefore tangent to grad S. If ℓ is the length of arc measured along the ray, the equations of the latter take the form

$$\frac{\frac{dx}{d\ell}}{\frac{\partial S}{\partial x}} = \frac{\frac{dy}{d\ell}}{\frac{\partial S}{\partial y}} = \frac{\frac{dz}{d\ell}}{\frac{\partial S}{\partial z}} = \frac{1}{n} \tag{9.7}$$

These equations are obtained by making use of the relationship

$$\left(\frac{dx}{d\ell}\right)^2 + \left(\frac{dy}{d\ell}\right)^2 + \left(\frac{dz}{d\ell}\right)^2 = 1 \tag{9.8}$$

We now eliminate S by the following manipulation :

$$\begin{aligned}\frac{d}{d\ell}\left(n\frac{dx}{d\ell}\right) &= \frac{d}{d\ell}\left(\frac{\partial S}{\partial x}\right) = \frac{\partial^2 S}{\partial x^2}\frac{dx}{d\ell} + \frac{\partial^2 S}{\partial x \partial y}\frac{dy}{d\ell} + \frac{\partial^2 S}{\partial x \partial z}\frac{dz}{d\ell} \\ &= \frac{1}{n}\frac{\partial S}{\partial x}\frac{\partial^2 S}{\partial x^2} + \frac{1}{n}\frac{\partial S}{\partial y}\frac{\partial^2 S}{\partial x \partial y} + \frac{1}{n}\frac{\partial S}{\partial z}\frac{\partial^2 S}{\partial x \partial z} \\ &= \frac{1}{2n}\frac{\partial}{\partial x}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right] \\ &= \frac{1}{2n}\frac{\partial n^2}{\partial x}\end{aligned} \tag{9.9}$$

Combining with similar equations involving y and z yields

$$\boxed{\frac{d}{d\ell}\left(n\frac{d\bar{r}}{d\ell}\right) = \text{grad } n} \tag{9.10}$$

It is clear that $\frac{d\bar{r}}{d\ell}$ is the unit vector $\bar{u}_\ell$ along the ray. Once the rays are found, $S(\bar{r})$ follows from

$$\frac{dS}{d\ell} = n$$

and a straightforward integration. Thus,

$$S(P_1) = S(P_o) + \int_{\ell_o}^{\ell_1} n \, d\ell \tag{9.11}$$

In a homogeneous volume $\underline{n}$ is constant, hence grad $n = 0$. Eq. (9.10) then implies that $\bar{u}_\ell$ is a constant, which in turn implies that the rays

are straight lines. In an inhomogeneous region, however, (9.10) gives

$$n\frac{d\bar{u}_\ell}{d\ell} + \bar{u}_\ell\frac{dn}{d\ell} = \text{grad } n \tag{9.12}$$

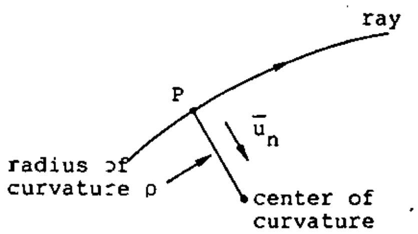


Fig. 9.1

The vector $\frac{d\bar{u}_\ell}{d\ell}$ is perpendicular to the ray, as

$$\frac{d\bar{u}_\ell}{d\ell} = \frac{1}{\rho}\bar{u}_n \tag{9.13}$$

It follows that

$$\frac{d\bar{u}_\ell}{d\ell} = \left(\frac{\text{grad } n}{n}\right)_\perp \tag{9.14}$$

where the subscript $\perp$ denotes projection on a plane perpendicular to the ray. A possible graphical construction of the ray follows from (9.14). Assume, indeed, that the direction of the ray is known in P_o (unit vector $\bar{u}_o$). The unit vector in a neighbouring point P_1 may be obtained from (9.14) by the operation

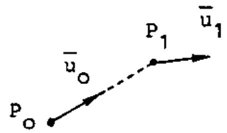


Fig. 9.2

$$\bar{u}_1 = \bar{u}_o + P_oP_1\left(\frac{\text{grad } n}{n}\right)_\perp \tag{9.15}$$

This relationship allows a point by point construction of the ray (Fig. 9.2). Eqs. (9.12) and (9.13) also yield

$$\bar{u}_n \cdot \text{grad}(\log_e n) = \frac{1}{\rho} \tag{9.16}$$

It is clear, from (9.16), that the rays are curved in the direction of high indices $\underline{n}$.

Ray tracing can go further, and generate the laws governing the amplitude and polarization of the fields along a ray. These laws are obtained by equating terms of higher order in (1/k) on both sides of (9.4). The detailed calculations are beyond the scope of the present notes.

10. Faraday effect

Macroscopic parameters of a cloud of charges

Let the cloud consist of particles of charge q and mass m, distributed with density n. One of the characteristics of the cloud is the plasma frequency, given by

$$f_p = \frac{1}{2\pi}\sqrt{\frac{nq^2}{m\varepsilon_o}} \quad (10.1)$$

A few values for an electron cloud are

$n\,(cm^{-3})$	10^8	10^{11}	10^{12}	10^{14}	10^{16}
f_p	90MHz	2,85GHz	9GHz	90GHz	900GHz

When the cloud is immersed in a D.C. magnetic field $\bar{b}_o$, the particles are subjected to a circular motion of angular frequency

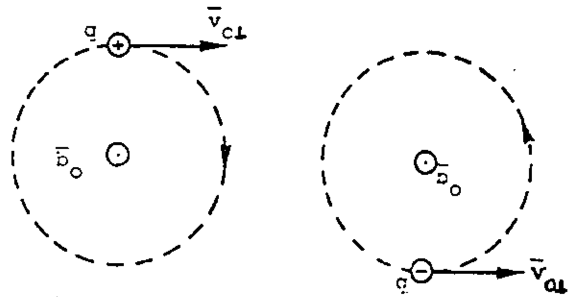


Fig. 10.1

$$\omega_c = \frac{qb_o}{m} \quad (10.2)$$

This is the cyclotron frequency. It has a sign, from which the rotation sense may be deduced. For an electron q = -e.

A few values for an electron cloud are (10000 Gauss = 1 T).

b_o (Gauss)	143	357	1070	3570	12500
f_c (GHz)	0.4	1	3	10	35

Given these parameters, it is possible to show that a charged cloud in a $\bar{b}_o$ behaves like an anisotropic medium in which $\bar{D} = \bar{\bar{\varepsilon}}.\bar{E}$, where

$$\bar{\bar{\varepsilon}} = \begin{pmatrix} \varepsilon & j\varepsilon' & 0 \\ -j\varepsilon' & \varepsilon & 0 \\ 0 & 0 & \varepsilon'' \end{pmatrix} \tag{10.3}$$

If we neglect the collisions, and the associated losses, the parameters are

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_o} = 1 + \frac{\omega_p^2}{\omega_c^2 - \omega^2}$$
$$\varepsilon_r' = \frac{\varepsilon'}{\varepsilon_o} = -\frac{\omega_c \omega_p^2}{\omega(\omega_c^2 - \omega^2)} \tag{10.4}$$
$$\varepsilon_r'' = \frac{\varepsilon''}{\varepsilon_o} = 1 - \frac{\omega_p^2}{\omega^2}$$

For the electron $\omega_c = -\frac{eb_{oz}}{m}$. The z-axis is directed along $\bar{b}_o$.

Plane wave propagation

For a wave propagating in the z-direction Maxwell's equations become

$$-\frac{\partial E_y}{\partial z} = -j\omega\mu_o H_x$$
$$\frac{\partial E_x}{\partial z} = -j\omega\mu_o H_y$$
$$-\frac{\partial H_y}{\partial z} = j\omega\varepsilon E_x - \omega\varepsilon' E_y = j\omega D_x \tag{10.5}$$
$$\frac{\partial H_x}{\partial z} = \omega\varepsilon' E_x + j\omega\varepsilon E_y = j\omega D_y$$

To uncouple these 4 equations with 4 unknowns we replace E_x, E_y by the linear combinations

$$A = \frac{1}{2}(E_x - jE_y) \tag{10.6}$$
$$B = \frac{1}{2}(E_x + jE_y)$$

In terms of these components:

$$\bar{E} = E_x\bar{u}_x + E_y\bar{u}_y = A(\bar{u}_x + j\bar{u}_y) + B(\bar{u}_x - j\bar{u}_y) \tag{10.7}$$

The electric field has clearly been split into two circularly polarized components. For the magnetic field, analogously,

$$C = \frac{1}{2}(H_x - jH_y)$$
$$D = \frac{1}{2}(H_x + jH_y) \tag{10.8}$$
$$\bar{H} = H_x\bar{u}_x + H_y\bar{u}_y = C(\bar{u}_x + j\bar{u}_y) + D(\bar{u}_x - j\bar{u}_y)$$

When these components are inserted in (10.5) two systems of (uncoupled) equations are obtained. For the (A,C) couple :

$$\frac{dA}{dz} = -j\omega\mu_o(jC)$$
$$-\frac{d(jC)}{dz} = j\omega(\varepsilon - \varepsilon')A \tag{10.9}$$

Elimination of C gives

$$\boxed{\frac{d^2A}{dz^2} + \omega^2\mu_o(\varepsilon - \varepsilon')A = 0} \tag{10.10}$$

In a similar fashion we obtain

$$\boxed{\frac{d^2B}{dz^2} + \omega^2\mu_o(\varepsilon + \varepsilon')B = 0} \tag{10.11}$$

Let us assume that $b_{oz} > 0$, i.e. that the wave propagates in the positive direction of $\bar{b}_o$. For such case the rotation sense of the "A" wave is that of the ions. This "ionic" wave has a propagation constant k_i given by

$$k_i^2 = \omega^2\mu_o(\varepsilon-\varepsilon') = k_o^2\left[1 - \frac{\omega_p^2}{\omega(\omega+|\omega_c|)}\right] \qquad (10.12)$$

The "B" wave is similarly an "electronic" wave, with constant

$$k_e^2 = \omega^2\mu_o(\varepsilon+\varepsilon') = k_o^2\left[1 - \frac{\omega_p^2}{\omega(\omega-|\omega_c|)}\right] \qquad (10.13)$$

It is clear that the wave is propagated when $k^2 > 0$ (passband), but that it is attenuated when $k^2 < 0$ (stop band).

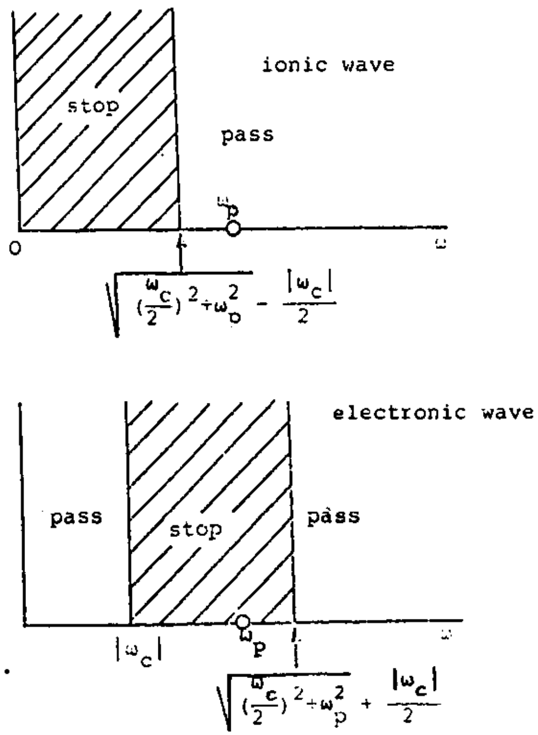


Fig. 10.2

Faraday effect

This effect arises because the two basic waves, A and B, have different propagation constants. Let us assume that both waves propagate, and that $\bar{E}$ is linearly polarized in the x-direction at z = 0. Thus,

$$\bar{E}(0) = \bar{u}_x = \underbrace{\tfrac{1}{2}(\bar{u}_x+j\bar{u}_y)}_{\text{A-wave}} + \underbrace{\tfrac{1}{2}(\bar{u}_x-j\bar{u}_y)}_{\text{B-wave}} \qquad (10.14)$$

Farther down the z-axis this field has become

$$\bar{E}(z) = \tfrac{1}{2}(\bar{u}_x+j\bar{u}_y)e^{-jk_iz} + \tfrac{1}{2}(\bar{u}_x-j\bar{u}_y)e^{-jk_ez} \qquad (10.15)$$

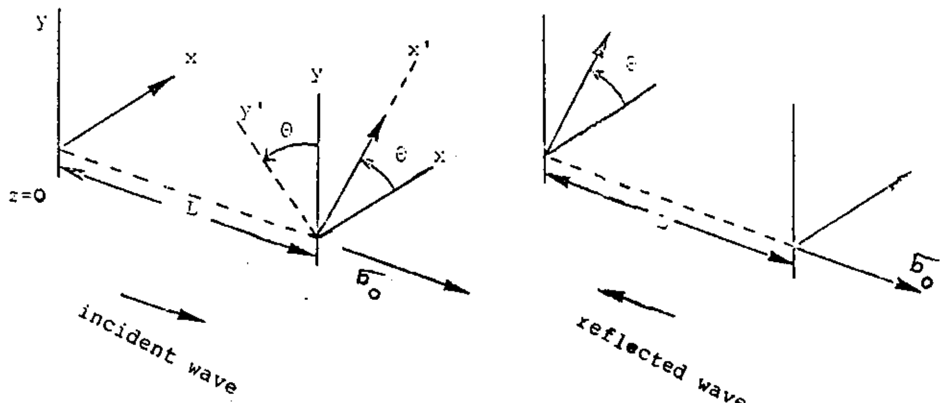


Fig. 10.3

Such a field is again linearly polarized, but in a new direction x' forming an angle θ with x. This is shown by applying the following coordinate transformation

$$\begin{aligned}\bar{u}_x &= \bar{u}_{x'}\cos\theta - \bar{u}_{y'}\sin\theta \\ \bar{u}_y &= \bar{u}_{x'}\sin\theta + \bar{u}_{y'}\cos\theta\end{aligned} \qquad (10.16)$$

The electric field is now

$$\bar{E}(z) = \frac{1}{2}\,\bar{u}_{x'}\left[e^{+j\theta}\,e^{-jk_i z} + e^{-j\theta}\,e^{-jk_e z}\right] + \frac{j}{2}\,\bar{u}_{y'}\left[e^{+j\theta}\,e^{-jk_i z} - e^{-j\theta}\,e^{-jk_e z}\right] \qquad (10.17)$$

It can be written as

$$\bar{E}(z) = \bar{u}_{x'}\, e^{-j\frac{k_e+k_i}{2}z} \qquad (10.18)$$

provided we set

$$\boxed{\theta = \frac{k_i-k_e}{2}\,z} \qquad (10.19)$$

This is the angle which characterizes the Faraday rotation. If we look at the reflected wave in Fig. (10.3) we see that it propagates against the magnetic field, hence that $b_{oz} < 0$. As a result, the angle θ is the negative of (10.19) with respect to the direction of propagation. It therefore has the same direction in space, a property which has interesting technological applications.

11. Far field

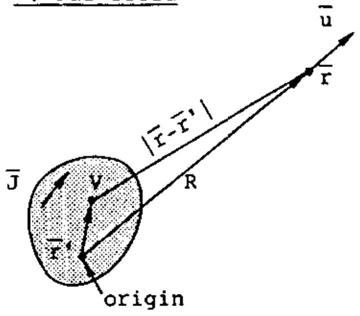


Fig. 11.1

Fig. 7.1 shows time harmonic currents radiating in free space. The complex vector potential $\bar{A}(\bar{r})$, derived from (2.8), is of the form

$$\bar{A}(\bar{r}) = \frac{\mu_o}{4\pi}\iiint \frac{\bar{J}(\bar{r}')\,e^{-jk_o|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|}\,dV' \qquad (11.1)$$

At large distances, in a direction of unit vector $\bar{u}$,

$$|\bar{r}-\bar{r}'| \approx R - \bar{u}.\bar{r}' \qquad (11.2)$$

Inserting this value in (7.1) gives the far field expression

$$\lim_{R\to\infty} \bar{A} = \frac{e^{-jk_oR}}{R}\,\underbrace{\frac{\mu_o}{4\pi}\iiint \bar{J}(\bar{r}')\,e^{jk_o\,\bar{u}.\bar{r}'}\,dV'}_{\text{direction dependent vector } \bar{N}(\bar{u})} \qquad (11.3)$$

The $\bar{E}$ and $\bar{H}$ fields, obtained from (1.17) and (2.1), are

$$\bar{E}(\bar{r}) = \bar{F}(\bar{u})\,\frac{e^{-jk_oR}}{R}$$

$$\bar{H}(\bar{r}) = \frac{1}{R_{co}}\,(\bar{u}\times\bar{F})\,\frac{e^{-jk_oR}}{R} \qquad (11.4)$$

where $\bar{F}$ is the transverse vector

$$\bar{F} = \bar{u}\times(\bar{u}\times\bar{N}) \qquad (11.5)$$

These very important formulas will be discussed further in the lectures on Antenna Theory.

When the dimensions of the source are small with respect to the wavelength λ, the exponential in $\bar{N}(\bar{u})$ may be usefully expanded as

$$e^{jk_o\bar{u}.\bar{r}'} = 1 + jk_o\bar{u}.\bar{r}' + \frac{1}{2}(jk_o)^2(\bar{u}.\bar{r}')^2 + \dots \qquad (11.6)$$

Inserting this value in (11.3) gives

$$\lim_{R\to\infty} \bar{A} = \frac{e^{-jk_oR}}{R}\frac{\mu_o}{4\pi}\left[\iiint \bar{J}\,(\bar{r}')dV' + jk_o \iiint(\bar{u}.\bar{r}')\bar{J}(\bar{r}')dV' + \text{terms in } k_o^2\right] \tag{11.7}$$

Suitable manipulation of this equation leads to the far fields

$$\bar{E} = \frac{R_{co}}{4\pi}\frac{e^{-jk_oR}}{R}\left[-k_o^2 c\,\bar{u} \times (\bar{u} \times \bar{P}_e) - k_o^2\bar{u} \times \bar{P}_m + \text{terms in } k_o^3\right]$$

$$\bar{H} = \frac{1}{4\pi}\frac{e^{-jk_oR}}{R}\left[k_o^2 c\,\bar{u} \times \bar{P}_e - k_o^2\,\bar{u} \times (\bar{u} \times \bar{P}_m) + \text{terms in } k_o^3\right] \tag{11.8}$$

We recognize in these formulas the contributions of, first, an electric dipole moment

$$\bar{P}_e = \frac{1}{j\omega}\iiint\bar{J}\,dV = \iiint_V \rho\,\bar{r}\,dV + \iint_S \rho_S\,\bar{r}\,dS \tag{11.9}$$

and, second, a magnetic dipole moment

$$\bar{P}_m = \frac{1}{2}\iiint_V \bar{r} \times \bar{J}\,dV \tag{11.10}$$

The discussion can be carried further, and the terms in k_o^3 shown to consist of contributions from e.g. quadrupole moments. The fields of the dipoles will be mentioned again in the lectures on Antenna Theory.

12. Time harmonic sources. Directivity

A time harmonic vector is elliptically polarized, i.e. its tip describes an ellipse. Polarization is discussed in Sec. 5 of the Refresher Course. A time harmonic field $\bar{a}(t)$ may be represented by a complex vector $\bar{A} = \bar{a}_r + j\,\bar{a}_i$, through which the actual time dependence may be restored by the simple operation

$$\bar{a}(t) = \text{Re}\,[\bar{A}\,e^{j\omega t}] = \bar{a}_r\cos\omega t - \bar{a}_i\sin\omega t \tag{12.1}$$

The main axes of the ellipse are often chosen to be the x and y axes. For such choice (Fig. 12.1)

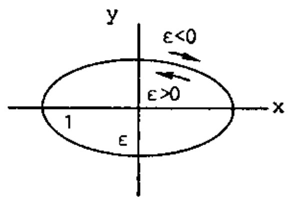


Fig. 12.1

$$\bar{A} = F(\bar{u}_x - j\varepsilon\,\bar{u}_y) \tag{12.2}$$

The sign of ε determines the sense in which the ellipse is described. The value ε = 0 corresponds to a linear polarization, and $|\varepsilon| = 1$ to a circular polarization.

The average value of $|\bar{a}|^2$ is given by

$$<|\bar{a}|^2>_{av} = \frac{1}{2}\,\bar{A}.\bar{A}^* = \frac{1}{2}\,|\bar{A}|^2 = \frac{1}{2}\,(1+\varepsilon^2)FF^* \tag{12.3}$$

Under time harmonic conditions the power budget takes the form

$$\frac{1}{2}\,\text{Re}\iiint_V \bar{E}_a.\bar{J}_f^*dV = \frac{1}{2}\iiint\frac{|\bar{J}_f|^2}{\sigma}\,dV + \frac{1}{2}\,\text{Re}\iint_S(\bar{E} \times \bar{H}^*).d\bar{S} \tag{12.4}$$

where $\bar{J}_f = \sigma(\bar{E} + \bar{E}_a)$ denotes the total free-charge current. It is seen that the average power delivered by the applied field (left-hand member) is spent in the form of

- an average Joule effect in the conductors (first term, second member)
- an average radiated power (second term, second member).

The vector $\bar{E} \times \bar{H}^*$ is the complex vector of Poynting. Half its real part represents the average power flux per m^2.

To write the retarded potentials (1.13) in complex form we notice that time retardation becomes <u>phase</u> retardation. If T is the period, the phase shift due to propagation takes the form

$$\frac{|\bar{r}-\bar{r}'|}{c}\,\frac{1}{T}\,2\pi = \frac{2\pi}{\lambda}\,|\bar{r}-\bar{r}'| = k_o|\bar{r}-\bar{r}'| \qquad (12.5)$$

as $cT = \lambda$. This leads to the expression

$$\bar{A}(\bar{r}) = \frac{\mu_o}{4\pi}\iiint \frac{\bar{J}(\bar{r}')e^{-jk_o|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|}\,dV' \qquad (12.6)$$

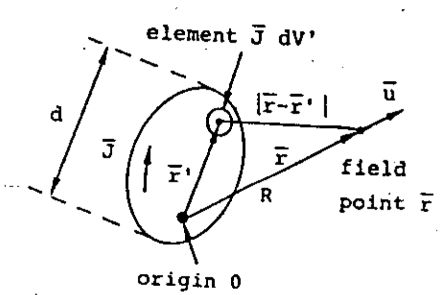


Fig. .2

It is clear, from this formula, that the contribution of an elementary source $\bar{J}$ dV' decreases with distance, and is phase-retarded through an angle $k_o|\bar{r}-\bar{r}'|$, where $k_o = \frac{\omega}{c} = \frac{2\pi}{\lambda}$.

An added distance of $(\lambda/2)$, for example, introduces an additional phase shift of 180°, i.e. creates a phase reversal. Two contributions which were initially in phase (constructive interference) may now, by way of shifts in direction or distance, end up in phase opposition (destructive interference). This interplay of phase and amplitude may be clarified by looking at the far-field. At distances R large with respect to the maximum dimension <u>d</u> of the source, $|\bar{r}-\bar{r}'|$ in a direction of unit vector $\bar{u}$ may be written as

$$|\bar{r}-\bar{r}'| = R - \bar{u}.\bar{r}' + \text{terms in } \frac{1}{R} \qquad (12.7)$$

As shown in Sec.11 of the Refresher Course, substitution of (12.7) in (12.6) leads to the following general form for the fields in a direction of unit vector $\bar{u}$:

$$\bar{E} = \bar{F}(\bar{u})\,\frac{e^{-jk_oR}}{R} \qquad \text{V m}^{-1}$$

$$\bar{H} = \frac{1}{R_{co}}\,(\bar{u} \times \bar{E})\,\frac{e^{-jk_oR}}{R} \qquad \text{A m}^{-1} \qquad (12.8)$$

In this equation $R_{co} = \sqrt{\mu_o/\varepsilon_o} = 377\Omega$ is the characteristic impedance of vacuum. It is seen that $\bar{E}$ is perpendicular to $\bar{H}$. In addition, as $\bar{F}$ is a transverse vector (i.e. a vector with polarization plane perpendicular to $\bar{u}$), $\bar{E}$ and $\bar{H}$ are perpendicular to $\bar{u}$. It is also seen that the R-dependence of the fields is of the form (e^{-jk_oR}/R), independently of the nature of the sources and the direction of observation. The latter only influences the coefficient $\bar{F}(\bar{u})$. Notice that a general criterion for the validity of the far-field approximation requires the distance to satisfy

$$R > \frac{2d^2}{\lambda} \qquad (12.9)$$

An example : the far field distance of a parabolic antenna of diameter 1.20m, transmitting at 10GHz (λ = 3cm), is 96 m.

From (2.8) the complex Poynting's vector is given by

$$\bar{E} \times \bar{H}^* = \frac{1}{R_{co}}\,\frac{1}{R^2}\,|\bar{F}|^2\,\bar{u} \qquad \text{W m}^{-2} \qquad (12.10)$$

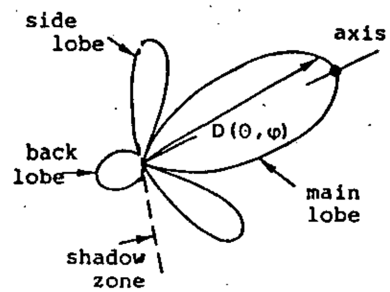


Fig. 12.3

It lies in the direction of observation $\bar{u}$. The power per steradian in the direction $\bar{u}$ is $|\bar{F}|^2/2R_{co}$, hence the total radiated power is of the form

$$\mathcal{P} = \frac{1}{2R_{co}}\iint|\bar{F}|^2 d\Omega = \frac{1}{2R_{co}}\iint|\bar{F}(\Theta,\varphi)|^2 \sin\Theta\, d\Theta\, d\varphi \qquad (12.11)$$

As the average power per steradian is $(\mathcal{P}/4\pi)$, the <u>directivity</u> D of the source may be defined as

$$D(\theta,\phi) = \frac{\text{power per steradian in } \bar{u}}{\text{average power per steradian}}$$

$$= \frac{|\bar{F}(\theta,\phi)|^2}{\frac{1}{4\pi}\iint |\bar{F}(\theta,\phi)|^2 \, d\Omega} \qquad (12.12)$$

Important parameters for an antenna are its maximum directivity and its side-lobe ratio (often expressed in dB). A three-dimensional plot of $|F(\bar{u})|$ is termed the field radiation pattern of the antenna, and that of $|\bar{F}|^2$ is the power radiation pattern. Fig.12.3 shows a source with a fairly structured radiation pattern, evidencing shadow zones and secundary lobes.

The narrowness of the main lobe increases the accuracy with which the angular position of a beacon may be determined (in azimuth or elevation). Early guidance and landing systems went one step further by using two beams.

13. Aperture antennas

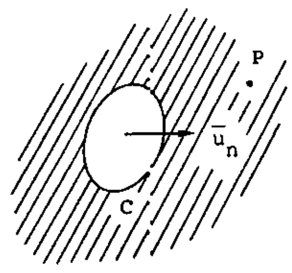


Fig. 13.1

Fig. 13 shows an aperture located in a perfectly conducting screen. On the screen $\bar{E}_{tang}$ vanishes, but in the aperture this component is different from zero. In fact, $(\bar{u}_n \times \bar{E})$ is the source of the field to the right of the screen, as can be seen from the formula

$$\bar{E}(\bar{r}) = \text{curl}\left[\frac{1}{2\pi}\iint_{ap} \bar{u}_n \times \bar{E}(\bar{r}')\,\frac{e^{-jk|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|}\,dS'\right] \qquad (13.1)$$

As in Sec. 12, the "far field" approximation can be applied to this formula, but such an exercise is left to the reader. The main problem is to determine $\bar{u}_n \times \bar{E}$. This task ideally requires solution of an integral equation. In many cases, however, a reasonable assumption can be made concerning $\bar{u}_n \times \bar{E}$. At very short λ, for example, $\bar{u}_n \times \bar{E}$ may be replaced by $\bar{u}_n \times \bar{E}_i$ everywhere but in the immediate vicinity of the sharp edge C. The contribution to the integral in (13.1) of this small region, just a few λ wide, is negligible. With the assumption

$$\bar{u}_n \times \bar{E} \approx \bar{u}_n \times \bar{E}_i \qquad (13.2)$$

it becomes possible to determine the far field produced, by, for example, a rectangular aperture illuminated by a plane wave at normal incidence. If the wave is polarized parallel with the broad

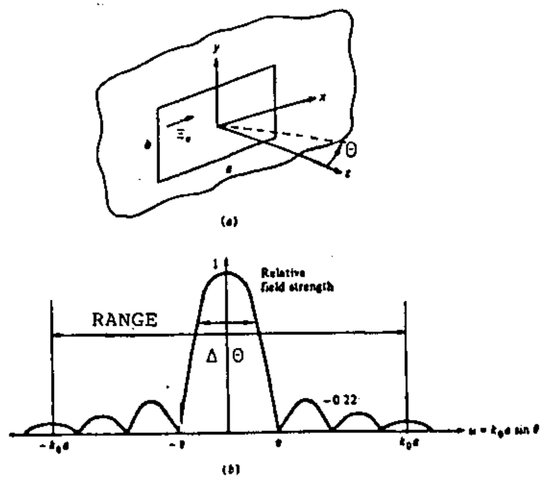


Fig. 13.2

side we set $\overline{E}_i = E\,\overline{u}_x$ (Fig. 13.2a). Using (13.1) yields a radiation pattern in the (x-z) plane of the general form shown in Fig. 13.2b.

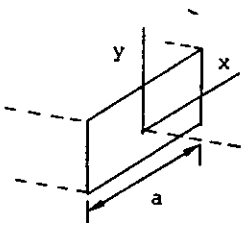


Fig. 13.3

When the rectangular aperture terninates a rectangular waveguide, a configuration often found in airborne antennas, $\overline{E}_i$ becomes the field of an incident dominant mode (Fig. 13.3)

$$\overline{E}_i = E \cos \frac{\pi x}{a}\, \overline{u}_y \qquad (13.3)$$

The diagram of Fig. 13.2b shows that the opening angle $\Delta\Theta$, defined by the 3dB points, is given by

$$\sin\left(\frac{\Delta\Theta}{2}\right) \approx 0.58\,\frac{\lambda}{a} \qquad (13.4)$$

For small (λ/a), this yields

$$\Delta\Theta \approx 1.16\,\frac{\lambda}{a} \qquad (13.5)$$

Such a relationship is quite general. In every plane containing the axis of the main beam :

$$\Theta \approx \frac{\lambda}{d} \qquad (13.6)$$

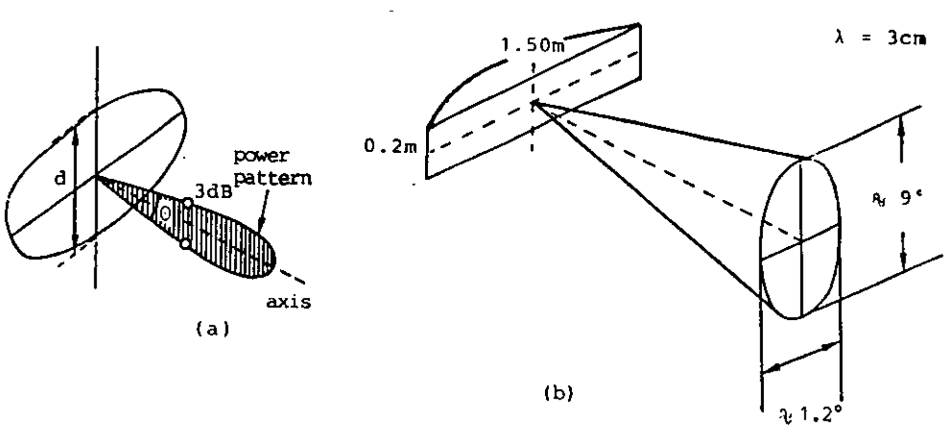


Fig. 13.4

where $\underline{d}$ is the "dimension" of the aperture in the plane under consideration. Fig. 13.4a shows the elliptical aperture of an antenna, and the opening angle in the vertical plane. Fig. 13.4b shows actual figures for the rectangular aperture of a "half-cheese" antenna, often used in conjunction with marine radar sets. As the horizontal dimension is 7.5 broader than the vertical one, the beam is 7.5 times thinner in azimuth than in elevation.

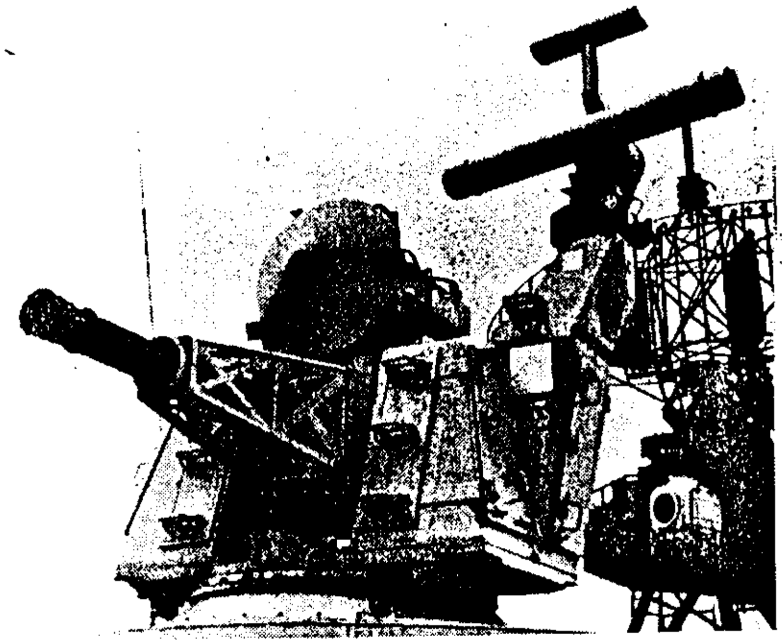


Fig. 13.5 X-band (3cm) search radar and X- and K_u band (3 and 1.8 cm) tracking radar (Hollandse Signaal "goalkeeper" system).

14. General equivalent circuit

Fig. 14.1 shows a horn antenna fed by a waveguide. For such a structure the terminals A and B are replaced by a terminal plane S. The transverse fields in S can be written, in monomode propagation, as

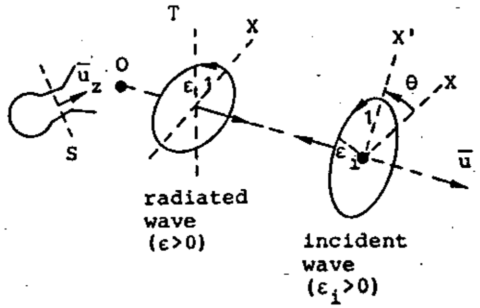


Fig. 14.1

$$\bar{E}_{tr} = V\bar{e}$$
$$\bar{H}_{tr} = I(\bar{u}_z \times \bar{e}) \qquad (14.1)$$

In these expressions $\bar{e}$ is the eigenvector of the mode. The ratio (V/I) is the generalized antenna impedance Z_a. The "generator" short circuit current is introduced through the formula

$$I_g\bar{e} = \bar{J}_s = \bar{u}_z \times \bar{H} \quad \text{on S} \qquad (14.2)$$

In this expression $\bar{J}_s$ is the current density induced on the short circuited S by an incident wave $\bar{E}_i,\bar{H}_i$. The value of I_g (a <u>receiving</u> parameter) can be deduced from the <u>transmitting</u> properties. To clarify this statement, let

$$\bar{E} = V\bar{F}\,\frac{e^{-jk_oR}}{R} \qquad (14.3)$$

be the far field of the transmitting antenna in a direction $\bar{u}$. The antenna is excited by a "voltage" V on S (Fig. 14.1). As $\bar{E}$ is proportional with V, a factor V has been explicited in (14.3). The dimensionless vector $\bar{F}$, a function of $\bar{u}$, is now independent of the level of the signal in the waveguide. The far field shown in Fig. 14.1 is elliptically polarized, ε being positive in the indicated rotation sense. Detailed calculations, based on a reciprocity property, show that, for the direction of incidence $-\bar{u}$,

$$I_g = \frac{4}{jk_oR_{co}} \frac{\bar{F}.\bar{E}_i(0)}{\iint\limits_S |\bar{e}|^2 ds} \tag{14.4}$$

The corresponding open circuit "voltage" is

$$V_g = Z_a I_g = \underbrace{\left[\frac{4\pi}{jk_oR_{co}} \frac{Z_a}{\iint\limits_S |\bar{e}|^2 ds} \bar{F} \right]}_{\bar{\ell}_{eff}} . \bar{E}_i(0) \tag{14.5}$$

For a coaxial line, for example,

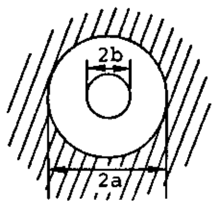


Fig. 14.2

$$\bar{e} = \frac{1}{\ell n(a/b)} \frac{\bar{u}_r}{r}$$

V = voltage between central conductor and shield (14.6)

$$\iint\limits_S |\bar{e}|^2 ds = \frac{2\pi}{\ell n(a/b)}$$

Good reception requires the signal-to-noise- ratio to exceed a threshold value, which varies from application to application. To evaluate the (S/N) ratio, it is necessary to know the signal power, i.e. the power which flows through S to the receiver. This power can be written as

$$P_{rec} = \frac{1}{2R_{co}} |\bar{E}_i|^2 S_{eff} \tag{14.7}$$

where ($|\bar{E}_i|^2/2R_{co}$) is the time averaged incident power density (in Wm^{-2}), and S_{eff} is the <u>antenna cross section</u>. A cross-section of $2m^2$, for example, means that, if the antenna is illuminated by a plane wave of power density $3\mu Wm^{-2}$, the power down the transmission line is $6\mu W$. The cross section (which is a "<u>receiver</u>" characteristic) is given by

$$S_{eff} = \frac{G\lambda^2}{4\pi} M P \tag{14.8}$$

The symbol G denotes the gain of the <u>transmitting</u> antenna. It is equal to $D\eta$, where D is the directivity, and η the efficiency, i.e. the ratio of the effectively radiated power to the power flowing to the antenna. An efficiency of 0.9, for example, means that 90% of the power is effectively radiated, while 10% is dissipated in the vicinity of the antenna. The factors M and P lie between zero and one. The factor M measures the mismatch between antenna and load impedances. It reaches unity when the load is matched, i.e. when $Z_L = Z_a^*$. The polarization factor P is given by

$$P = \frac{|\bar{F}_i . \bar{E}_i|^2}{|\bar{F}_i|^2 |\bar{E}_i|^2} = \frac{|\bar{\ell}_{eff} . \bar{E}_i|^2}{|\bar{\ell}_{eff}|^2 |\bar{E}_i|^2} \tag{14.9}$$

In terms of the parameters shown in Fig. 7.1 :

$$P = \frac{(\varepsilon_i - \varepsilon)^2}{(1+\varepsilon_i^2)\ (1+\varepsilon^2)} + \cos^2\theta \frac{(1-\varepsilon^2)\ (1-\varepsilon_i^2)}{(1+\varepsilon^2)\ (1+\varepsilon_i^2)} \tag{14.10}$$

where θ is the angle between the major axes of the ellipses. Factor P reaches unity when $\theta = 0$ and $\varepsilon = -\varepsilon_i$. Maximum extracted power therefore obtains when incident and radiated ellipses have the same shape, are similarly oriented, and are described in opposite senses in space (i.e. in the same sense with respect to the direction of propagation). Complete polarization mismatch (i.e. zero received power) obtains when the ellipses have the same shape, are perpendicular to each other, and are described in the same sense in space.

Application : microwave relay

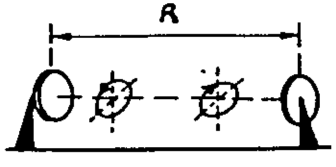


Fig. 14.3

The incident power density on 2 is

$$W_i = \frac{\mathcal{P}_1}{4\pi R^2} G \qquad Wm^{-2} \tag{14.11}$$

From (14.8) the power to the receiver in 2 is

$$\mathcal{P}_2 = \frac{G\lambda^2}{4\pi} \frac{\mathcal{P}_1}{4\pi R^2} G = \left(\frac{G\lambda}{4\pi R}\right)^2 \mathcal{P}_1 \tag{14.12}$$

For antennas with an aperture, there exists a coefficient K such that

$$S_{eff} = K\, S_{geom} \tag{14.13}$$

For a parabolic antenna K = 0.7. Let the diameter of the antenna be 2 m. Then, at λ = 5 cm

$$D = \frac{4\, S_{eff}}{\lambda^2} \approx 11.000 \tag{14.14}$$

Assuming an efficiency $\eta = 1$, and $\mathcal{P}_1$ = 1 W yields

$$\mathcal{P}_2 \approx \frac{2}{(R\,in\,km)^2}\ mW \tag{14.15}$$

15. Antenna arrays.

A typical antenna array is shown in Fig. 15.1. The signals from the various radiators add up in space. To avoid unnecessary complications we shall assume that the radiators are identical and identically-oriented, but the excitation voltages V_i in the cross sections S_i may be different. The points O_i are identically-located reference points, with respect to which the radiation vector is $V_i\bar{F}$. The total far field with respect to a common origin O is therefore

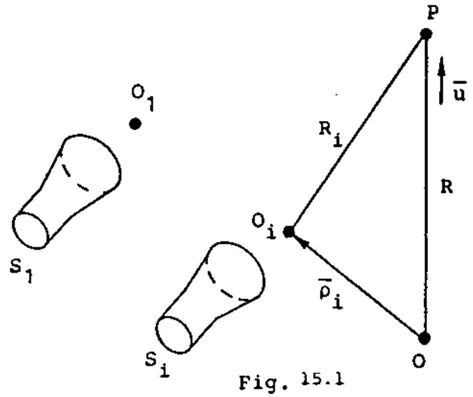


Fig. 15.1

$$\bar{E} = \bar{F}_{tot} \frac{e^{-jkR}}{R} = V_1\bar{F}\frac{e^{-jkR_1}}{R_1} + V_2\bar{F}\frac{e^{-jkR_2}}{R_2} + \dots V_N\bar{F}\frac{e^{-jkR_N}}{R_N}$$

$$= \frac{e^{-jkR}}{R} \underbrace{(V_1 e^{jk\bar{u}.\bar{\rho}_1} + V_2 e^{jk\bar{u}.\bar{\rho}_2} + \dots V_N e^{jk\bar{u}.\bar{\rho}_N})}_{\text{array factor } \mathcal{R}(\bar{u})} \bar{F} \tag{15.1}$$

It is seen that $\mathcal{R}(\bar{u})$ is independent of the nature of the individual radiators (which may be short dipoles, or the huge parabolas encountered in radioastronomic applications). The array factor, on the other hand, is a function of the geometry of the array, the relative excitation of the elements, and the direction of observation $\bar{u}$. The simplest example is perhaps that of a linear array of N elements (Fig. 15.2). Assume

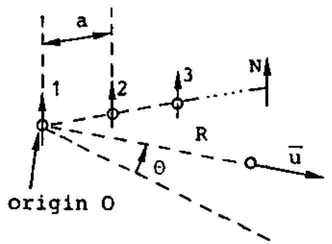


Fig. 15.2

that the feed voltages V_i have the same amplitude, but experience a phase shift $\underline{\alpha}$ from element $\underline{i}$ to element $\underline{i+1}$. For such case the array factor becomes

$$\mathcal{R}(\Theta) = e^{j(\frac{N-1}{2}\beta)} \frac{\sin\frac{N\beta}{2}}{\sin\frac{\beta}{2}} \qquad (15.2)$$

where $\beta=\alpha+ka \sin \Theta$.

The amplitude of $\mathcal{R}$ reaches a maximum (equal to N) for $\beta=0$, i.e. for a direction Θ_{max} given by

$$\sin \Theta_{max} = -\frac{\alpha}{ka} = -\frac{\alpha}{2\pi}\frac{\lambda}{a} \qquad (15.3)$$

The maximum in $\Theta=\Theta_{max}$ results from the constructive addition of the element contributions. The general variation of $|\mathcal{R}|$ is shown in Fig. 15.3 for several values of $\underline{N}$. The detailed Θ dependence for $\alpha=0$, $N=6$ and $a=(\lambda/2)$ is shown in Fig. 15.4, where

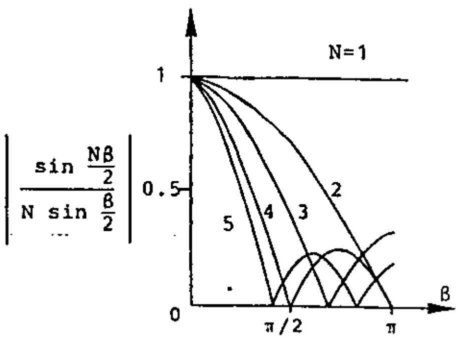


Fig. 15.3

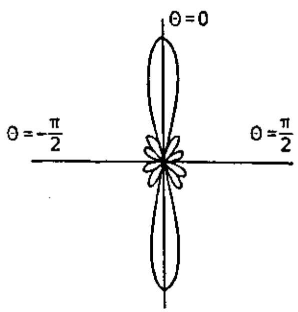


Fig. 15.4

the maximum $\mathcal{R}$ is obtained for $\Theta=0$, i.e. for a "broadside"

direction. Notice that $\mathcal{R}$ is rotationally symmetric with respect to the axis, hence that a complete plot of $\mathcal{R}$ is obtained by rotating the meridian cross-section shown in Fig. 15.4 around the axis.

It is clear, from (8.3), that the largest possible value of $\mathcal{R}$, i.e. N, can only be obtained if $(\alpha\lambda/2\pi a)$ does not exceed unity. It is also clear that the direction of maximum radiation can be shifted in space by varying $\underline{\alpha}$. Such a variation can be obtained electronically, a method which is sometimes preferable to mechanical scanning.

16 Scattering cross-sections

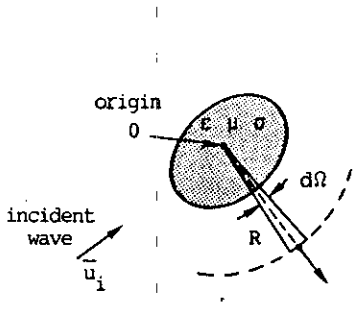


Fig . 16.1

A typical scattering configuration is shown in Fig.16.1 , which displays a target of arbitrary shape and constitutive parameters (ε,μ,σ), immersed in an incident electromagnetic wave of arbitrary time dependence. The determination of the scattered fields is a most difficult task, often made somewhat easier by assuming that the incident wave is plane and time-harmonic. Such a restriction is reasonable because, at large distances, the fields of an arbitrary source behave locally as those of a plane wave (see eq.11.4). Further, Fourier expansions with respect to time and space coordinates show that an arbitrary incident wave may be considered as the superposition of an infinite number of time-harmonic plane waves.

The power density in a progressive plane wave is obtained from (5.12) and (8.5) as

$$W_i = \frac{1}{2}\,\mathrm{Re}\iint_{\substack{\text{unit}\\\text{area}}} (\bar{E}\times\bar{H}^*).\bar{u}_i\,dS = \frac{1}{2R_{co}}\,|\bar{E}_i|^2 \quad \mathrm{W\ m^{-2}} \tag{16.1}$$

Illuminated by this wave the "scatterer" or "target" becomes a source of induced currents (volume or surface conduction currents, polarization currents ...), and acts as a secundary "antenna" producing a far field,

$$\bar{E}_{sc} = \bar{F}_{sc}(\bar{u})\,\frac{e^{-jkR}}{R} \qquad \mathrm{V\ m^{-1}}$$
$$\bar{H}_{sc} = \frac{1}{R_{co}}\,(\bar{u}\times\bar{F}_{sc})\,\frac{e^{-jkR}}{R} \qquad \mathrm{A\ m^{-1}} \tag{16.2}$$

Such a formula holds for every observation direction $\bar{u}$. From (5.12) the time averaged power radiated in an elementary solid angle dΩ centered on $\bar{u}$ is

$$d\mathcal{P}_{sc} = \frac{1}{2}\,\mathrm{Re}(\bar{E}_{sc}\times\bar{H}^*_{sc}).\bar{u}\underbrace{(R^2 d\Omega)}_{dS} = \frac{1}{2R_{co}}|\bar{F}_{sc}|^2\,d\Omega \quad \mathrm{W} \tag{16.3}$$

The total scattered power follows by summing over all solid angles (i.e. over 4π steradians). Thus,

$$\mathcal{P}_{sc} = \frac{1}{2R_{co}}\iint_4 |\bar{F}_{sc}|^2\,d\Omega \quad \mathrm{W} \tag{16.4}$$

A quantity independent of the power level, the <u>total scattering cross-section</u>, is obtained by dividing the power by the incident power density. Thus,

$$\sigma_{sc}(\bar{u}_i) = \frac{\mathcal{P}_{sc}}{W_i} = \frac{\iint_{4\pi}|\bar{F}_{sc}|^2}{|\bar{E}_i|^2} \quad \mathrm{m^2} \tag{16.5}$$

This cross-section is a function of frequency, of the direction of incidence $\bar{u}_i$, and of the state of polarization of the incident wave. To illustrate the concept : a σ_{sc} of 3 m^2 means that the target, illuminated by 1 kW per m^2, will scatter 3 kW.

The total scattering cross-section does not express how much power is scattered in any given direction $\bar{u}$. This directional sensitivity is expressed by the <u>bistatic cross-section</u> $\sigma_{bis}(\bar{u}|\bar{u}_i)$, which can be most conveniently defined by way of a numerical example. Let W_i = 1 kW m^{-2}, then σ_{bis} = 2 m^2 means that a power

$$d\mathcal{P}_{sc} = 2\left(\frac{d\Omega}{4\pi}\right)\ \mathrm{kW} \tag{16.6}$$

is scattered in an elementary solid angle dΩ centered on the direction $\bar{u}$. A very important particular case is that of the monostatic or <u>radar cross-section</u>. It is the bistatic cross-section relative to the backscattering direction ($-\bar{u}_i$). Thus,

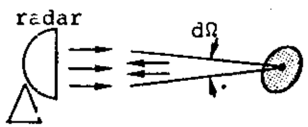


Fig. 16.2

$$\sigma_{rad}(\bar{u}_i) = \sigma_{bis}(-\bar{u}_i|\bar{u}_i) \tag{16.7}$$

With W_i = 0.1W m^{-2}, a σ_{rad} = 3 m^2 means that

$$d\mathcal{P}_{sc} = 0.3\left(\frac{d\Omega}{4\pi}\right) \quad W \qquad (16.8)$$

is scattered back in a solid angle $d\Omega$ (towards the radar set, see Fig. 16.2.

The previous considerations are operational definitions. They do not solve the real problem, which is to find $\overline{F}(\overline{u})$. We ignore this difficult assignment, and limit ourselves to a display of a few typical results.

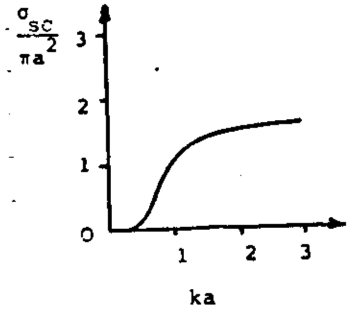


Fig. 16.3

Fig. 16.3 shows the total cross section of an iron sphere as a function of the radius a. The frequency is 7.1 10^{14} Hz (green light), and the curve is drawn as a function of the dimensionless parameter ka = $(2\pi a/\lambda)$. Fig. 16.4 shows σ_{rad} for a perfectly conducting sphere. Notice the successive

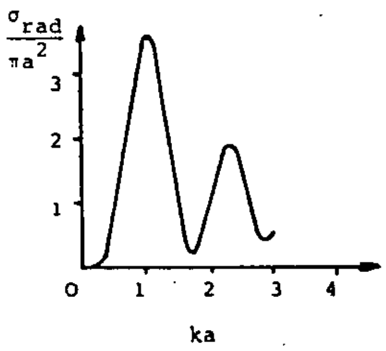


Fig. 16.4

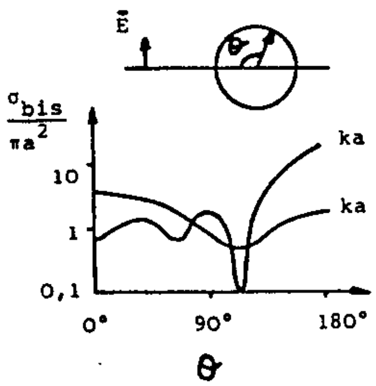


Fig. 16.5

"resonance" peaks. Fig. 16.5 shows, for the same sphere, the bistatic cross-section as a function of the angle of observation Θ.

17. Doppler effect

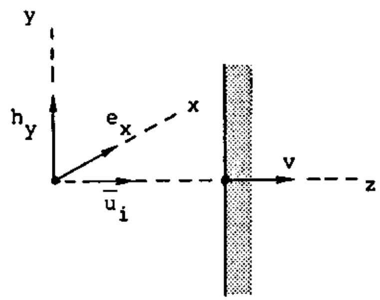


Fig. 17.1

Consider an incident plane wave propagating along the z-axis in the "laboratory" frame, i.e. in the axes of a (static) observer. The incident fields are

$$e_{ix} = E \cos(\omega t - \frac{\omega}{c} z)$$
$$h_{iy} = \frac{E}{R_{co}} \cos(\omega t - \frac{\omega}{c} z) \qquad (17.1)$$

This wave impinges at normal incidence on a perfectly conducting plane moving with velocity v. Let K' denote the axes in which the conductor is at rest. According to special relativity the fields in K', at velocities v<<c, are related to those in K by

$$e'_x = e_x - v\mu_o h_y$$
$$h'_y = h_y - v\varepsilon_o e_x \qquad (17.2)$$

Relativity also implies that the coordinates of an event (e.g. the measurement of a field) in K and K' are related by the Lorentz transformation

$$z = z' + vt'$$
$$t = t' + \frac{vz'}{c^2} \qquad (17.3)$$

Combining (17.1), (17.2) and (17.3) gives the transformed incident fields

$$e'_{ix} = E(1 - \frac{v}{c})\cos(\omega' t' - \frac{\omega'}{c} z')$$
$$h'_{iy} = \frac{E}{R_{co}} (1 - \frac{v}{c})\cos(\omega' t' - \frac{\omega'}{c} z') \qquad (17.4)$$

where

$$\omega' = \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}\, \omega \approx (1 - \frac{v}{c})\omega = \omega - (\frac{v}{c}\, \omega) \qquad (17.5)$$

This formula expresses the Doppler shift. More generally, for arbitrary velocities and a wave propagating in a direction making an angle α with the z-axis :

$$\omega' = \frac{\omega}{\sqrt{1-\frac{v^2}{c^2}}}\left(1 - \frac{v}{c}\cos\alpha\right) \tag{17.6}$$

The Doppler shift is the source of numerous technical applications, e.g. in the area of electronic navigation. This shift is also of fundamental importance for the operation of moving target indicator radars (MTI), burglar alarms etc... To clarify this statement consider the simple model shown in Fig. .1. In the K' axes the reflected wave is

$$e'_{rx} = -E\left(1 - \frac{v}{c}\right)\cos\left(\omega' t' + \frac{\omega'}{c} z'\right)$$

$$h'_{ry} = \frac{E}{R_{co}}\left(1 - \frac{v}{c}\right)\cos\left(\omega' t' + \frac{\omega'}{c} z'\right) \tag{17.7}$$

The reflected fields in K are obtained from those in K' through the "inverse" transformation formulas

$$e_x = e'_x + v\mu_o h'_y$$

$$h_y = h'_y + v\varepsilon_o e'_x$$

$$z' = z - vt \tag{17.8}$$

$$t' = t - \frac{vz}{c^2}$$

This gives

$$e_{rx} = -E\left(1 - 2\frac{v}{c}\right)\cos\left(\omega'' t + \frac{\omega''}{c} z\right)$$

$$h_{ry} = \frac{E}{R_{co}}\left(1 - 2\frac{v}{c}\right)\cos\left(\omega'' t + \frac{\omega''}{c} z\right) \tag{17.9}$$

Here ω'' is the angular frequency of the radar echo, which is twice Doppler shifted according to the formula

$$\omega'' = \omega\,\frac{1-\frac{v}{c}}{1+\frac{v}{c}} \sim \omega\left(1 - 2\frac{v}{c}\right) \tag{17.10}$$

18. The radar equation

Noise output

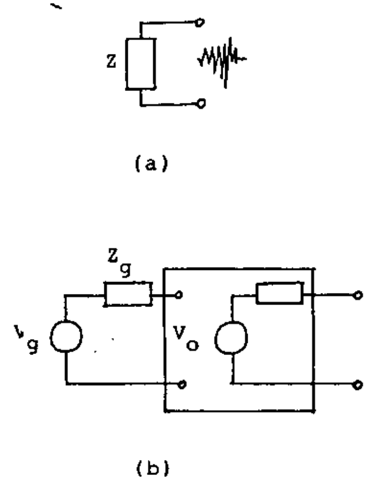


Fig. 18.1.

The thermal noise at the terminals of an impedance Z is given by (Fig. 9.1a)

$$\overline{dv^2} = 4kTR(\omega)df \tag{18.1}$$

The quadratic values add up. Let Q be the quadratic gain of the system (Fig.18.1b)

$$Q(\omega) = \frac{|V_o|^2}{|V_g|^2} \tag{18.2}$$

If no additional noise were created the quadratic fluctuation would be

$$\overline{dV_o^2} = 4kTR_g(\omega)Q(\omega)df \tag{18.3}$$

Because of additional noise sources, a noise factor F > 1 is introduced, hence

$$\overline{dv_o^2} = 4\ k\ T\ R_g(\omega)Q(\omega)F(\omega)df \tag{18.4}$$

If the input system has n_g times as much noise as Z_g (where n_g, the noise source factor, is > 1) :

$$\overline{dv_o^2} = 4\ k\ T\ R_g(\omega)Q(\omega)\underbrace{\left[F(\omega)+n_g(\omega)-1\right]}_{F_{eff}}df \tag{18.5}$$

This gives (Fig.18.2)

$$(\overline{v_o^2})_{noise} = 4\ k\ T\int_{\omega_1}^{\omega_2} R_g\ Q\ F_{eff}\ df$$

$$= 4\ k\ T\ R_g\ Q_{ref}\ (F_{eff})_{av}\ B \tag{18.6}$$

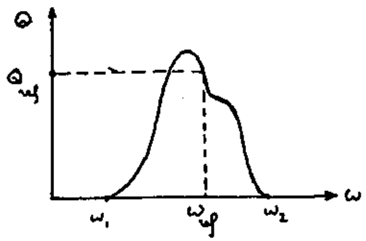


Fig. 18.2

where

$$B = \int_{f_1}^{f_2} \frac{Q}{Q_{ref}} \, df$$

$$F_{av} = \frac{1}{B} \int_{f_1}^{f_2} F \frac{Q}{Q_{ref}} \, df \qquad (18.7)$$

Minimum detectable signal

$$\frac{(\overline{v_o^2})_{signal}}{(\overline{v_o^2})_{noise}} = \frac{Q(\overline{v_g^2})}{(\overline{v_o^2})_{noise}} \geqslant \min.\left(\frac{S}{N}\right)_{output} = N \qquad (18.8)$$

The minimum signal to noise ratio at the output depends on the detection method, the equipment etc ...

Radar signal

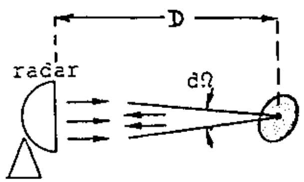


Fig. 18.3

From (.12) the power density incident on the target is

$$W_i = \frac{|\overline{E_i}|^2}{2R_{co}} = \frac{\mathcal{P}_{tr}}{4\pi D^2} G \qquad (18.9)$$

where G is the gain (Dη) of the antenna. The power scattered in a solid angle dΩ is (Fig.18 .3)

$$\mathcal{P} = W_i \, \sigma_{rad} \frac{d\Omega}{4\pi} \qquad (18.10)$$

The solid angle of concern is, from (14.8) ,

$$d\Omega = \frac{S_{eff}}{D^2} = \frac{1}{4\pi D^2} G \lambda^2 M P \qquad (18.11)$$

Therefore

$$\mathcal{P}_{rec} = \frac{G^2 \lambda^2}{64\pi^3 D^4} \sigma_{rad} M P \mathcal{P}_{tr} = \frac{|\overline{v_g^2}|}{4R_g} \qquad (18.12)$$

Radar equation

Averaged over all frequencies :

$$\boxed{\mathcal{P}_{rec} = \frac{G^2 \lambda^2}{64\pi^3 D^4} \sigma_{rad} M P \mathcal{P}_{tr} \geqslant k T B F_{eff} N} \qquad (18.13)$$

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List of symbols

$\bar{a}$ = magnetic potential (T m)

$\bar{b}$ = magnetic induction (T)

c = $(\varepsilon_o\mu_o)^{-0.5} = 3.10^8$ = velocity of light in vacuum (m s^{-1})

$\bar{d}$ = electric induction (C m^{-2})

$\bar{e}$ = electric field (V m^{-1})

$\bar{e}_a$ = impressed electric field (V m^{-1})

$\bar{e}_i, \bar{h}_i$ = incident fields

$\bar{h}$ = magnetic field (A m^{-1})

$\bar{j}$ = volume current density (A m^{-2})

$\bar{j}_a$ = applied volume current density (A m^{-2})

$\bar{j}_s$ = surface current density (A m^{-1})

$k_o = \frac{\omega}{c} = \frac{2\pi}{\lambda}$ = wave number in vacuum (m^{-1})

$\bar{m}_e$ = electric polarization density (C m^{-2})

$\bar{m}_m$ = magnetic polarization density (A m^{-1})

n = $(\varepsilon\mu)^{0.5}$ = index of refraction

$\bar{u}_a$ = unit vector in direction <u>a</u>

D = directivity (dimensionless)

$\bar{F}$ = radiation vector (V)

G = gain of an antenna (dimensionless)

I = current (A)

M = mismatch factor (dimensionless)

P = polarization factor (dimensionless)

$\bar{P}_e$ = electric dipole moment (C m)

$\bar{P}_m$ = magnetic dipole moment (A m^2)

R = distance to the origin (m)

$R_{co} = (\mu_o/\varepsilon_o)^{0.5} = 120\pi$ = characteristic resistance of vacuum (Ω)

S_{eff} = effective cross-section of an antenna (m^2)

W = electromagnetic energy density (J m^{-3})

$Z_a = R_a + jX_a$ = antenna impedance (Ω)

Z_c = characteristic impedance of a medium (Ω)

Z_L = a load impedance (Ω)

$\mathcal{E}$ = electromagnetic energy (J)

$\mathcal{P}$ = a power (W)

$\mathcal{R}$ = array factor (dimensionless)

$\varepsilon_o = \frac{1}{36\pi} 10^{-9}$ F m^{-1}

λ = wavelength in vacuum (m)

$\mu_o = 4\pi\ 10^{-7}$ H m^{-1}

ν = frequency (Hz)

σ = conductivity (S m^{-1})

σ_{bis} = bistatic cross-section (m^2)

σ_{rad} = radar cross-section (m^2)

σ_{sc} = total scattering cross-section (m^2)

ρ = volume charge density (C m^{-3})

ϕ = electric potential (V)

Φ = magnetic flux (Wb)

Ω = solid angle (sr)

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