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Algebraic Multigrid Methods

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This talk covers:

- description of multigrid principles
- AMG approach

- scalar problems
- system problems
- "element" interpolation
- future challenges
- conclusions

Algebraic multigrid (AMG):

- uses multigrid principles
- automatically defines mg components
- solves problems hard for standard mg
  - irregular domains
  - unstructured grids
  - discontinuous coefficients
  - discrete problems
- suited to structural & flow problems
- requires no multigrid expertise

AMG is <u>not</u> a general matrix solver

- applies to classes of problems
- research is application driven
- sense of smoothness
- use available information

Multigrid Principles

Problem: $LU = f$ on Ω

Components required

- A sequence of <u>grids</u> $\Omega^1, \Omega^2, \ldots \Omega^m$
  - Ω^1 — finest grid
  - Ω^m — coarsest grid
- Operators $L^1, L^2, \ldots, L^M$
- Grid transfer operators:
  - I_{k+1}^{k} Interpolation
  - I_{k}^{k+1} Restriction
- Relaxation method for each level.

Multigrid cycle (ν_1, ν_2) V-cycle $MG^k(u^k; f^k)$

If $k = m$, set $u^k = (L^k)^{-1} f^k$

Otherwise

Relax ν_1 times on $L^k u^k = f^k$

$$\left.\begin{array}{l} \text{Set } u^{k+1} = 0, \; f^{k+1} = I_k^{k+1}(f^k - L^k u^k) \\ \text{Perform } MG^{k+1}(u^{k+1}; f^{k+1}) \\ \text{Set } u^k \leftarrow u^k + I_{k+1}^{k} u^{k+1} \end{array}\right\} \text{Coarse-grid correction}$$

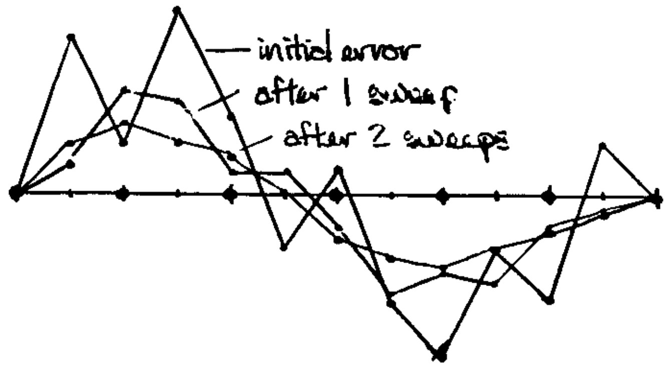
Relax ν_2 times on $L^k u^k = f^k$

The Cycle – why it works –

Let $L = \frac{d^2}{dx^2}$

Solve $L^k U^k = F^k$ on $[0,1]$ $U(0) = U(1) = 0$

$e^k = U^k - u^k$



Relaxation smooths error quickly (converges slowly on smooth error)

$e^k = U^k - u^k$ and $r^k = F^k - L^k u^k$ smooth

Residual equation

$$L^k e^k = r^k$$

Transfer to Ω^{k+1}

$$L^{k+1} U^{k+1} = I_k^{k+1} r^k$$

U^{k+1} approximates e^k, so

$u^k + I_{k+1}^k U^{k+1}$ approximates U^k.

Reasons for efficiency:

- Relaxation smoothes error
  ⇒ residual equation can be solved on a coarser grid.

General Conditions for an efficient MG solution –

(1) Error not reduced by relaxation must be approximated by some function in $R(I_{k+1}^k)$

(2) Coarse-grid correction must eliminate error in $R(I_{k+1}^k)$.

Relaxation in "Geometric" multigrid must be chosen carefully –

Algebraic Multigrid —

- Fix relaxation (point Gauss-Seidel)
- Choose coarse grids and interpolation to satisfy (1)
- Define other components to satisfy (2).

Setup & Solution phases separate

Outline of Setup Phase for $L^1 u^1 = f^1$ on Ω^1

1) Set $k=1$

2) Partition Ω^k into C^k and F^k. Set $\Omega^{k+1} = C^k$.
   Define I_{k+1}^k
   Set $I_k^{k+1} = (I_{k+1}^k)^T$
   $L^{k+1} = I_k^{k+1} L^k I_{k+1}^k$ } Variational or Galerkin formulation

3) If Ω^{k+1} is "small", Stop
   Otherwise, set $k \leftarrow k+1$ & go to 2.

The Energy Functional and the Goal of Computation

$Au = f$ A s.p.d U exact sol. $e = U - u$

Energy norm $\|u\|_A = \langle Au, u\rangle^{1/2}$ $\langle u, v\rangle = \sum_j u_j v_j$

- Solving $Au = f \Leftrightarrow$ minimizing $\langle Au - 2f, u\rangle$

$\langle Au - 2f, u\rangle = \sum_i \left(\sum_j a_{ij} u_j - 2f_i\right) u_i$

To minimize $0 = \frac{\partial}{\partial u_i}$ $i = 1, \dots, n$

$0 = \sum_j a_{ij} u_j + \sum_j a_{ji} u_j - 2f_i$

or, A symmetric $\Rightarrow \sum a_{ij} u_j = f_i \Rightarrow Au = f$

- Minimizing $\langle Au - 2f, u\rangle \Leftrightarrow$ min. $\|e\|_A$

$$\langle Au - 2f, u\rangle + \langle AU, U\rangle = \langle Au - 2AU, u\rangle + \langle AU, U\rangle$$
$$= \langle Au, u\rangle - 2\langle AU, u\rangle + \langle AU, U\rangle$$
$$= \langle Au, u\rangle - \langle Au, U\rangle - \langle AU, u\rangle + \langle AU, U\rangle$$
$$= -\langle Au, e\rangle + \langle AU, e\rangle$$
$$= \langle Ae, e\rangle = \|e\|_A^2$$

Energy Minimization and the Coarse Grid Problem

Given I_c^f (interpolation)

Goal: Eliminate error in $\mathcal{R}(I_c^f)$ How?

$$\min_{v^c} \langle A(u+I_c^f v^c)-2f,\ u+I_c^f v^c\rangle$$

$$= \underbrace{\langle Au-2f,u\rangle}_{\text{constant}} + \langle Au-2f,\ I_c^f v^c\rangle$$

$$+ \underline{\langle A I_c^f v^c, u\rangle} + \langle A I_c^f v^c,\ I_c^f v^c\rangle$$

$$\hookrightarrow = \langle Au,\ I_c^f v^c\rangle$$

$$= c + 2\langle Au-f,\ I_c^f v^c\rangle + \langle A I_c^f v^c,\ I_c^f v^c\rangle$$

$$= c + \langle A I_c^f v^c - 2(f-Au),\ I_c^f v^c\rangle$$

$$= c + \langle (I_c^f)^T A I_c^f v^c - 2(I_c^f)^T(f-Au),\ v^c\rangle$$

$\Leftrightarrow$ solve

$$\underbrace{(I_c^f)^T A I_c^f}_{A^c} v^c = \underbrace{(I_c^f)^T}_{I_f^c \text{ (restriction)}}(f-Au)$$

Note: $\langle Bu,v\rangle = \langle u, B^T v\rangle$

$\langle u,v\rangle = \langle v,u\rangle$

$\langle u,v+w\rangle = \langle u,v\rangle + \langle u,w\rangle$

Energy Minimization and Gauss-Seidel

Let $d \in \mathbb{R}^n$. Consider the problem

$$\min_{\alpha} \langle A(u+\alpha d)-2f,\ u+\alpha d\rangle$$

$$= \langle Au-2f,u\rangle + \alpha\langle Au-2f,d\rangle$$

$$+ \alpha\langle Ad,u\rangle + \alpha^2\langle Ad,d\rangle$$

$$= c + 2\alpha\langle Au-f,d\rangle + \alpha^2\langle Ad,d\rangle$$

To minimize, set $\frac{d}{d\alpha} = 0$

$$0 = \frac{d}{d\alpha} = 2\langle Au-f,d\rangle + 2\alpha\langle Ad,d\rangle$$

$$\alpha = -\frac{\langle Au-f,d\rangle}{\langle Ad,d\rangle} = \frac{\langle r,d\rangle}{\langle Ad,d\rangle}$$

Let $\delta_i^f = [0,0,\ldots,1,0\ldots0]^T$ 1 in ith place

- For $d = \delta_i^f$, $i=1,2,\ldots,n$ This is Gauss-Seidel
- For $d = I_c^f \delta_i^c$, $i=1,2,\ldots,n_c$.

This is Gauss-Seidel on the coarse grid

Given L^k – How do we choose Ω^{k+1} and I^k_{k+1} ?

Assume L^k is "positive type" –
ie $l_{ii} > 0$, $l_{ij} \le 0$ for $i \ne j$ and $\sum_j l_{ij} \ge 0$.
(not necessary, but convenient)

For each $i \in \Omega^k$, define

$$S_i = \{ j \in \Omega^k : -l_{ij} \ge \alpha \max_{k \ne i} -l_{ik} \}$$ "strong" connections

(usually $\alpha = .25$)

Choose C^k, F^k, so that:

* $\boxed{\text{For each } i \in F^k, \text{ each } j \in S_i \text{ is either in } C \text{ or } S_j \cap C_i \ (= S_i \cap C) \ne \phi.}$

2-stage process –
- Quick C/F choice
- 2nd pass – enforce * by introducing C-points.

Choosing the coarse grid.

To Partition Ω into C and F – 2 part process

1. For each $i \in \Omega$, let λ_i be the number of points j which strongly depend on i.
2. Pick i with maximal λ_i (i not already in C or F.) Put i in C.
3. For each j which strongly depends on i (j not already in C or F), put j in F, then increment λ_k for each k on which j strongly depends.
4. If $\Omega = C \cup F$ stop. Otherwise go to 2.

Definition of interpolation:

Let $C_i = S_i \cap C$

$D_i^s = S_i \cap F$

D_i^w = everything else (i.e. "weak" connections)

Form of interpolation

Let v^{k+1} be defined on Ω^{k+1}

$$\left(I_{k+1}^{k} v^{k+1}\right)_i \begin{cases} v_i^{k+1} & \text{if } i \in C \\ \sum_{j \in C_i} \omega_{ij} v_j^{k+1} & \text{if } i \in F \end{cases}$$

Relaxation is slow if and only if the residual is small compared to the error.

i.e. if $i \in F$

$$l_{ii} e_i \approx -\sum_{j \neq i} l_{ij} e_j$$

$$= -\underbrace{\sum_{j \in C_i} l_{ij} e_j}_{\text{C-points}} - \underbrace{\sum_{j \in D_i^s} l_{ij} e_j}_{\text{Strong}} - \underbrace{\sum_{j \in D_i^w} l_{ij} e_j}_{\text{Weak}}$$

For weak connections — set $e_j = e_i$

For strong connections, set

$$e_j = \left(\sum_{k \in C_i} l_{jk} e_k\right) \Big/ \sum_{k \in C_i} l_{jk}$$

Solve for e_i to get ω_{ij}'s

Results

Laplace Operator

Stencil	Convergence per cycle	Complexity	Times
$\frac{1}{h^2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}$	.054	2.21	.29 - Cycle 1.63 - Setup
$\frac{1}{2h^2}\begin{bmatrix} -1 & & -1 \\ & 4 & \\ -1 & & -1 \end{bmatrix}$	.067	2.12	.27 - Cycle 1.52 - Setup
$\frac{1}{8h^2}\begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix}$	.078	1.30	.26 - Cycle 1.83 - Setup
$\frac{1}{20h^2}\begin{bmatrix} -1 & -4 & -1 \\ -4 & 20 & -4 \\ -1 & -4 & -1 \end{bmatrix}$	.109	1.30	.26 - Cycle 1.83 - Setup

Results Finite difference - symmetric 1/64

$-\epsilon u_{xx} - u_{yy}$	$\epsilon =$ .001	.084 /cycle
	.01	.093
	.1	.098
	.5	.069
	1	.056
	2	.079
	10	.087
	100	.093
	1000	.083
$-\nabla(d\nabla u)$		.069

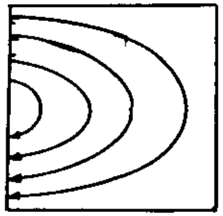
10	100
1	1000

Finite Element - Symmetric

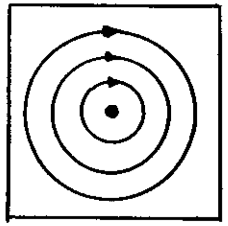
Δu with local refinements	.124
Δu ~~with~~	.063
BINACAOOR	.18

Non-symmetric problems (upwind differencing)

$$-\epsilon \Delta u + a(x,y)\, u_x + b(x,y)\, u_y = f$$

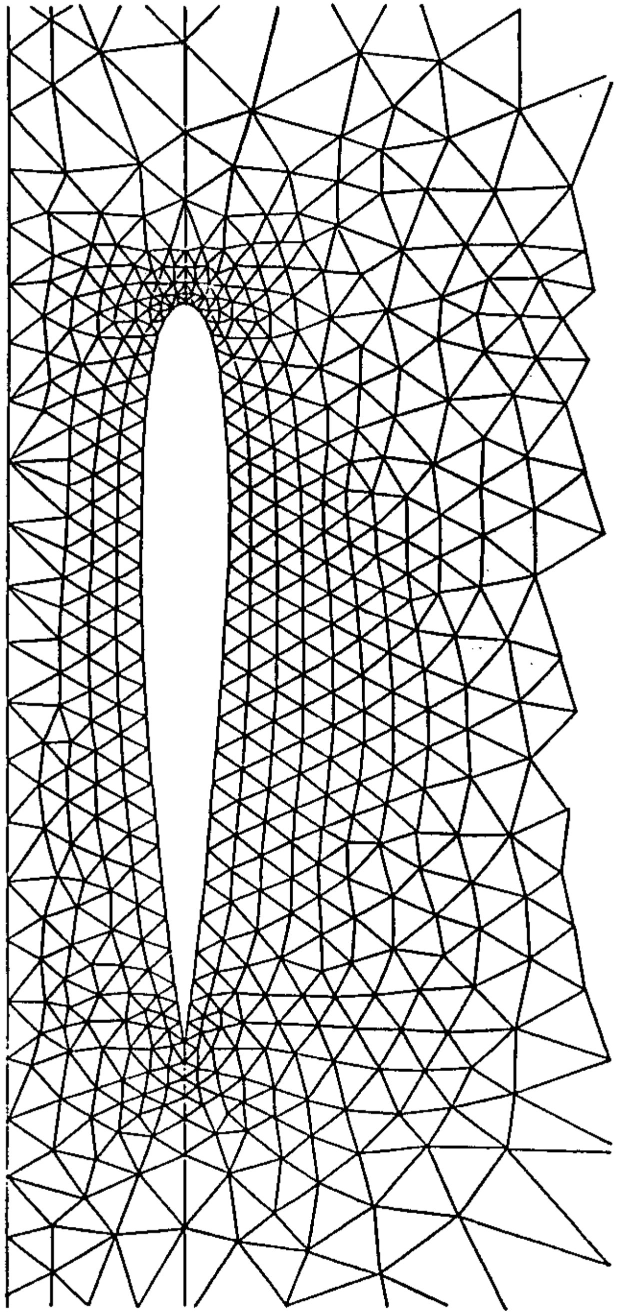


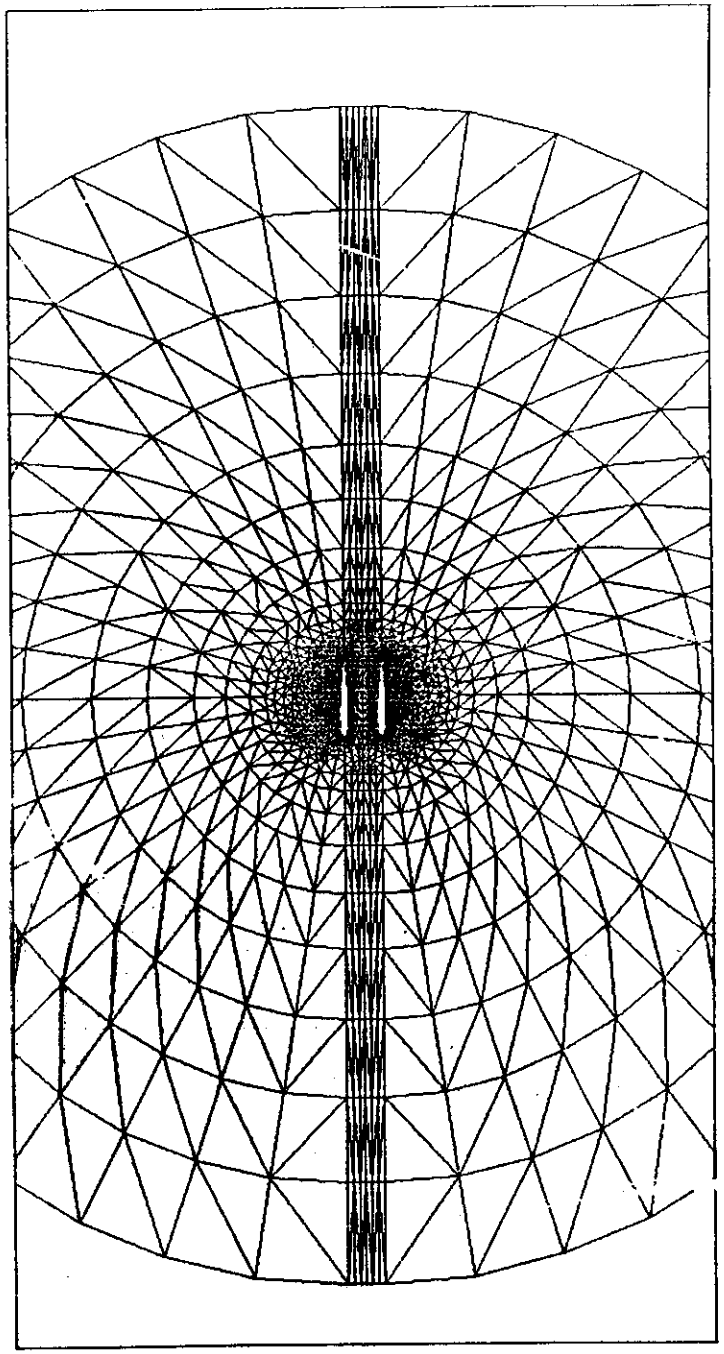
$\epsilon = .1$	.069
$\epsilon = .001$	.050
$\epsilon = .00001$	.029



$\epsilon = .1$	.062
$\epsilon = .001$	.076
$\epsilon = .00001$	.102

$a = \cos\theta$	$\theta =$	0	.00002
$b = \sin\theta$		$\pi/8$	.0003
		$\pi/4$	.00008
		$3\pi/8$	.0007
		$\pi/2$	.00002





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Problems

Solve Poisson equation on unstructured finite-element discretizations of:

(1) Domain around NACA0012 airfoil

(2) Domain around 2 NACA0012 airfoils

(3) Domain between 2 spheres
$\{ \bar{x} \in \mathbb{R}^3 : 2 \le \|\bar{x}\| \le 10 \}$

	Problem 1	2	3	
# of points	1020	1702	2052	
grid complexity	1.81	1.74	1.61	
L-complexity	2.67	2.63	3.01	
Convergence factor	.094	.167	.128	(asymptotic)
Time/cycle*	.10	.16	.34	

* On an IBM 3083.
Last problem – cycle time estimated

Approaches for "System" Problems (>1 unknown)

Example: Plane-stress

$$\left.\begin{aligned} u_{xx} + \tfrac{1-\nu}{2} u_{yy} + \tfrac{1+\nu}{2} v_{xy} &= f \\ \tfrac{1+\nu}{2} u_{xy} + \tfrac{1-\nu}{2} v_{xx} + v_{yy} &= g \end{aligned}\right\} \text{ on } \Omega \subset \mathbb{R}^2$$

yields discrete system

$$\begin{bmatrix} A_{11}^1 & A_{12}^1 \\ A_{21}^1 & A_{22}^1 \end{bmatrix} \begin{bmatrix} u^1 \\ v^1 \end{bmatrix} = \begin{bmatrix} f^1 \\ g^1 \end{bmatrix}$$

- AMG as described does not work well. (Relaxation does not make error in u and v "equal".)

"Block" Approach

Block Gauss-Seidel

$$u^1 \leftarrow (A_{11}^1)^{-1}(f^1 - A_{12}^1 v^1)$$
$$v^1 \leftarrow (A_{22}^1)^{-1}(g^1 - A_{21}^1 u^1)$$

Repeat until convergence

Use "scalar" AMG to "solve" (1 or more cycles)

- separate coarsening
- separate interpolation $I_{k+1}^{k\,(1)}$ $I_{k+1}^{k\,(2)}$

Problem: This does not work. Outer iteration is very slow ($v = .3$ 1x2 problem .75)

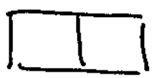
Solution: Allow interaction between u and v on all levels.

$$A^k = \begin{bmatrix} A^k_{11} & A^k_{12} \\ A^k_{21} & A^k_{22} \end{bmatrix} \qquad I^k_{km} = \begin{bmatrix} I^{k(1)}_{km} & 0 \\ 0 & I^{k(2)}_{km} \end{bmatrix}$$

Continue as usual

$$I^{km}_k = (I^k_{km})^T \qquad A^{km} = I^{km}_k A^k I^k_{km} \qquad k = 1, 2, \ldots$$

This is called the "unknown" approach.

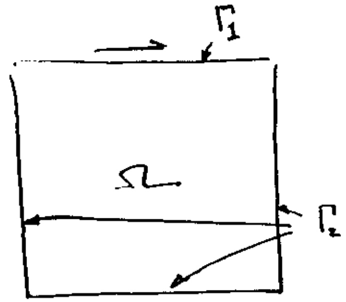


The Model Problem

- Stokes flow in a driven cavity
- Vorticity - stream function formulation

u - stream function

v - vorticity



$$-\Delta u - v = 0$$
$$-\Delta v = 0 \qquad \text{on } \Omega$$

$$u = 0 \quad \text{on } \Gamma = \Gamma_1 \cup \Gamma_2$$

$$\vec{\nabla} u \cdot \vec{n} = 1 \text{ on } \Gamma_1 \qquad \vec{\nabla} u \cdot \vec{n} = 0 \text{ on } \Gamma_2$$

Discretization of the Problem

- uniform quadrilateral mesh
  $h = 1/8,\ 1/16,\ 1/32,\ 1/64$
- bilinear finite element

Application of AMG

- Separate coarsening for u, v
- Gauss-Seidel relaxation
- (1,1) V-cycles

Results (asymptotic convergence)

1/8	.22
1/16	.39
1/32	.42
1/64	.44

Approaches for Systems

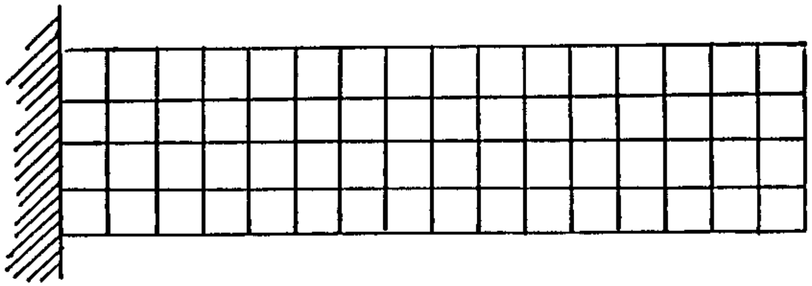
- Elasticity and structures
- Fluid flow problems

"Standard" approach

- Separate interpolation for functions
- Separate coarsening

Works well on

- VLSI design problem
- 2 & 3-D Elasticity (some domains)



Fails on cantilever beam

Why?

- Very ill-conditioned problem
- Has very slowly converging error

$\Rightarrow$ Separate interpolation was not accurate enough.

"Block" approach

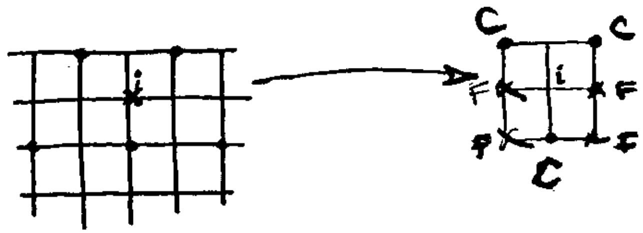
- Blocks correspond to points
- Block Gauss-Seidel
- Point coarsening
- Block interpolation

Problem: Proper distribution difficult

$$A_{ii}\vec{e}_i = -\sum_{k\in C} A_{ik}\vec{e}_k \underbrace{- \sum_{j\in F} A_{ij}\vec{e}_j}_{?}$$

Element Interpolation

Consider the cantilever beam discretized using finite elements:



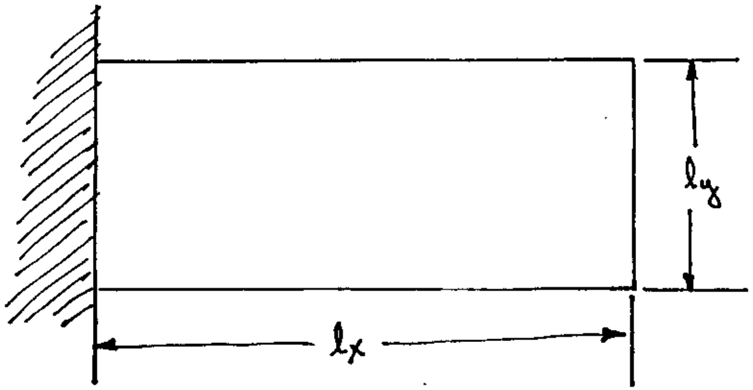
To compute interpolation to point i:

- Consider only elements adjacent to i
- For fixed (but arbitrary) values at C-points, minimize the energy.

2-D Elasticity – Plane Stress

$$u_{xx} + \frac{1-\nu}{2} u_{yy} + \frac{1+\nu}{2} v_{xy} = f_1$$

$$\frac{1+\nu}{2} u_{xy} + \frac{1-\nu}{2} v_{xx} + v_{yy} = f_2$$

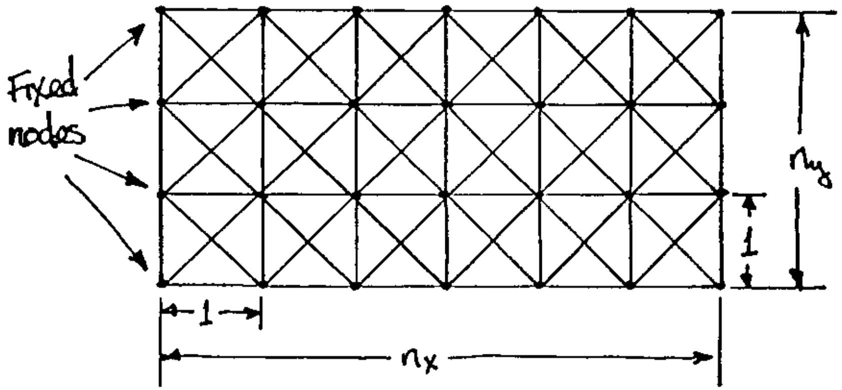


- Finite element discretization (bilinear test functions)

l_x	l_y	h	Rates (V-cycle) element int.	linear int
1	1	1/32	.20	.20
1	1/4	1/8	.25	.86
1	1/8	1/16	.26	.96
1	1/32	1/64	.26	.98

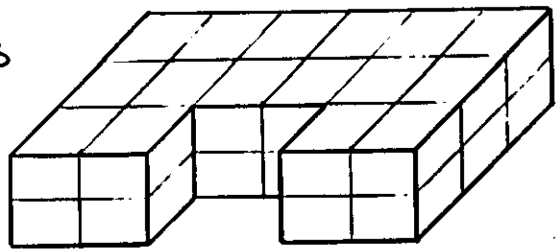
2-D Beam Structures

- Hinged joints
- Beams subject to compression/extension
- Equations derived from Hooke's Law.

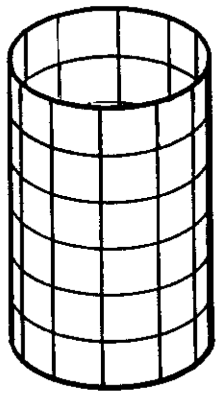


n_x	n_y	V-cycle Rates element int.	linear int.
16	16	.26	.20
32	8	.27	.64
64	4	.28	.97
128	2	.28	.93 +

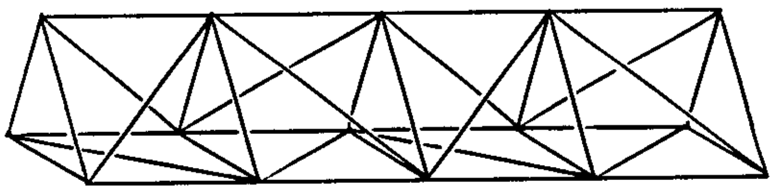
- Solids



- Shells



- Beam Structures

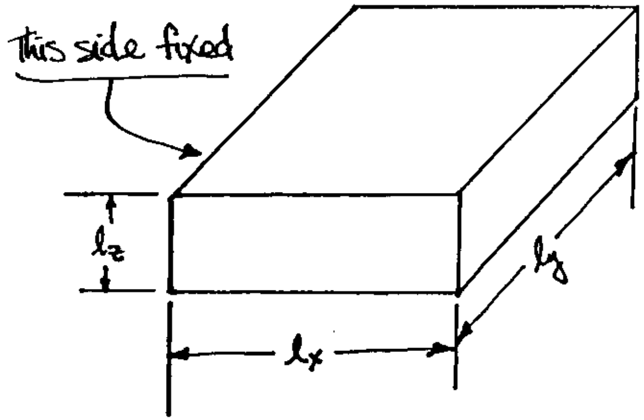


3-D Elasticity - Solids

$$u_{xx} + \frac{1-\nu}{2}(u_{yy} + u_{zz}) + \frac{1+\nu}{2}(v_{xy} + w_{xz}) = f_1$$

$$\frac{1+\nu}{2} u_{xy} + \frac{1-\nu}{2}(v_{xx} + v_{zz}) + v_{yy} + \frac{1+\nu}{2} w_{yz} = f_2$$

$$\frac{1+\nu}{2}(u_{xz} + v_{yz}) + \frac{1-\nu}{2}(w_{xx} + w_{yy}) + w_{zz} = f_3$$

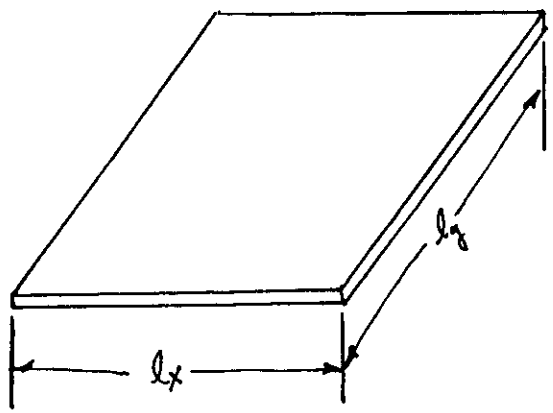


- Finite elements (trilinear test functions)

l_x	l_y	l_z	h	V-cycle (element int.)
1	1	1	1/8	.28
1	1	1/4	1/4	.41
1	1	1/8	1/8	.41
1	1	1/16	1/16	.42

3-D Elasticity - Shells

- Equations modified to model physics on thin plates.



- Finite elements - 6 degrees of freedom per node.

Preliminary results (element int.)

$l_x = l_y = 1 \quad h = 1/16 \quad$ V-cycle rate $= .41$

Remarks

- Must save A_α's
- Extends to coarser grids

$$A^c_\alpha = (I^f_c)^T A_\alpha I^f_c$$

- Easy to use for systems
- Can make computation cheaper

$$A^c = (I^f_c)^T A\, I^f_c$$