INTERNATIONAL ATOMIC ENERGY AGENCY
UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

WINTER COLLEGE ON "MULTILEVEL TECHNIQUES IN COMPUTATIONAL PHYSICS"

Physics and Computations with Multiple Scales of Lengths
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H4.SMR 539/11

Non-Scalar Systems

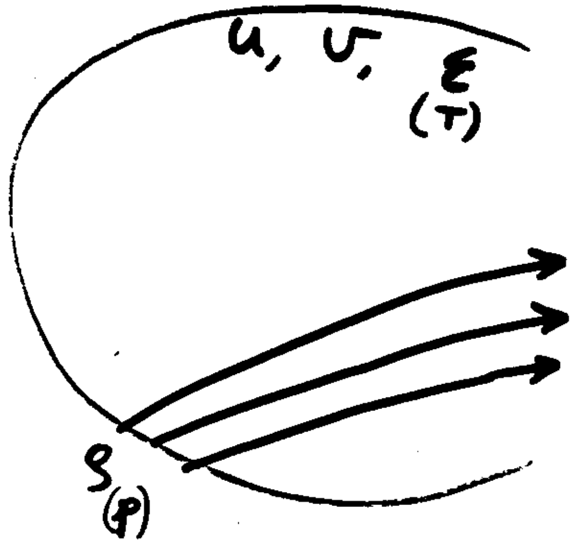
A. Brandt
The Weizmann Institute of Science
Rehovot. Israel

Compressible Navier-Stokes: 2D

- $-\mu\Delta u + \rho u u_x + \rho v u_y - (\lambda+\mu)(u_x+v_y)_x + p_x = 0$
- $-\mu\Delta v + \rho u v_x + \rho v v_y - (\lambda+\mu)(u_x+v_y)_y + p_y = 0$
- $-\kappa\Delta\varepsilon + \rho u \varepsilon_x + \rho v \varepsilon_y + p(u_x+v_y)$
  $\quad -\mu(v_x+u_y)^2 - \lambda(u_x+v_y)^2 - 2\mu(u_x^2+v_y^2) = 0$
- $(\rho u)_x + (\rho v)_y - \nu\Delta\rho = 0$
- $p = p(\varepsilon, \rho)$

Non-elliptic BVP.
unknown $u, v, \varepsilon, \rho, p$

B.C.:



Inviscid case: $\lambda, \mu, \kappa \ll \rho \ell \max(|u|, |v|)$

ℓ = length at which (u, v, ε) change. → Euler eq.

Usually there are viscous layers.

Incomressible:

$$\begin{pmatrix} Q & 0 & \partial_x \\ 0 & Q & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ p \end{pmatrix} = 0$$

$Q = -\frac{1}{Re}\Delta + u\partial_x + v\partial_y$.

det ↑ $= (-\Delta) Q$ elliptic.

large Re: Sing. Pert.

"True" MG Efficiency

- ~ .1 error reduction per cycle
- 1-FMG solver to $O(h^2)$ accuracy
  ~ 10 minimal work units
- Predicted theoretically

Simple elliptic
mode analysis 1972-3
rigorous $\square$ 1977

General elliptic systems 1979-84
rigorous: exact ($\Rightarrow$ opt.)
general , norms, cycles 1985-90

SD transonic 1976
etc. SOUTH

General non-elliptic $\underline{O(h^2)}$ 1990
mode analysis YAVNEH

h-principal $L_h u = f$ $\qquad\qquad$ $\det L$

Cauchy-Riemann $\quad \begin{pmatrix} \partial_x & \partial_y \\ \partial_y & -\partial_x \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0 \qquad -\Delta$

2D Stokes $\quad \begin{pmatrix} -\Delta & 0 & \partial_x \\ 0 & -\Delta & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ p \end{pmatrix} \qquad \Delta^2$

2D Incompressible Navier-Stokes $\quad \begin{pmatrix} Q & 0 & \partial_x \\ 0 & Q & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ p \end{pmatrix} \qquad -Q\Delta$

$$Q = -\frac{1}{R}\Delta + \underline{u}\cdot\underline{\nabla}$$

2D Euler $\quad \begin{pmatrix} \rho\underline{u}\cdot\underline{\nabla} & 0 & 0 & 0 & \partial_x \\ 0 & \rho\underline{u}\cdot\underline{\nabla} & 0 & 0 & \partial_y \\ \rho\partial_x & \rho\partial_y & \underline{u}\cdot\underline{\nabla} & 0 & 0 \\ p\partial_x & p\partial_y & 0 & \rho\underline{u}\cdot\underline{\nabla} & 0 \\ 0 & 0 & -p_\rho & -p_\varepsilon & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ \rho \\ \varepsilon \\ p \end{pmatrix} \qquad \rho^3(\underline{u}\cdot\underline{\nabla})^2 \times \left[(\underline{u}\cdot\underline{\nabla})^2 - a^2\Delta\right]$

$$a^2 = p_\rho + \frac{p}{\rho^2} p_\varepsilon$$

Compressible Navier-Stokes (on the viscous scale) $\qquad k\mu(2\mu+\lambda)\,\Delta^3(\underline{u}\cdot\underline{\nabla})$

Central Cauchy-Riemann $\qquad \Delta^{2h}$

Central (Navier-) Stokes $\qquad Q^h\Delta^{2h}$

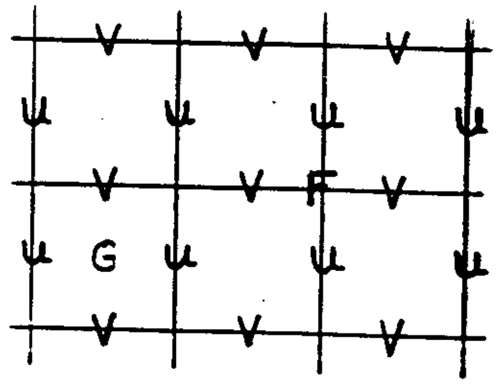
Cauchy-Riemann Equations

① $U_y - V_x = F$
② $U_x + V_y = G$
One boundary condition

$$L\underline{u} = \begin{pmatrix} \partial_y & -\partial_x \\ \partial_x & \partial_y \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}$$

$$\det \uparrow = \partial_x^2 + \partial_y^2 = \Delta$$

Difference Equations :

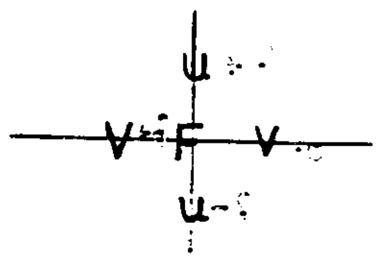


Eq. 1 defined at grid nodes
Eq. 2 defined at cell centers

$\partial_x^h = (T_x^{1/2} - T_x^{-1/2})/h$,

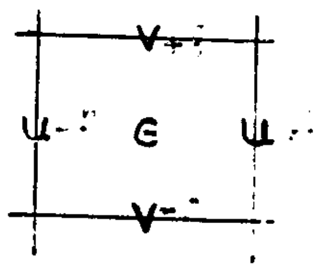
$\det L^h = \Delta^h$ 5-point

Relaxation on Equation 1 :



preserves $R_2 = G - u_x - v_y$
smoothes $R_1 = F - u_y + v_x$

Relaxation on Equation 2 :



Smoothing Factor : $\bar{\mu} = .5$

Red-Black : $\bar{\mu} = .25$

Central Differencing of Cauchy-Riemann Eq.

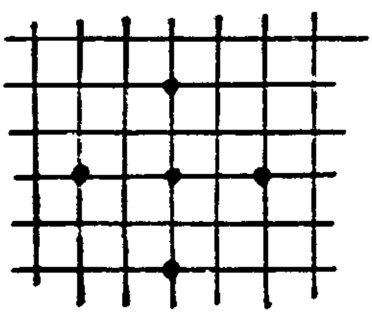
$$U_y - V_x = F$$
$$U_x + V_y = G$$

$$\frac{U^h(x,y+h) - U^h(x,y-h)}{2h} - \frac{V^h(x+h,y) - V^h(x-h,y)}{2h} = F^h(x,y)$$

$$\frac{U^h(x+h,y) - U^h(x-h,y)}{2h} + \frac{V^h(x,y+h) - V^h(x,y-h)}{2h} = G^h(x,y)$$

$$L^h \begin{pmatrix} U^h \\ V^h \end{pmatrix} = \begin{pmatrix} \frac{T_y - T_y^{-1}}{2h} & -\frac{T_x - T_x^{-1}}{2h} \\ \frac{T_x - T_x^{-1}}{2h} & \frac{T_y - T_y^{-1}}{2h} \end{pmatrix} \begin{pmatrix} U^h \\ V^h \end{pmatrix} = \begin{pmatrix} F^h \\ G^h \end{pmatrix}$$

$$\det L^h = \frac{1}{4h^2}\left(T_x^2 + T_x^{-2} + T_y^2 + T_y^{-2} - 4 \right) = \Delta^{2h}$$



$$\begin{matrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{matrix}$$

unstable modes:

a b a b a b
c d c d c d
a b a b a b
c d c d c d
a b a b a b
c d c d c d

Positive type but "Quasi-elliptic"

Distributive Relaxation $L\underline{u}=\underline{f}$

Cauchy-Riemann: $L = \begin{pmatrix} \partial_y & -\partial_x \\ \partial_x & \partial_y \end{pmatrix}$

Relaxing Eq. 1 is equivalent to changing a function w_1 at the relaxed vertex, with

$$u = \partial_y^h w_1, \qquad v = -\partial_x^h w_1 .$$

Relaxing Eq. 2 – changing w_2 at the relaxed center

$$u = \partial_x^h w_2, \qquad v = \partial_y^h w_2$$

$\Longleftrightarrow$ GS relaxation for $\underline{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, where

$$u = \partial_y w_1 + \partial_x w_2, \qquad v = -\partial_x w_1 + \partial_y w_2$$

or $\underline{u} = M\underline{w}$, $\quad M = \begin{pmatrix} \partial_y & \partial_x \\ -\partial_x & \partial_y \end{pmatrix}$

$\Longleftrightarrow$ GS for $LM\underline{w} = \underline{f}$, $\qquad LM = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}$

$$\therefore \bar{\mu}_L = \bar{\mu}_\Delta = \begin{cases} .5 & : \text{Lexicographic}^2 \\ .25 & : RB^2 \quad \text{(one or two sweeps/cycle)} \end{cases}$$

Generally $\bar{\mu}_L = \bar{\mu}_{\det L}$ can be obtained for any choice of relaxation for $\det L$

Method: M_{ij} = cofactor of L_{ji}

$\Rightarrow \quad LM = (\det L)\, I$

Relaxation of Systems $L\underline{u} = \underline{f}$

Guiding Principle: Principal $\det L = \ell_1 \ell_2 \cdots$

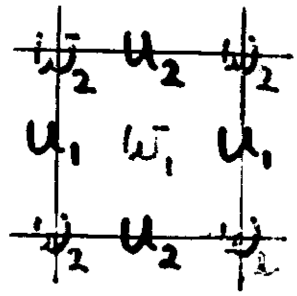
$\Rightarrow \bar{\mu}_L = \max[\bar{\mu}_{\ell_1}, \bar{\mu}_{\ell_2}, \ldots]$ can be obtained

Theorem (Brandt - Joseph). $\det L = \ell_1 \ell_2$

$\Rightarrow \exists L_i$ such that $L = L_1 L_2$, $\det L_i = \ell_i$.

Relaxation of $L_1 L_2 \underline{u} = \underline{f}$

by alternately relaxing $L_1 \underline{v} = \underline{f}$, $L_2 \underline{u} = \underline{v}$



Example: Elasticity

$$\begin{pmatrix} \mu\Delta + \lambda\partial_{xx} & \lambda\partial_{xy} \\ \lambda\partial_{xy} & \mu\Delta + \lambda\partial_{yy} \end{pmatrix} = \begin{pmatrix} \partial_x & \partial_y \\ \partial_y & -\partial_x \end{pmatrix} \begin{pmatrix} \mu+\lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} \partial_x & \partial_y \\ \partial_y & -\partial_x \end{pmatrix}$$

$\Rightarrow \bar{\mu} = \bar{\mu}_\Delta$ even for $\mu \ll \lambda$.

usually: find

$$LM = \begin{pmatrix} \ell_{i_1} & & & 0 \\ & \ell_{i_2} & & \\ & & \ddots & \\ & & & \ell_{i_q} \end{pmatrix}$$

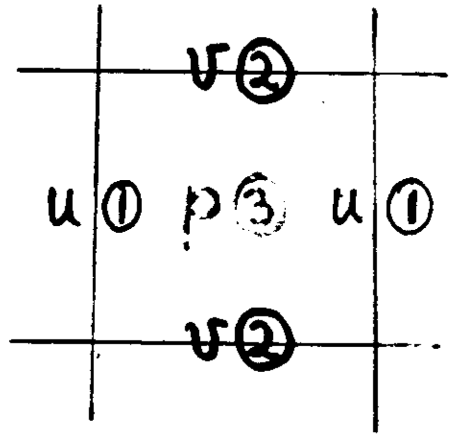
Incompressible Navier Stokes: 2D

① $Qu + p_x = f_1$
② $Qv + p_y = f_2$
③ $u_x + v_y = f_3$

$$L\underline{u} = \begin{pmatrix} Q & 0 & \partial_x \\ 0 & Q & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix}\begin{pmatrix} u \\ v \\ p \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

$Q = -\frac{1}{R}\Delta + \tilde{u}\partial_x + \tilde{v}\partial_y$ $\qquad \det L = -\Delta Q$

two boundary conditions



Differencing:

$\partial_x^h = \left(T_x^{1/2} - T_x^{-1/2}\right)/h.$ $\qquad \det L^h = -\Delta^h Q^h$

h-ellipticity of L^h depends on Q^h.

Relaxation:

distribution $M = \begin{pmatrix} 1 & 0 & -\partial_x \\ 0 & 1 & -\partial_y \\ 0 & 0 & Q \end{pmatrix} \Rightarrow LM = \begin{pmatrix} Q & 0 & 0 \\ 0 & Q & 0 \\ \partial_x & \partial_y & -\Delta \end{pmatrix}$

cofactors of last row in L, divided by their common divi

Relaxation of L is reduced to that of Q

Box GS on coarsest grids

Higher-order smoothing boundary conditions

FAS

FMG solution to $O(h^2)$ in one $W(2,0)$.

Non staggered grids. Conservative schemes

Relaxation:

① GS for $Qu = f_1 - p_x$

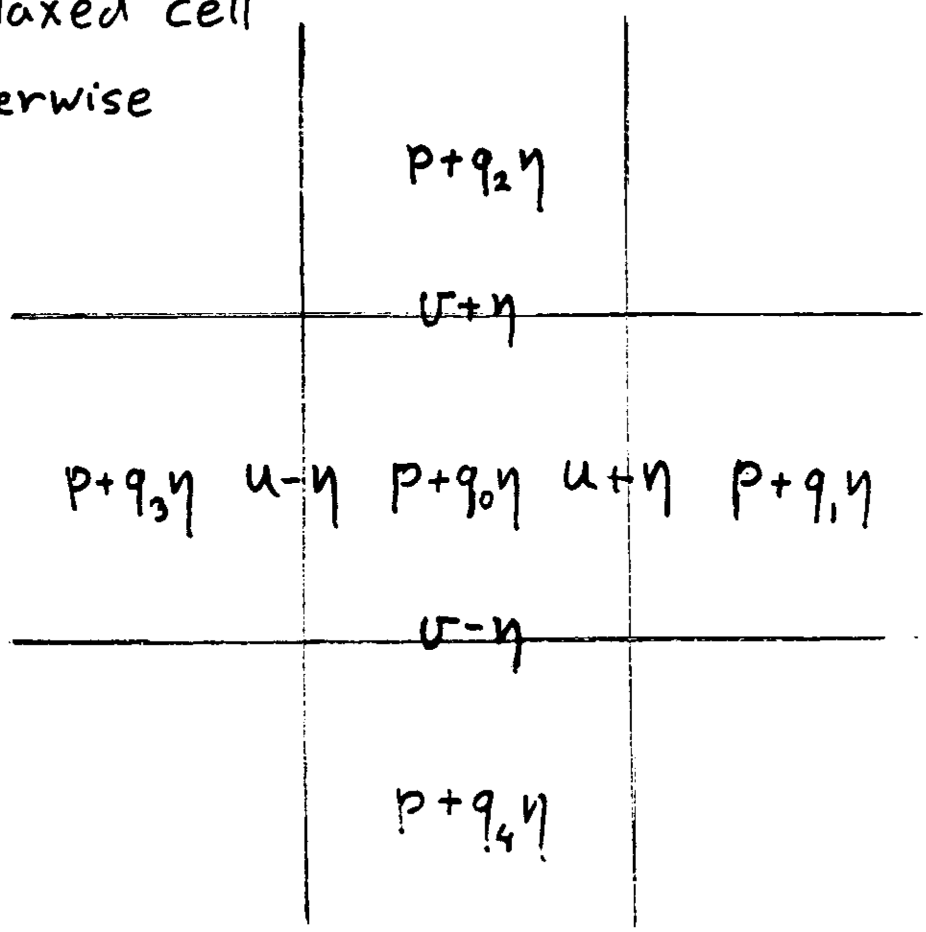
② GS for $Qv = f_2 - p_y$

③ GS for $u_x + v_y = f_3$

with $\delta u = -\omega_x$, $\delta v = -\omega_y$, $\delta p = Q\omega$

$\Rightarrow \ \delta(Qu + p_x) \approx 0, \quad \delta(Qv + p_y) \approx 0$

$$\delta\omega = \begin{cases} h\eta & \text{: relaxed cell} \\ 0 & \text{otherwise} \end{cases}$$



η chosen to satisfy $u_x + v_y = f_3$

Convection Diffusion $Q = -\frac{1}{R}\Delta + \sum u_i \partial_i$

Large hR: $Q^h = -\sum \beta_i h \partial_{ii}^h + \sum u_i \partial_i^{2h}$

AV (artificial viscosity)

For stability (i.e., in relaxation) $\beta_i = O(|u_i|)$

i.e., stream-wise AV is $O(h|\underline{u}|)$. semi h-elliptic.

Intended alignment: for good approximation of cross-stream h-f or discontinuities:

(1) grid aligned with the flow throughout

(2) Q^h that avoid straddling the discontinuity

Relaxation: red-black Gauss-Seidel $\frac{\beta_i}{|u_i|} > .5$

Block relaxation: only for the corresponding intended alignment, hence simple.

Slow smoothing in regions of accidental alignment is for "characteristic" components, which anyway are not approximated at all in other regions

Residuals transfer: $\beta_i = 0 \Rightarrow O(h^2)$.

Downstream bias • Avoid straddling discontinuitie

Interpolation: Avoid straddling discont.