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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS
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WINTER COLLEGE ON "MULTILEVEL TECHNIQUES IN COMPUTATIONAL PHYSICS"

Physics and Computations with Multiple Scales of Lengths (21 January - 1 February 1991)

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Multigrid Methods for Transport Models

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Multi Grid Methods
for
Transport Models

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Applications

- Satellite Electronics Shielding
- Radiation Effects on Electronics
- Solar Wind Models
- Nuclear Reactor Design
- Radiation Detector Design
- Radiation Effects on Materials

Linear Boltzman Transport Equations

$N(\underline{x}, \vec{\Omega}, E, T)$ Density of Particles

$\underline{x}$	position
$\vec{\Omega}$	direction
E	energy
T	time

$$\frac{\partial N}{\partial T} = -\vec{\Omega} \cdot v \nabla N - \sigma_t v N + S(\underline{x}, \vec{\Omega}, E, T)$$

S - Sources $\sigma_t dt$ = Collision Probability

Time Independent

$\Psi = vN$ flux

$$\vec{\Omega} \cdot \nabla \Psi + \sigma_t \Psi = S(\underline{x}, \vec{\Omega}, E)$$

Sources

i) Scattering Source

Particles at position $\underline{x}$ that scatter

from		to
E'	$\rightarrow$	E
$\vec{\Omega}'$	$\rightarrow$	$\vec{\Omega}$

due to a collision

$$S_s = \int dE' \int d\vec{\Omega}' \, \Sigma_s(\underline{x}, \vec{\Omega}' \to \vec{\Omega}, E' \to E) \, \Psi(x, \vec{\Omega}', E$$

ii) Fission Source

$$S_f = \int dE' \int d\vec{\Omega}' \, \Sigma_f(\underline{x}, \vec{\Omega}' \to \vec{\Omega}, E' \to E) \, \Psi(x, \vec{\Omega}', E'$$

iii) Other Sources

$$Q(\underline{x}, \vec{\Omega}, E)$$

Multi Group

Discrete Energies

$\psi^{(k)}(x, \vec{\Omega})$ - Flux with energy $E^{(k)}$

$$\vec{\Omega} \cdot \nabla \psi^{k} + \sigma_t \psi^{k} = \sigma_s \int d\vec{\Omega}' \sum_s^{k} (x, \vec{\Omega}' \to \vec{\Omega}) \psi^{k}(x, \vec{\Omega}') + Q$$

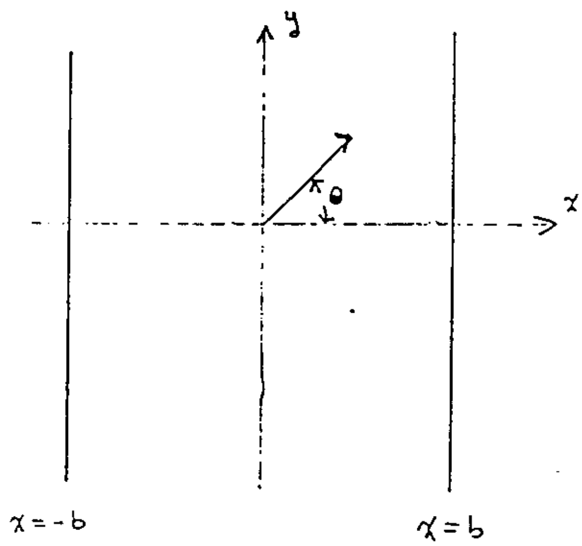
$\sigma_a = \sigma_t - \sigma_s$ probability particle will
- be absorbed
- be scattered with energy other than $E^{(k)}$.

- Particles lost become source for other energy levels
- Without fission particles generally lose energy
- Sequentially solve for $E^{(k)}, E^{(k-1)}, \ldots, E^{(1)}$

3-D Single Group

$$\vec{\Omega}\cdot\nabla\psi + \sigma_t\psi = \sigma_s\int d\vec{\Omega}'\,\Sigma_s(\underline{x},\vec{\Omega}'\to\Omega)\,\psi(x,\vec{\Omega}') + Q$$

Slab Geometry



$\mu = \cos(\theta)$

Anisotropic

$$\mu\frac{\partial\psi}{\partial x} + \sigma_t\psi = \sigma_s\frac{1}{2}\int_{-1}^{1}\Sigma_s(x,\mu'\to\mu)\,\psi(x,\mu')\,d\mu' + Q$$

Isotropic

$$\mu\frac{\partial\psi}{\partial x} + \sigma_t\psi = \sigma_s\frac{1}{2}\int_{-1}^{1}\psi(x,\mu')\,d\mu' + Q$$

Most Difficult

$\sigma_s = \sigma_t$ No Particles lost

$\sigma_t b >> 1$ Slab optically dense

$$\frac{\mu}{\sigma_t}\frac{\partial \psi}{\partial x} + \psi = \frac{1}{2}\int_{-1}^{1} \Sigma_s(x, \mu' \to \mu)\, \psi(x, \mu')\, d\mu' + \hat{Q}$$

$-b \le x \le b$ $\psi(-b, \mu) = g_1(\mu)$ $\mu > 0$

$\psi(b, \mu) = g_2(\mu)$ $\mu < 0$

Transformation: $x = bz$

$$\frac{\mu}{\sigma_t b}\frac{\partial \hat{\psi}}{\partial z} + \hat{\psi} = \frac{1}{2}\int_{-1}^{1} \Sigma_s(z, \mu' \to \mu)\, \hat{\psi}(z, \mu')\, d\mu' + \hat{Q}$$

$-1 \le z \le 1$ $\hat{\psi}(-1, \mu) = g_1(\mu)$ $\mu > 0$

$\hat{\psi}(1, \mu) = g_2(\mu)$ $\mu < 0$

Singularly Perturbed Problem

$$\hat{\psi} = \frac{1}{2}\int_{-1}^{1} \Sigma_s^1(z, \mu' \to \mu)\, \hat{\psi}(z, \mu')\, d\mu' + Q \qquad (*)$$

Infinite Number of Solutions

Suppose

$$\xi(z, \mu) = \frac{1}{2}\int_{-1}^{1} \Sigma_s^1(z, \mu')\, \xi(z, \mu')\, d\mu'$$

Then for any $f(z)$

$$\underbrace{\xi(z, \mu)\, f(z)} = \frac{1}{2}\int_{-1}^{1} \Sigma_s^1(z, \mu')\, \underbrace{\xi(z, \mu') f(z)}\, d\mu'$$

and

$$\hat{\psi} + \xi(z, \mu)\, f(z)$$

satisfies $(*)$.

- Lower frequency in space more nearly singular
- In many Applications the operator is nearly singular on all frequencies of interest

Motivation

- Many Important Applications depend upon efficient solution of Single Group Transport Model
- Single Group Problem is illconditioned in Optically Dense Media

Numerical Solution

- Problem Formulation
- Discrete Approximation
- Solution Algorithm
- Software Development

Discrete Approximation Angle

S_N Equations

$$\mu \frac{\partial \psi}{\partial x} + \sigma_t \psi = \sigma_s \frac{1}{2}\int_{-1}^{1} \Sigma_s (x, \mu' \to \mu)\, \psi(x,\mu')\, d\mu' + Q$$

Let

$$K \cdot = \frac{1}{2} \int_{-1}^{1} \Sigma_s (x, \mu' \to \mu) \cdot d\mu'$$

By Construction

EigenVectors of K are

Legendre Polynomials 1-D

Spherical Harmonics 2,3-D

S_N - Assume the variation in Angle of $\psi(x,\mu)$ can be expressed by the first N eigenvectors of K

1-D S_N Equations

Legendre Polynomials $\{P_\ell(\mu)\}_{\ell=0}^{N-1}$

Quadrature Points $\{\mu_1, \dots, \mu_N\}$

weights $\{\omega_1, \dots, \omega_N\}$

Let

$$\psi_m(x) = \psi(x, \mu_m)$$

Moments:

$$\phi_\ell(x) = \tfrac{1}{2}\int_{-1}^{1} \psi(x,\mu) P_\ell(\mu)\, d\mu = \sum_{m=1}^{N} \psi_m(x) P_\ell(\mu_m)\, \omega_m$$

By Assumption

$$\phi_\ell(x) = 0 \qquad \ell \geq N$$

Inverse Transformation

$$\psi_m(x) = \sum_{\ell=0}^{N-1} (2\ell+1)\, \phi_\ell(x)\, P_\ell(\mu_m)$$

1-D S_N Equations

$$\mu_m \frac{\partial}{\partial x}\psi_m(x) + \sigma_t \psi_m(x) = S_m(x) + Q_m \qquad m = 1, \ldots, N$$

$$S_m = \sum_{\ell=0}^{N-1} (2\ell+1)\, \sigma_\ell\, \phi_\ell(x)\, P_\ell(\mu_m)$$

$$\phi_\ell(x) = \sum_{m=1}^{N} \psi_m(x)\, P_\ell(\mu_m)\, \omega_m$$

Algebraiclly

$$\underline{\Psi}(x) = (\psi_1(x), \ldots, \psi_N(x))^T$$

$$D \frac{\partial \underline{\Psi}}{\partial x} + \sigma_t \underline{\Psi} = T^{-1} \Sigma T \underline{\Psi}(x) + \underline{Q}$$

$$D = \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_N \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} \sigma_0 & & \\ & \ddots & \\ & & \sigma_{N-1} \end{pmatrix}$$

$\sigma_t \geq \sigma_0$

($\sigma_t = \sigma_0$ Pure Scattering)

1-D S_N Equations Isotropic

$$\sigma_\ell = 0 \qquad \ell > 0$$

$$\sigma_0 = \sigma_t \qquad \text{pure scattering}$$

$$\Sigma = \begin{pmatrix} \sigma_0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

$$T^{-1} \Sigma \, T = \sigma_0 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} (\omega_1 \; \omega_2 \; \cdots \; \omega_N)$$

$$= \sigma_0 \, \underline{1} \, \underline{\omega}^T = R$$

$$D \frac{\partial}{\partial x} \underline{\psi} + \sigma_t \underline{\psi} = \sigma_t R \, \underline{\psi} + \underline{Q}$$

Solution Algorithm

- In Optically Dense Medium, Transport is Diffusive in Nature

- Diffusion Synthetic Acceleration (DSA)
  - Preconditions by Inverse of Diffusion Operator
  - Very successful for Isotropic Scattering

- Why Multigrid?
  - DSA Fails for Anisotropic Scattering and Inhomogeneous Material
  - Multigrid Excellent Technique for Diffusion-like Problems

Isotropic Scattering

- Multigrid in space

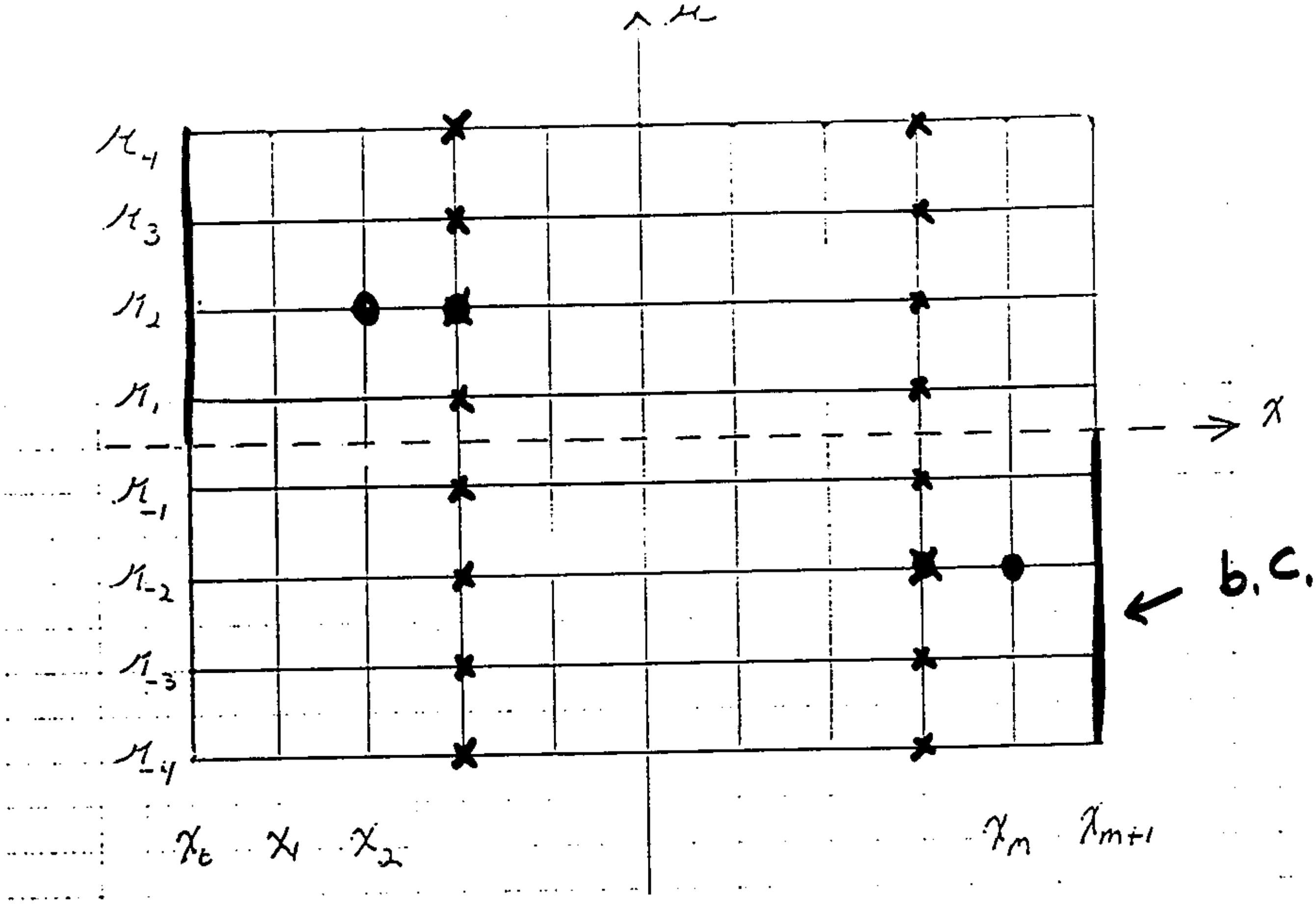
Anisotropic Scattering

- Multigrid in Angle
- S_2 Isotropic Problem on Coarse Grid

Multi Grid Depends upon

- Effective Smoothing
- Effecient Grid Transfer
- Inexpensive Solution on Coarsest Grid

Upwind Difference



$$\psi_{0,j} = g_{1j} \qquad (j = 1, \cdots, N/2)$$

$$\varepsilon \mu_j \frac{\psi_{ij} - \psi_{i-1,j}}{h} + \psi_{ij} = \sum_{k=-4}^{4} w_k \psi_{ik} + q_{ij}$$

$$\varepsilon \mu_j \frac{\psi_{i+1,j} - \psi_{ij}}{h} + \psi_{ij} = \sum_{k=-4}^{4} w_k \psi_{ik} + q_{ij}$$

$$\psi_{m+1,j} = g_{2j} \qquad (j = -1, \cdots, -N/2)$$

Write

$$\frac{\varepsilon}{h} D\left(\underline{\psi}_{i,+} - \underline{\psi}_{i-1,+} \right) + \underline{\psi}_{i,+} = R(\psi_{i,+} + \psi_{i,-}) + \underline{q}_{i,+}$$

$$\frac{\varepsilon}{h} D\left(\psi_{i,-} - \psi_{i+1,-} \right) + \psi_{i,-} = R(\psi_{i,+} + \psi_{i,-}) + \underline{q}_{i,-}$$

where

$$\underline{\psi}_{i,+} = \begin{pmatrix} \psi_{i,1} \\ \vdots \\ \psi_{i,\frac{N}{2}} \end{pmatrix} \qquad \underline{\psi}_{i,-} = \begin{pmatrix} \psi_{i,-1} \\ \vdots \\ \psi_{i,-\frac{N}{2}} \end{pmatrix}$$

$$D = \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_{\frac{N}{2}} \end{pmatrix}$$

$$R = \underline{1}\,\underline{w}^T \qquad \begin{aligned} \underline{1} &= (1, 1, \cdots, 1)^T \\ w &= (w_1, w_2 \cdots, w_{\frac{N}{2}})^T \\ &\left(\underline{1}^T \cdot \underline{w} = \tfrac{1}{2} \right) \end{aligned}$$

x-line Relaxation (Transport Sweep)

$$\frac{\varepsilon}{h} D\left(\underline{\psi}_{i,+}^{\ell+1} - \underline{\psi}_{i-1,+}^{\ell+1} \right) + \underline{\psi}_{i,+}^{\ell+1} = R\left(\underline{\psi}_{i,+}^{\ell} + \underline{\psi}_{i,-}^{\ell} \right) + \underline{b}_{i,+}$$

$$\frac{\varepsilon}{h} D\left(\underline{\psi}_{i,-}^{\ell+1} - \underline{\psi}_{i+1,-}^{\ell+1} \right) + \underline{\psi}_{i,-}^{\ell+1} = R\left(\underline{\psi}_{i,+}^{\ell} + \underline{\psi}_{i,-}^{\ell} \right) + \underline{b}_{i,-}$$

For $\frac{\varepsilon}{h} \gg 1$ Effective Smoother

For $\frac{\varepsilon}{h} < 1$ Ineffective Smoother

For Some Problems

$h = 10^{-3}$

$\varepsilon = 10^{-6}$

$\frac{\varepsilon}{h} = 10^{-3}$

On Finest Grid!

x-line Relaxation

$$\frac{\varepsilon}{h} D \left(\underline{\psi}_{i,+}^{l+1} - \underline{\psi}_{i-1,+}^{l} \right) + \underline{\psi}_{i,+}^{l+1} = R \left(\underline{\psi}_{i,+}^{l+1} + \underline{\psi}_{i,-}^{l+1} \right) + \underline{g}_{i,+}$$

$$\frac{\varepsilon}{h} D \left(\underline{\psi}_{i,-}^{l+1} - \underline{\psi}_{i+1,-}^{l} \right) + \psi_{i,-}^{l+1} = R \left(\underline{\psi}_{i,+}^{l+1} + \underline{\psi}_{i,-}^{l+1} \right) + \underline{g}_{i,-}$$

Rewrite as

$$\begin{bmatrix} I + \frac{\varepsilon}{h} D - R & -R \\ -R & I + \frac{\varepsilon}{h} D - R \end{bmatrix} \begin{pmatrix} \psi_{i,+}^{l+1} \\ \psi_{i,-}^{l+1} \end{pmatrix} = \begin{pmatrix} \frac{\varepsilon}{h} D \underline{\psi}_{i-1,+}^{l} + \underline{g}_{i+} \\ \frac{\varepsilon}{h} D \underline{\psi}_{i+1,-}^{l} + \underline{g}_{i-} \end{pmatrix}$$

- Blocks Easily Invertible

  Diagonal plus Rank one

Sherman Morrison Formula

$$(I - \underline{v}\,\underline{w}^T)^{-1} = (I + \xi\,\underline{v}\,\underline{w}^T)$$

$$\xi = \frac{1}{1 - \underline{w}^T \underline{v}}$$

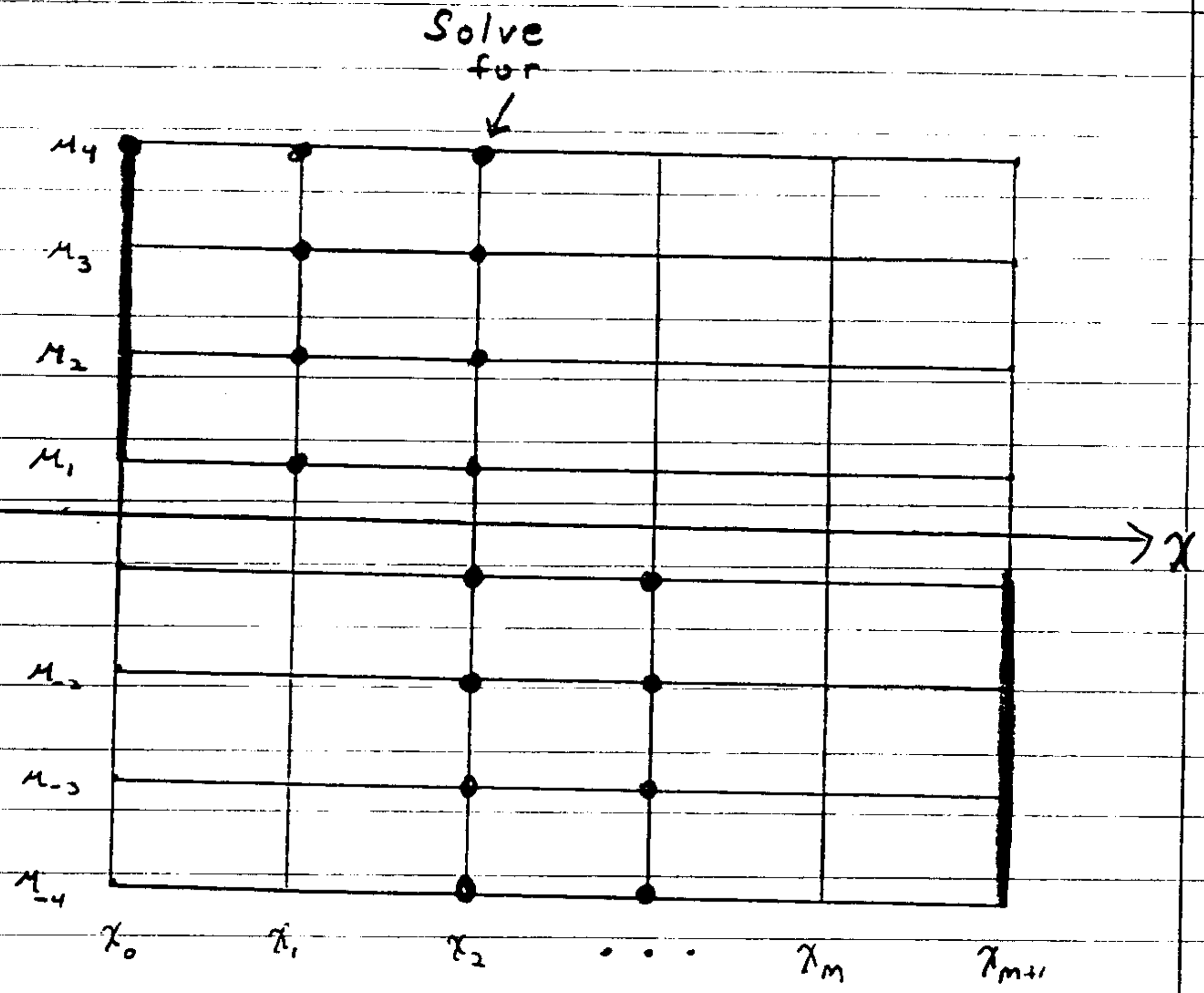
Multiply by: $(I - \underline{v}\,\underline{w}^T)(I + \xi\,\underline{v}\,\underline{w}^T) = I$

Solve for ξ.

General Formula

$$(A_0 - \underline{v}\,\underline{w}^T)^{-1} = A_0^{-1} + \xi A_0^{-1}\underline{v}\,\underline{w}^T A_0^{-1}$$

$$\xi = \frac{1}{1 - \underline{w}^T A_0^{-1} \underline{v}}$$



μ-line Relaxation

Solve for • $\psi(x_2, \mu_j)$

Using • $\psi(x_3, \mu_j)$ $\mu_j < 0$

$\psi(x_1, \mu_j)$ $\mu_j > 0$

Multigrid Algorithm

Linear Interpolation
Injection

$\Big\}$ upwind Diff. on Coarse Grid

Red/Black x-line Relaxation

Convergence Factor

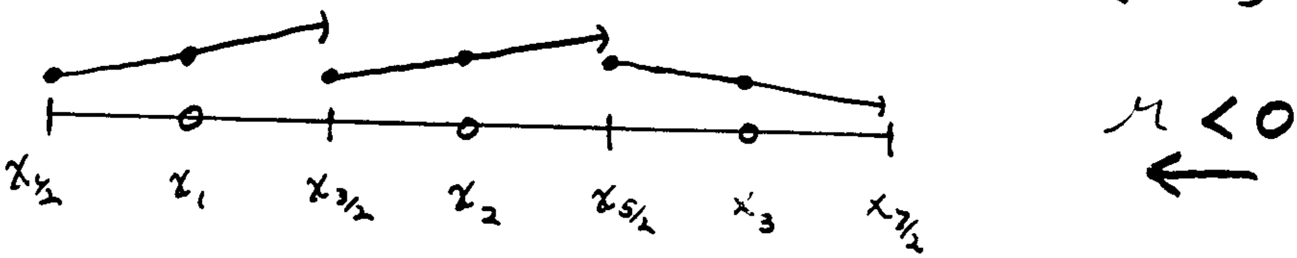
2-Grid $\quad r = O\left(\frac{\varepsilon}{h}\right)$ (proof)

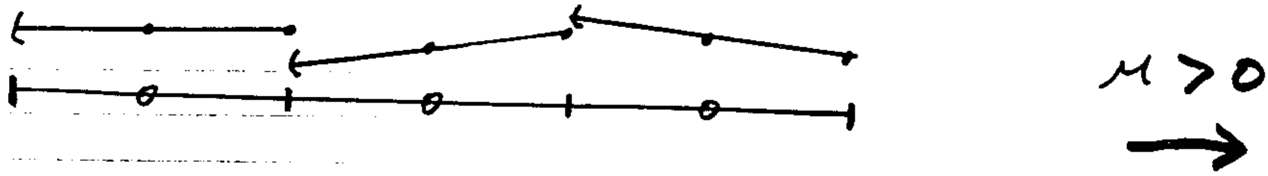
Multigrid (Numerical proof)

V(1,0) cycle

$\frac{\varepsilon}{h}$	r
.1	.136
.01	.0136
.001	.00140

Modified Linear Discontinuous Difference (MLD)

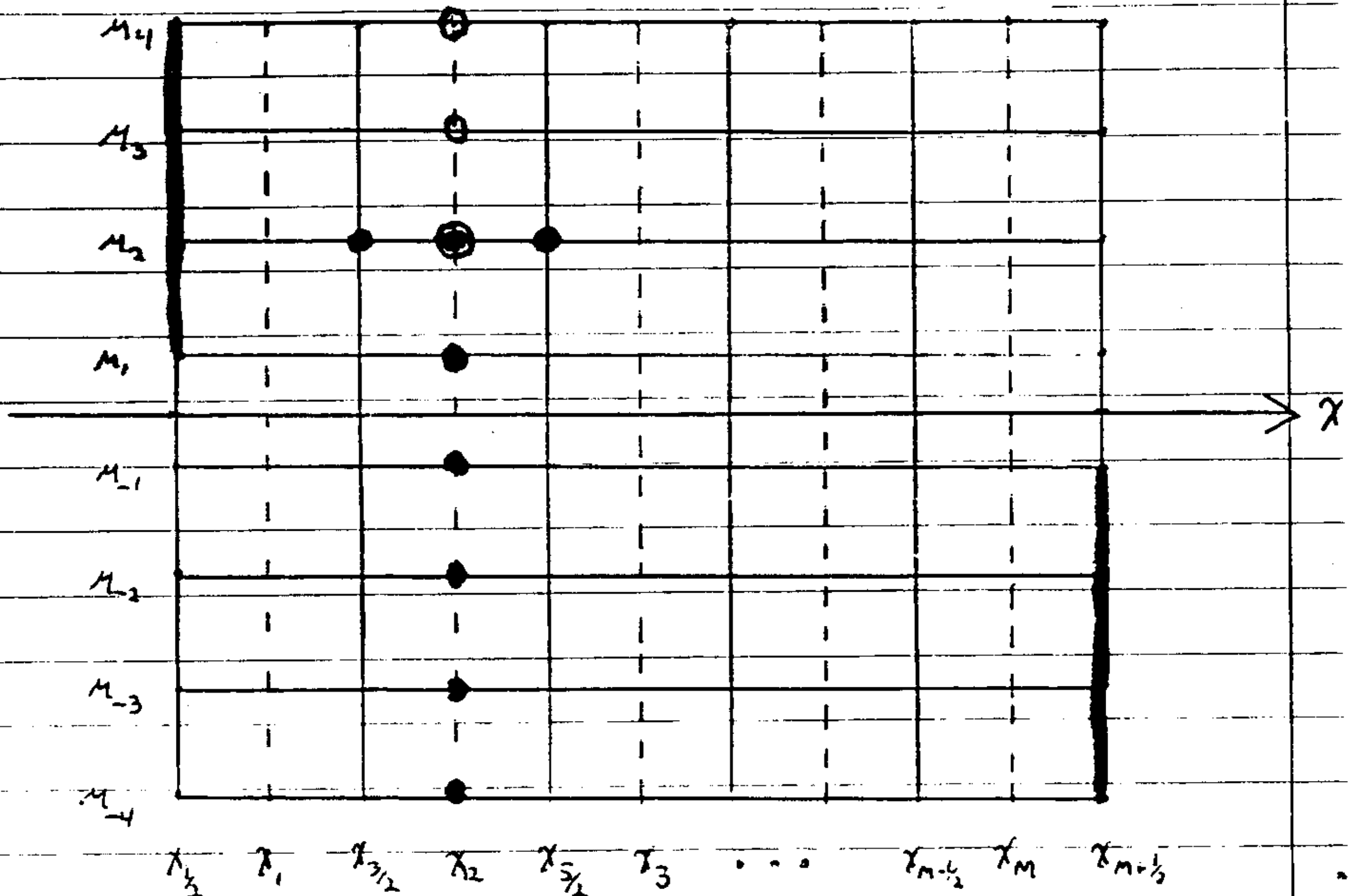




$$
\begin{array}{cccccc|l}
\ddots & & & & & & \psi^+_{1/2} \\
 & I+\frac{2\varepsilon}{h}D - R & -2R & -\frac{2\varepsilon}{h}D & R & & \psi^-_{1/2} \\
-\frac{\varepsilon}{h}D & & I-R & -R & & & \psi^+_{1} \\
 & & -R & I-R & & -\frac{\varepsilon}{h}D & \psi^-_{1} \\
 & R & -\frac{2\varepsilon}{h}D & -2R & I+\frac{2\varepsilon}{h}D - R & & \psi^+_{3/2} \\
 & & & & & \ddots & \psi^-_{3/2}
\end{array}
$$

$$\underline{\underline{A}}\, \underline{\Psi} = \underline{Q}$$

Cell Center Equations



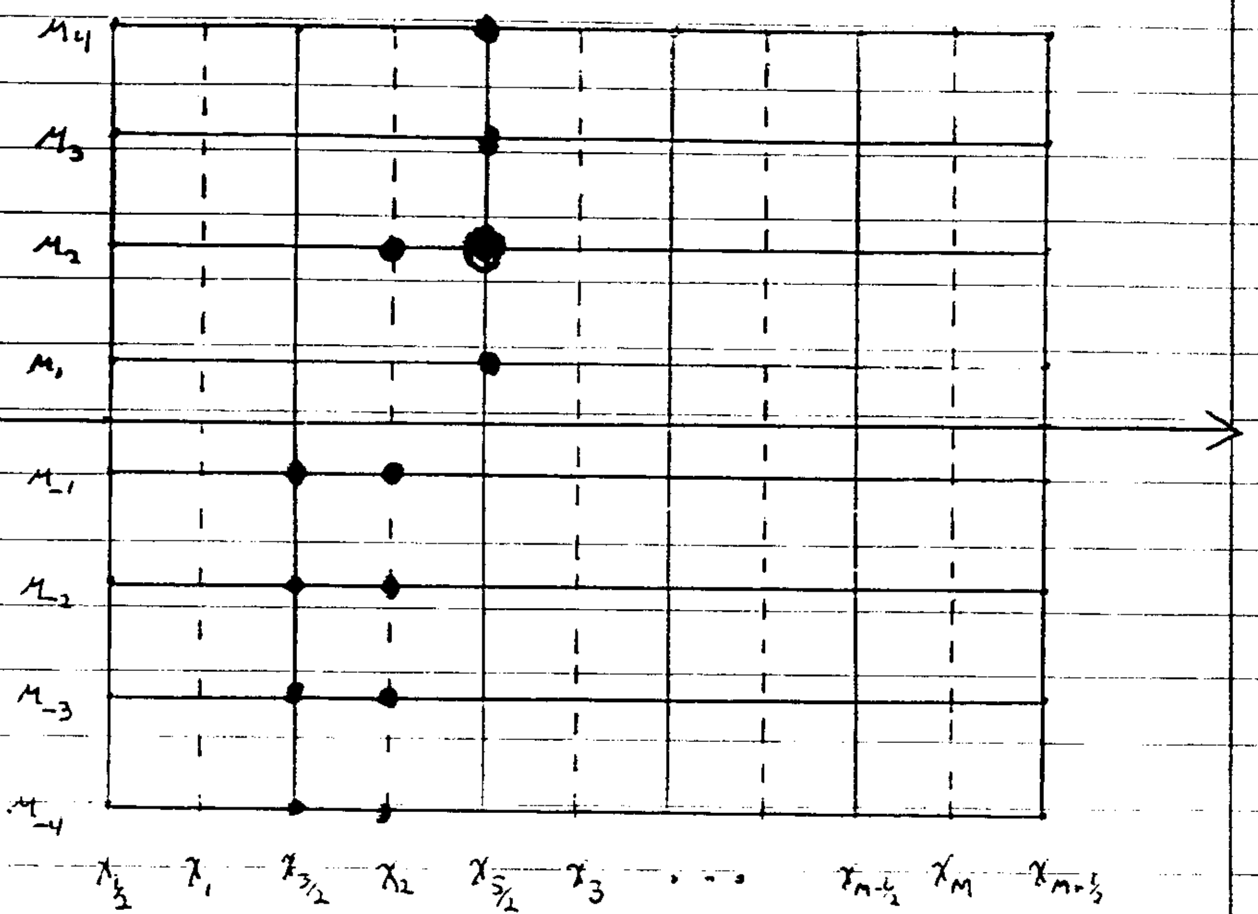
$\boxed{\mu_j > 0}$

$$|\mu_j|\varepsilon \frac{\psi_{i+1/2,j} - \psi_{i-1/2,j}}{h} + \psi_{i,j} = \sum_k w_k \psi_{ik} + q_{ij}$$

$\boxed{\mu_j < 0}$

$$-|\mu_j|\varepsilon \frac{\psi_{i+1/2,j} - \psi_{i-1/2,j}}{h} + \psi_{ij} = \sum_k w_k \psi_{ik} + q_{ij}$$

Cell Edge Equations



$\boxed{\mu_j > 0}$

$$\mu_j \,\varepsilon\, \frac{\psi_{i+1/2,j} - \psi_{i,j}}{h/2} + \psi_{i+1/2,j} = \sum_{k>0} \omega_k \psi_{i+1/2,k}$$

$$+ \sum_{k<0} \omega_k \left(2\psi_{i,k} - \psi_{i-1/2,k} \right) + q_{i,j}$$

Why Modified Linear Discontinuous

More Accurate

Upwind $O(h)$

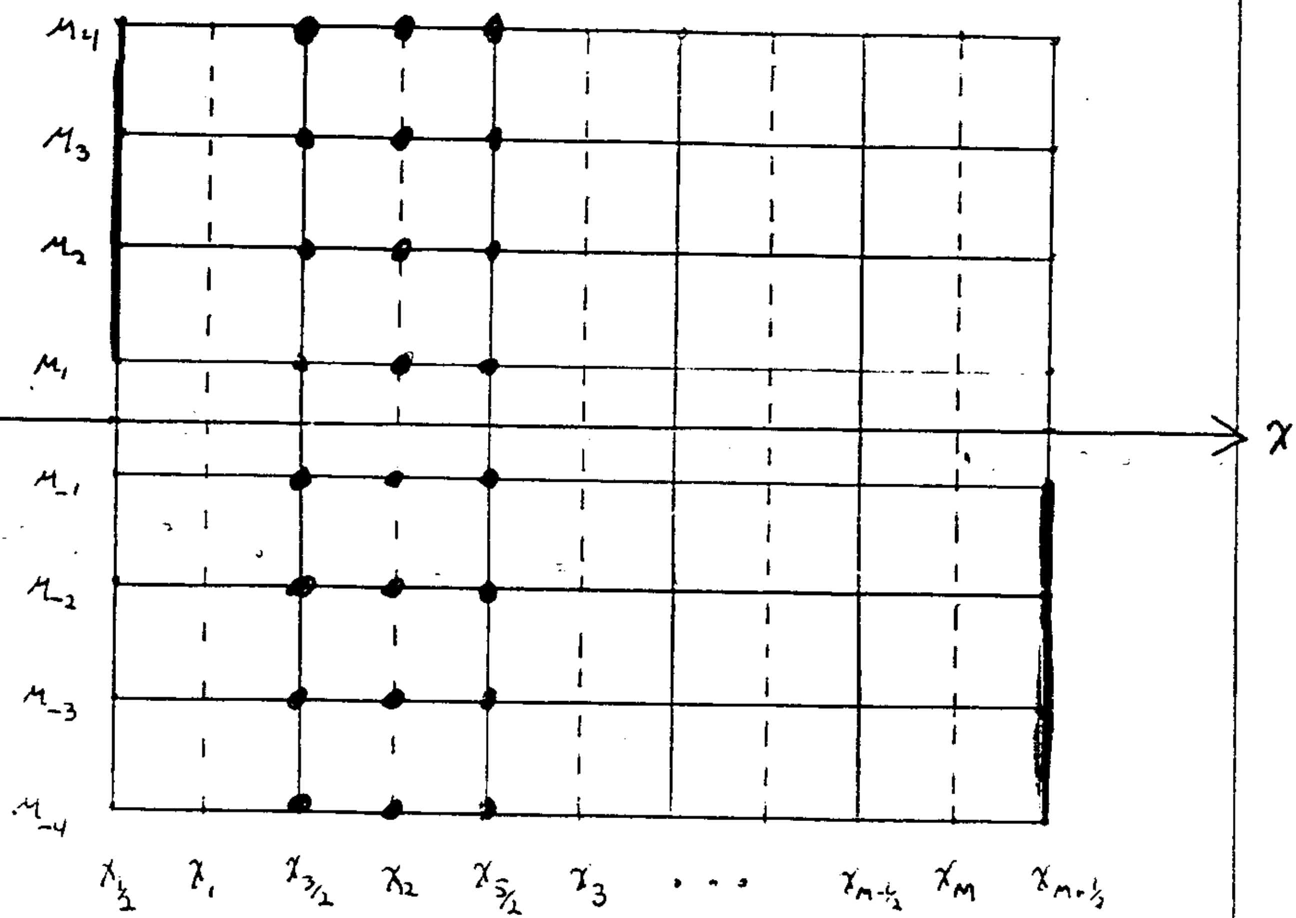
MLD $O(h^2)$

Diffusion Limit

MLD reproduces diffusion as $\frac{\varepsilon}{h} \to 0$

Upwind does not

Block Equations



Given values at ● $\left(\begin{array}{ll} \psi(x_{3/2}, \mu_j) & \mu_j > 0 \\ \psi(x_{5/2}, \mu_j) & \mu_j < 0 \end{array} \right)$

Solve simultaneously for values at ●

Modified Linear Discontinuous

Red/Black μ-line Relaxation

$A_0 + R_0$ $\qquad$ R_0 rank 2

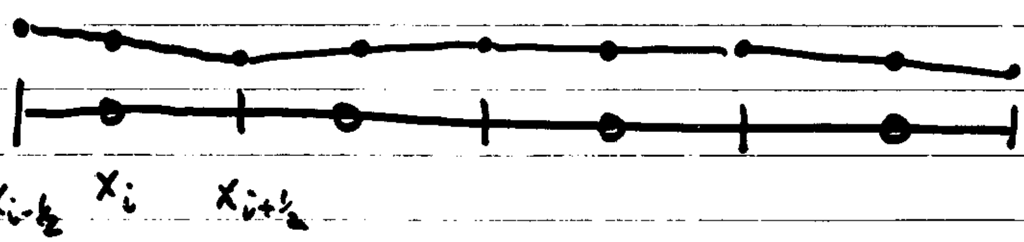
A_0^{-1} Easy

$$(A_0+R_0)^{-1} = \left(A_0^{-1} + \xi\, A_0^{-1} R_0 A_0^{-1}\right)$$

- Effective smoother for all values of $\frac{\varepsilon}{h}$
- For $\frac{\varepsilon}{h} \ll 1$ error becomes

Piecewise linear across cells

Independent of angle (μ)



$$e_i = \frac{e_{i-\frac{1}{2}} + e_{i+\frac{1}{2}}}{2} + O\left(\frac{\varepsilon}{h}\right)$$

Coarse Grid Operator

For $\frac{\varepsilon}{h} > 1$ use MLD

For $\frac{\varepsilon}{h} < 1$ use Upwind

Grid		Operator	
h	$\frac{\varepsilon}{h}$	MLD	
$2h$	$\frac{\varepsilon}{2h}$	MLD	
$4h$	$\frac{\varepsilon}{4h}$	MLD	
$\vdots$			
$2^{p-1}h$	$\frac{\varepsilon}{2^{p-1}h}$	MLD	
		$\downarrow$	<u>Switch</u>
$2^{p}h$	$\frac{\varepsilon}{2^{p}h} < .1$	UW	
		$\downarrow$	
		UW	

- If $\frac{\varepsilon}{h} < .1$ on Finest Grid then Switch to Upwind on first Coarse Grid

Convergence Rates

V(1,0) Cycle

$\frac{\varepsilon}{h}$ (Finest Grid)	r
40	.95
10	.97
2.6	.110
1.3	.116
.66	.114
.32	.097
.16	.086
.04	.045
.01	.069
.001	.077
.0001	.078
.00002	.080

Anisotropic Scattering

Multi Grid in Space (Jim Morel)

- S_N Equations
- LD Differencing
- 2 Block μ-line Relaxation
- Element interpolation / restriction

V-cycle rate .4 - .6

- independent of ε/h
- large material discontinuities

Problem

Expensive Inversion of Full Blocks

S_{16} $N = 16$

Block 64 90,000 operations

MultiGrid IN Angle

S_N Equations

$$D \frac{\partial}{\partial x} \underline{\psi} + \sigma_t \underline{\psi} = T^{-1} \Sigma T \underline{\psi} + \underline{q}$$

$$D = \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_N \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \sigma_0 & & \\ & \ddots & \\ & & \sigma_{N-1} \end{pmatrix} \qquad \sigma_t \geq \sigma_0$$

Shifted x-line Sweep (Transport)

$$D \frac{\partial}{\partial x} \underline{\psi}^{k+1} + (\sigma_t - \alpha) \underline{\psi}^{k+1} = T^{-1} (\Sigma - \alpha I) T \, \underline{\psi}^{k} + \underline{q}$$

- Solve Transport Equation
- Stable for $(\sigma_t - \alpha) > 0$

Residual

$$r^{k+1} = T^{-1} (\Sigma - \alpha I) T (\underline{\psi}^{k+1} - \underline{\psi}^{k})$$

Multi Grid in Angle

Two Representations

Flux Representation

Moment Representation

$$\underline{\phi} = T \underline{\psi} \qquad M = T D T^{-1}$$

$$M \frac{\partial}{\partial x} \underline{\phi}^{k+1} + (\sigma_t - \alpha)\, \phi^{k+1} = (\Sigma - \alpha I)\, \phi^{k} + \tilde{q}$$

$$M = \begin{pmatrix} 0 & 1 & & & & \\ \frac{1}{3} & 0 & \frac{2}{3} & & & \\ & & \ddots & & & \\ & & \frac{\ell}{2\ell+1} & 0 & \frac{\ell+1}{2\ell+1} & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

$$\Sigma - \alpha I = \begin{pmatrix} (\sigma_0 - \alpha) & & & & \\ & \ddots & & & \\ & & (\sigma_{\frac{N-1}{2}} - \alpha) & & \\ & & & (\sigma_{\frac{N+1}{2}} - \alpha) & \\ & & & & \ddots \\ & & & & & (\sigma_{N-1} - \alpha) \end{pmatrix}$$

Multi Grid in Angle

Moment Representation

$$\underset{\sim}{\tilde{r}}^{k+1} = (\Sigma' - \alpha I)(\underset{\sim}{\phi}^{k+1} - \underset{\sim}{\phi}^{k})$$

Choose

$$\alpha = \frac{\sigma_{\frac{N+1}{2}} + \sigma_{N-1}}{2}$$

Smoothing Rate ($\sigma_T = \sigma_0$)

$$\rho = \frac{\sigma_{\frac{N+1}{2}} - \sigma_{N-1}}{2\sigma_0 - (\sigma_{\frac{N+1}{2}} + \sigma_{N-1})}$$

On moments $\frac{N+1}{2} \dots N-1$

Multi Grid IN Angle

Coarse Grid Equations (Moment)

Interpolation $\quad P = \begin{bmatrix} I_{N/2} \\ 0 \end{bmatrix}$

Restriction $\quad Q = \begin{bmatrix} I_{N/2}, & 0 \end{bmatrix}$

$$\underbrace{(QMP)}_{\tilde{M}} \frac{\partial}{\partial x}\phi + \sigma_t I_{N/2}\phi = \underbrace{Q\Sigma P}_{\tilde{\Sigma}}\phi + P\underline{\tilde{r}}$$

Moment equations associated with a $S_{N/2}$ problem with $P\underline{\tilde{r}}$ as a source term

Multi Grid In Angle

Coarse Grid Operator Flux Rep.

Quadrature points $\{\tilde{\mu}_1, \ldots, \tilde{\mu}_{N/2}\}$
weights $\{\tilde{\omega}_1, \ldots, \tilde{\omega}_{N/2}\}$

$$\tilde{\psi}_m(x) = \sum_{\ell=0}^{N/2-1} (2\ell+1)\,\phi_\ell(x)\,P_\ell(\tilde{\mu}_m)$$

$$\phi_\ell(x) = \sum_{m=1}^{N/2} \tilde{\psi}_m(x)\,P_\ell(\tilde{\mu}_m)\,\tilde{\omega}_m$$

$$\tilde{D}\,\frac{\partial \tilde{\underline{\psi}}}{\partial x} + \sigma_t\,\underline{\tilde{\psi}} = \tilde{T}^{-1}\tilde{\Sigma}\,\tilde{T}\,\underline{\tilde{\psi}} + \tilde{T}^{-1}P\,\underline{\tilde{r}}$$

No simple relation between

ψ and $\hat{\psi}$

Multi Grid in Angle

$$S_N \rightarrow S_{N/2} \rightarrow \quad \rightarrow S_2$$

S_2 Moment Representation

$$\begin{pmatrix} 0 & 1 \\ 1/3 & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial x}\phi_0 \\ \frac{\partial}{\partial x}\phi_1 \end{pmatrix} + (\sigma_t - \alpha)\begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} = \begin{pmatrix} \sigma_0 - \alpha & 0 \\ 0 & \sigma_1 - \alpha \end{pmatrix}\begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} + \begin{pmatrix} \hat{r}_0 \\ \hat{r}_1 \end{pmatrix}$$

Choose $\alpha = \sigma_1$

Isotropic S_2 Equation

Multi Grid in Angle

Fokker Plonk Scattering

Smoothing Rate $\leq .6$

Coarsest Grid Isotropic S_2