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# Multigrid Methods for Incompressible Navier-Stokes Equations

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# Multigrid Methods for incompressible Navier-Stokes equations

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- The equations
- aspects of discretization
  - checkerboard instability
  - convection - diffusion problem
  - staggered - nonstaggered grids
- relaxation schemes: distributive Gauß-Seidel
  box - relaxation
  collective relaxations
- finite volume approach and flux splitting on non-staggered grids

# Incompressible Navier-Stokes, primitive variables — 2D —

$$\text{(i)} \quad u_t - \frac{1}{Re}\Delta u + u u_x + v u_y + p_x = 0$$

$$\text{(ii)} \quad v_t - \frac{1}{Re}\Delta u + u v_x + v v_y + p_y = 0$$

$$\text{(iii)} \quad u_x + v_y = 0$$

- $Re$ : Reynoldsnumber
- $u, v$ : cartesian velocity components
  $p$ : pressure

(i), (ii) : momentum equations

(iii) : Continuity equations

- **In conservative Form:**

$$u_t - \frac{1}{Re}\Delta u + (u^2)_x + (uv)_y + p_x = 0$$

$$v_t - \frac{1}{Re}\Delta v + (uv)_x + (v^2)_y + p_y = 0$$

$$u_x + v_y = 0$$

Stokes - system: highly viscous flow

$$-\Delta u + p_x = 0$$

$$-\Delta v + p_y = 0$$

$$u_x + v_y = 0$$

- Dirichlet boundary condition (i.e. $u, v$ given)

  $\Longrightarrow$ well posed b.v.p.

  (on bounded domains, boundary sufficiently smooth)

- $p$ determined only up to a constant

- more complicated b.c. needed for the various flow situations; e.g. outflow conditions (→ literature, still matter of research)

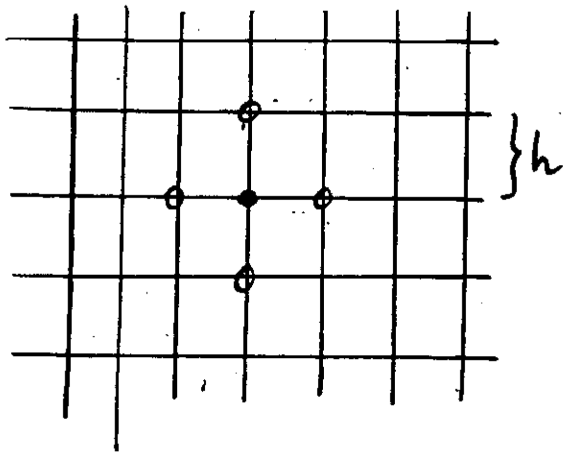
# Discretization

## A first attempt:

$\Omega = [0,1]^2$

Stokes equations

Square cartesian grid, $h$




$\boxed{\text{Central differencing for all derivatives}}$

$$-\Delta^h = \frac{1}{h^2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix} \qquad \text{5 point Laplacian}$$

$$\Longrightarrow \quad L^h \underline{u}^h = \begin{cases} -\Delta^h u^h + \frac{1}{2h}\begin{bmatrix} -1 & 0 & 1 \end{bmatrix} p^h & = 0 \\ -\Delta^h v^h + \frac{1}{2h}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} p^h & = 0 \\ \frac{1}{2h}\begin{bmatrix} -1 & 0 & 1 \end{bmatrix} u^h + \frac{1}{2h}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} v^h & = 0 \end{cases}$$

- discrete pressure is determined only up to a constant : OK
  (could lead to compatibility problems)
- pressure values at boundary gridpoints?
  E.g. extrapolation from interior nodes

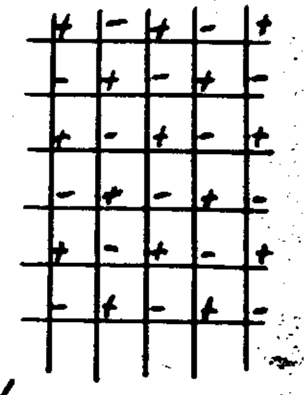
## Observation:

on an infinite grid, the system $L^h \underline{u}^h = 0$ has highly oscillating, nontrivial solutions

E.g. set $u^h = 0$, $v^h = 0$;

$p^h$ = "checkerboard mode"

i.e.: $p^h_{ij} = (-1)^{i+j}$



## Or, more generally:

Compute the (Fourier-) symbol of operator $L^h$:

let $\underline{u}^h = A\, e^{i\vartheta_1 x/h} \cdot e^{i\vartheta_2 y/h}$, $A = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$ Amplit

then: $L^h \underline{u}^h = \tilde{L}^h(\vartheta_1, \vartheta_2) \cdot \underline{u}^h$

with $$\boxed{\tilde{L}^h = \frac{1}{h^2} \begin{pmatrix} 4 - 2\cos\vartheta_1 - 2\cos\vartheta_2 & 0 & hi\sin\vartheta_1 \\ 0 & 4 - 2\cos\vartheta_1 - 2\cos\vartheta_2 & hi\sin\vartheta_2 \\ hi\sin\vartheta_1 & hi\sin\vartheta_2 & 0 \end{pmatrix}}$$

"Symbol" of $L^h$

$$\Longrightarrow \quad \det \tilde{L}^h(\vartheta_1, \vartheta_2) = 0$$

$$\Longleftrightarrow \begin{cases} \vartheta_1 = \vartheta_2 = 0 & \text{, constant component} \\ \boxed{\begin{array}{l} \vartheta_1 = \pi, \ \vartheta_2 = \pi \\ \vartheta_1 = 0, \ \vartheta_2 = \pi \\ \vartheta_1 = \pi, \ \vartheta_2 = \cancel{\pi}\,0 \end{array}} & \text{checkerboard mode} \end{cases}$$

high frequencies !

There are high frequencies which are annihilated by the discrete operator.

$\longrightarrow$ Non-Elliptic

Unstable : checkerboard instability

For $L^h \underline{u}^h = F^h$ consider a relaxation of the

$$A^h u^h + B^h \bar{u}^h = F^h$$

with $u^h$: initial approximation

$\bar{u}^h$: approximation after relaxation sweep

$A^h, B^h$: splitting of $L^h$ : $L^h = A^h + B^h$

(see: talk on smoothing analysis)

<u>Then</u>: Smoothing rate:

$$\mu = \max \left\{ |\lambda(\vartheta)| : \det\left( \lambda(\vartheta) \tilde{A}^h(\vartheta) + \tilde{B}^h(\vartheta) \right) = 0 \right\}$$

with "max" taken over all high frequencies.

- If there is a $\overbrace{\vartheta = (\vartheta_1, \vartheta_2)}^{\text{high frequency}}$ with $\det \tilde{L}^h(\vartheta) = 0$

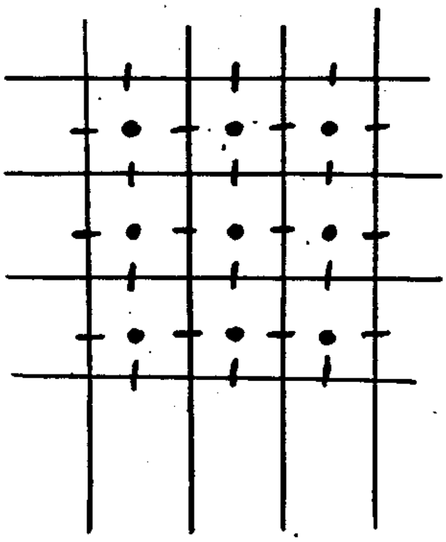
$\Longrightarrow \mu \geq 1$ (for all splittings $A^h, B^h$)

<u>Note</u>: $\tilde{L}^h(\vartheta) = \tilde{A}^h(\vartheta) + \tilde{B}^h(\vartheta)$

i.e. The smoothing properties of any relaxation will be bad in that case!

## Staggered Grids: for solving the checkerboard problem

- suggest by Harlow/Welsh
- used in multigrid by Brandt/Dinar ...



idea: put the discrete unknowns at different locations.

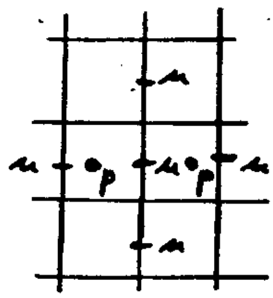
u-velocities: midpoints of vertical cell
u-momentum faces —

v-velocities: midpoints of horizontal cell
v-momentum faces |

pressure: cell centers •
Continuity

u-momentum (at vertical cell faces):

$$-\Delta^h u^h + \frac{1}{h}[-1\ 1]\, p^h = 0$$



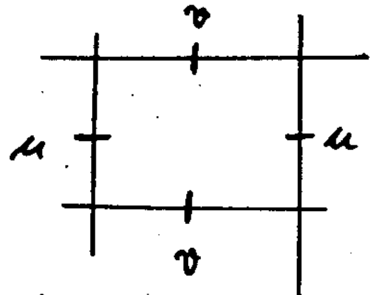

v-momentum (at horizontal cell faces):

$$-\Delta^h v^h + \frac{1}{h}\begin{bmatrix}1\\-1\end{bmatrix} p^h = 0$$

similar

Continuity (at cell centers):

$$\frac{1}{h}[-1\ 1]\, u^h + \frac{1}{h}\begin{bmatrix}1\\-1\end{bmatrix} v^h = 0$$




- Short differences for all first derivatives!

- The staggered grid discretization is <u>stable</u>!

Symbol of the staggered grid system $L^h$:

$$\tilde{L}^h(\vartheta_1,\vartheta_2) = \frac{1}{h^2}\begin{pmatrix} 4-2\cos\vartheta_1-2\cos\vartheta_2 & 0 & 2hi\sin\frac{\vartheta_1}{2} \\ 0 & 4-2\cos\vartheta_1-2\cos\vartheta_2 & 2hi\sin\frac{\vartheta_2}{2} \\ 2hi\sin\frac{\vartheta_1}{2} & 2hi\sin\frac{\vartheta_2}{2} & 0 \end{pmatrix}$$

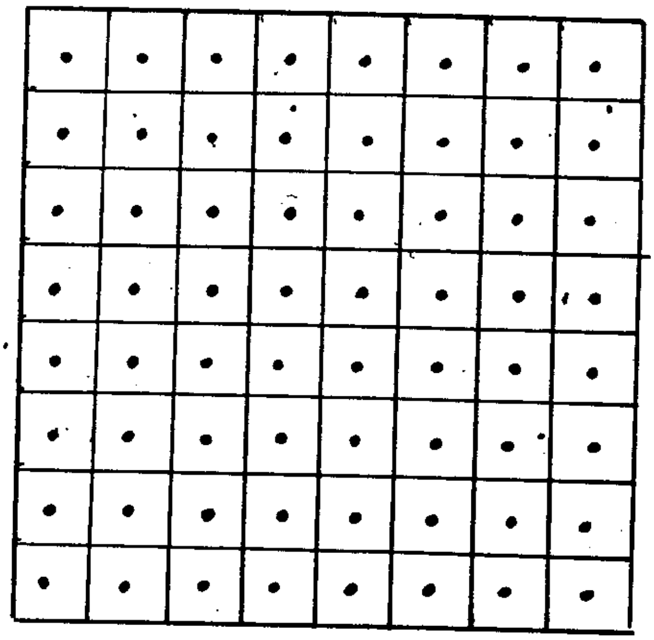
$$\Longrightarrow \det\tilde{L}^h(\vartheta_1,\vartheta_2) = -\frac{1}{h^4}4\left(4-2\cos\vartheta_1-2\cos\vartheta_2\right)\left(\sin^2\frac{\vartheta_1}{2}+\sin^2\frac{\vartheta_2}{2}\right)$$

$$\Longrightarrow \det\tilde{L}^h(\vartheta_1,\vartheta_2) = 0 \iff \underbrace{\vartheta_1=\vartheta_2=0}_{\text{Constant component}}$$

$\leadsto$ No checkerboard instability

$\Longrightarrow$ System is elliptic!

Staggered grids: Remarks



- "Short" differences for all 1st derivatives +
- No pressure values needed at boundaries. +
- On grid 2h the locations of unknown are shifted. ⇒ a technical problem +−
  not a principal one
- On general curvilinear meshes the staggered grid approach gets much more complicated. (co- or contra-variant formulation needed).

## The convection-diffusion problem:

Consider u-momentum:

$$\underbrace{-\frac{1}{Re}\Delta u}_{\text{diffusion}} + \underbrace{u u_x + v u_y}_{\text{Convection}} + p_x$$

- Re large → convection dominates

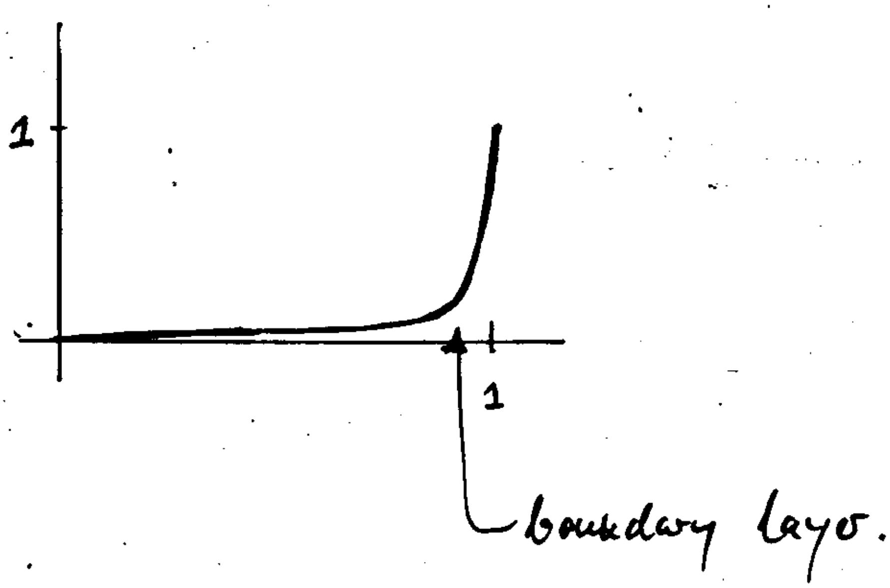
## Model problem: (linearized)

$$-\varepsilon \Delta u + a u_x + b u_y = 0, \quad 1 >> \varepsilon > 0$$

Or simply in 1D: $\quad -\varepsilon u'' + u' = 0$

$$\Omega = [0,1], \quad u(0) = 0, \quad u(1) = 1$$

1D continuous solution: $\quad u(x) = \dfrac{1 - e^{x/\varepsilon}}{1 - e^{1/\varepsilon}}$



1D discrete solution: central differencing

grid: $G^h = \{kh : k = 0, \dots, n\}$ $h = 1/n$

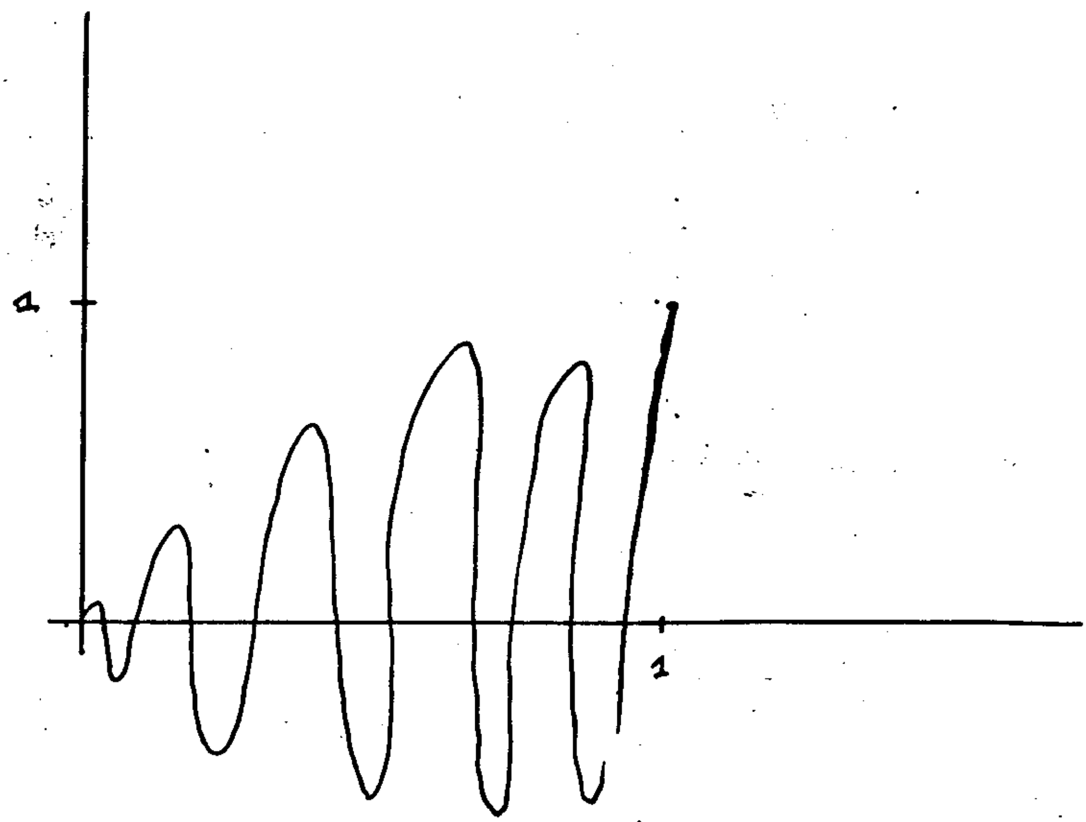
$$\Rightarrow \quad -\frac{\varepsilon}{h^2}\begin{bmatrix} 1 & -2 & 1 \end{bmatrix}_k u^h + \frac{1}{2h}\begin{bmatrix} -1 & 0 & 1 \end{bmatrix}_k u^h = 0$$

for $k = 1, \dots, n-1$

$u^h_0 = 0$, $u^h_n = 1$. boundary values

Then: if $\varepsilon < h/2$

$$u^h_k = \frac{1-q^k}{1-q^n} \quad \text{with } q = \frac{2\varepsilon + h}{2\varepsilon - h} < -1$$



- central differencing not suited for convection dominated flows.

<u>Consider Gauss-Seidel Smoothing</u>

$$-\varepsilon u'' + u' = 0$$

central differences $\Rightarrow$ $L^h u^h = \frac{1}{h^2}\left[-\varepsilon - \frac{h}{2} \quad 2\varepsilon \quad -\varepsilon + \frac{h}{2}\right] u^h = 0$

- not diagonally dominant if $\varepsilon < h/2$

<u>Smoothing Analysis:</u>

$$\mu(\vartheta) = \left| \frac{(\varepsilon - h/2)\, e^{i\vartheta}}{2\varepsilon - (\varepsilon + h/2)\, e^{-i\vartheta}} \right|$$ amplification factor of frequency $\vartheta$

e.g. $\vartheta = \pi$ : $\mu(\pi) = \left| \frac{\varepsilon - h/2}{3\varepsilon + h/2} \right| \xrightarrow[\varepsilon \to 0]{} 1$ ↯

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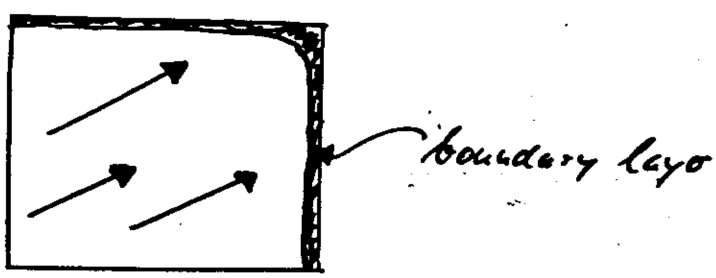
Consider 2D case: $-\varepsilon \Delta u + a u_x + b u_y = 0$ + (D)-boundary conditions




$\varepsilon > 0$: convection-direction given by vector $\binom{a}{b}$.
this is: "downwind" direction.
Upwind: opposite

$\varepsilon < 0$: Vice Versa

# Upwind differencing, hybrid schemes

1D case $\quad -\varepsilon u'' + a u' = \dots \qquad \varepsilon > 0$

- $a > 0$ : convection from "left to right"
  $a < 0$ : convection from "right to left"

Discretization: (on a uniform 1D grid $h$)

- diffusive part : $\quad -\varepsilon u'' \Leftarrow \frac{1}{h^2}\left[-\varepsilon \;\; 2\varepsilon \;\; -\varepsilon\right]_h u^h$

  Standard central difference

- convective part : $\quad$ let $\mu \in [0,1]$

$\longrightarrow \qquad a u' \Leftarrow \mu \frac{a}{h}\left[-1 \; 1 \; 0\right] u^h + (1-\mu)\frac{a}{h}\left[0 \; -1 \; 1\right] u^h$

- linear combination of backward and forward differences
- $\mu = 1/2$ : central difference
- $\mu = 1$ : upwind difference ($a>0$), otherwise downwind.

Choice of $\mu$: $\quad \mu$ such that :

(1) the resulting difference scheme (diffusion + convection)

$$L_\mu^h = \frac{1}{h^2}\left[-\varepsilon - \mu a h \quad 2\varepsilon + (2\mu-1)h \quad -\varepsilon + (1-\mu) a h\right]_h$$

is diagonally dominant, with negative off-diagonal terms.

(2) $\left|\mu - \frac{1}{2}\right|$ should be minimal.

$$\Rightarrow \quad \mu = \begin{cases} 1 - 1/Pe & \text{if } Pe > 2 \\ 1/2 & \text{if } |Pe| \leq 2 \\ -1/Pe & \text{if } Pe < -2 \end{cases}$$

Where $Pe := \frac{ah}{\varepsilon}$ "Peclet-Number"

- for $\mu \neq 1/2$ the scheme is only 1st order.
- the scheme is stable : no oscillating solutions
- Note: $L_\mu^h = \underbrace{L_{1/2}^h}_{\text{central difference}} - \underbrace{ah\left(\tfrac{1}{2}-\mu\right)\tfrac{1}{h^2}\left[-1 \;\; 2 \;\; -1\right]_h}_{\Uparrow}$

$ah\left(\tfrac{1}{2}-\mu\right)\partial''$

"artificial viscosity"

i.e: $L_\mu^h$ corresponds to central differences of modified operator $-\varepsilon_h u'' + a u' = \ldots$

With $\varepsilon_h = \begin{cases} \frac{|a|h}{2} & \text{if } |Pe| > 2 \\ \varepsilon & \text{otherwise} \end{cases}$

- Alternative Approach: explicit artificial viscosity

Replace: $\varepsilon$ by $\varepsilon_h := \max\left(\varepsilon, \beta \frac{|a|h}{2}\right)$ with $\beta \geq 1$

Note: hybrid scheme $\hat{=}$ artificial viscosity with $\beta = 1$

## Higher Dimensions: $\quad -\varepsilon \Delta u + a u_x + b u_y = \ldots$

- apply the above schemes separately in each coordinate direction.
- note: the amount of artificial viscosity is then different in different coordinate directions.

  "anisotropic" artificial viscosity.

- many other schemes known; in particular, with conservative Finite Volume approaches.

## Remark on Smoothing:

- Smoothing properties of conventional relaxation schemes applied to the above hybrid discretizations can still be bad.

e.g. if $|Pe| > 2$: $\quad L_h^h = \frac{1}{h^2}\left[-ah \quad ah \quad 0\right]$

- Gauss-Seidel "from left to right": Direct Solver

  "downstream relaxation"

- Gauss-Seidel "from right to left": bad convergence

  Smoothing rate $= 1$

$\Rightarrow$ To get "good" smoothing properties independent of the "numbering of gridpoints", one need a bit more artificial viscosity. (e.g. explicit art. visc. with $\beta = 2$)

- Instead of downstream relaxation one could use "Symmetric Sweeps"

## Relaxation of the Staggered grid system:

### Distributive Relaxation:

Consider the linearized (frozen) Navier-Stokes-System:

$$\tilde{Q}\underline{u} + \nabla p = \underline{F} \quad , \quad \text{momentum}$$

$$\nabla \cdot \underline{u} = 0 \quad \text{continuity}$$

With $\tilde{Q} := -\frac{1}{Re}\Delta + \tilde{\underline{u}}\cdot\nabla$ ; $\tilde{\underline{u}}, \tilde{p}$ last iterates

Then: One relaxation sweep consists of 2 steps

(1) Relaxation of momentum equations with pressure $\tilde{p}$
$\longrightarrow$ new velocities $\underline{u}^*$

(2) Aim: new iterates $\hat{\underline{u}} := \underline{u}^* + \delta u$
$\hat{p} := \tilde{p} + \delta p$

Such that (i) continuity is satisfied
(ii) momentum defects (from (1)) are unchanged

$$\Rightarrow \hat{d} := F - \tilde{Q}\hat{u} - \nabla\hat{p} \overset{!}{=} d^* \quad \text{(momentum defects from (1))}$$

$$(*) \quad \nabla \cdot \hat{\underline{u}} \overset{!}{=} 0$$

Let $\chi$ be a grid function, defined by $\delta\underline{u} = \nabla\chi$ $\quad \delta p = -\tilde{Q}\chi$

then from (*) $\Rightarrow$ $\boxed{\nabla^2\chi = -\nabla\cdot u^*}$

Then the new momentum defects are:

$$\hat{d} = d^* - (\tilde{Q}\nabla - \nabla\tilde{Q})\chi$$

## The distributive Gauss-Seidel scheme of Brandt/Dinar

① as above $\rightarrow u^*$ (conventional relaxation of momentum equations with pressure values from last iterates)

② - sweep over all cell centers

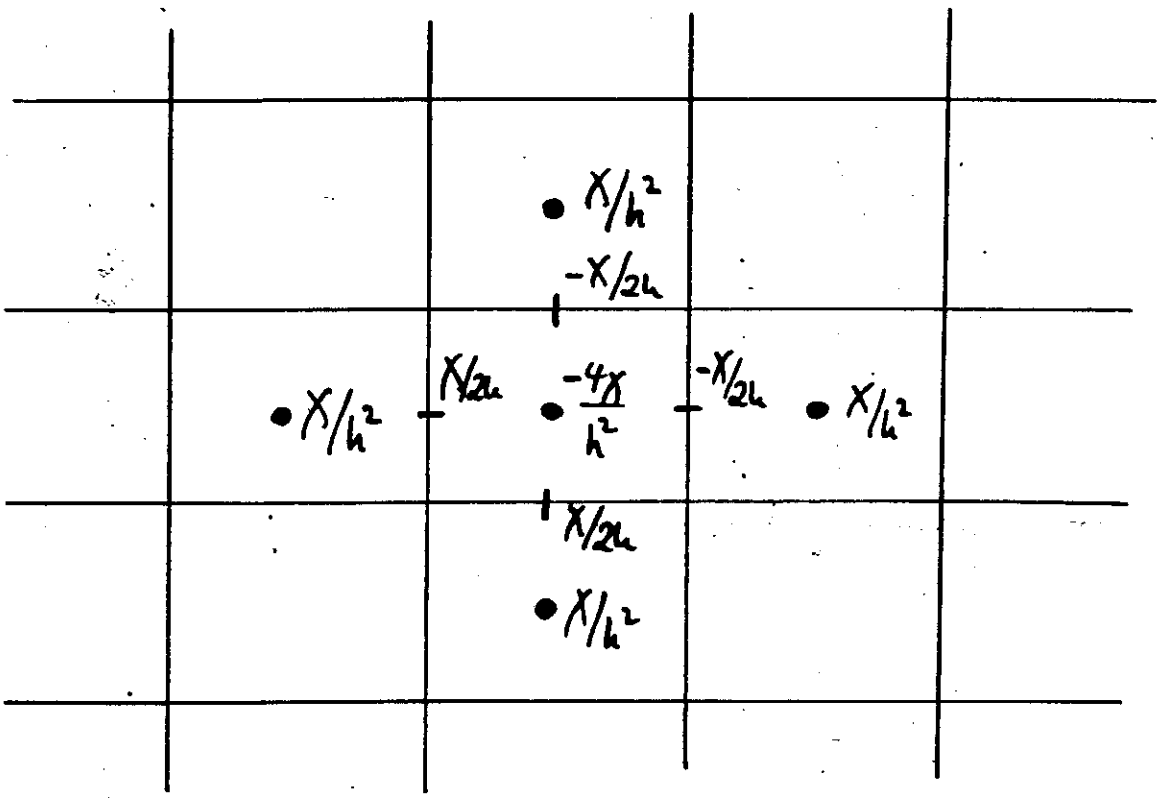
- at each cell center compute a $\chi$, non vanishing only at this cell center:

$$\Rightarrow \chi = -\frac{h^2}{4} D$$

with $D$ = current defect of continuity at this cell center

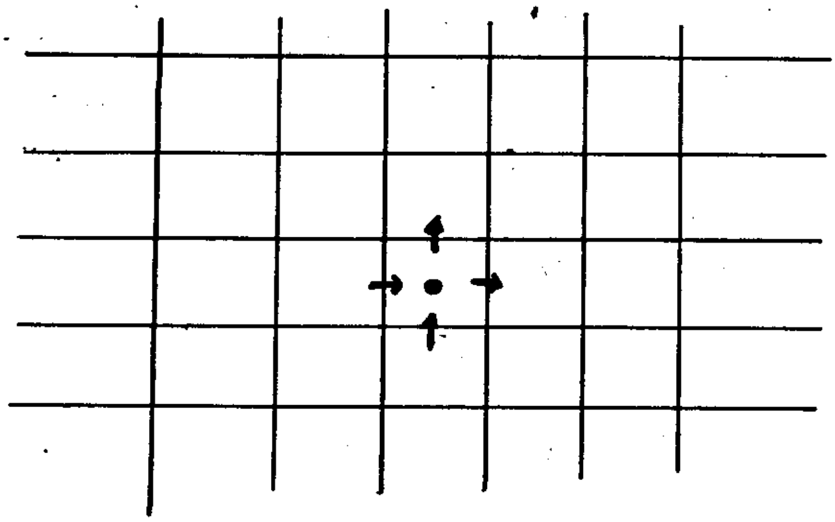
- compute changes $\delta \underline{u} := \nabla \chi$, $\delta p = -\tilde{Q}\chi$

__Changes for Stokes:__ $\tilde{Q} = -\Delta$ (i.e without convection)



# An alternative: Box Relaxation — S. P. Vanka

- Simultaneous treatment of velocities and pressure.



- Sweep over all cells

  at each cell compute simultaneously

  - new u-velocities at vertical cell faces
  - new v-velocities at horizontal cell faces
  - new pressure at cell center

  $\Longrightarrow$ 5x5 (nonlinear) system per cell.

## Non-staggered grids

and

## Flux-Splitting Methods

- Flux-splitting : discretization concept for the Euler-part (convection)

  ⟹ "Vector positive type" equations

⟹ collective relaxations
 pointwise
 linewise

## Flux-difference splitting for the incompressible Navier-Stokes equations

System of equations; conservative form:

$$\frac{\partial}{\partial x} f + \frac{\partial}{\partial y} g = \frac{\partial}{\partial x} f_v + \frac{\partial}{\partial y} g_v$$

where

$$f = \begin{pmatrix} u^2+p \\ uv \\ c^2 u \end{pmatrix} \qquad g = \begin{pmatrix} uv \\ v^2+p \\ c^2 v \end{pmatrix} \quad \text{convective fluxes}$$

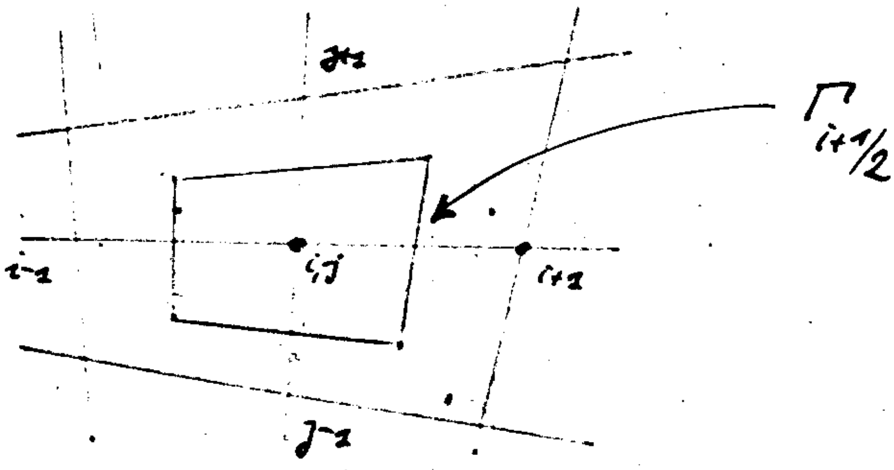
$$f_v = \begin{pmatrix} \nu u_x \\ \nu v_x \\ 0 \end{pmatrix} \qquad g_v = \begin{pmatrix} \nu u_y \\ \nu v_y \\ 0 \end{pmatrix} \quad \text{viscous fluxes}$$

### Discretization:

- Convective part: Flux difference Splitting
  → "vector positive type" equations
  → allows usage of collective relaxations

- viscous part: Standard finite volume method
  eg. Peyret's method

## Discretization of convective part:




$$\int_\Omega f_x + g_y \, d\Omega = \sum_{\Gamma_*} \underbrace{\int_{\Gamma_*} f n_x^* + g n_y^* \, d\Gamma^*}_{F_{\Gamma_*}}$$

Consider $\Gamma_{i+1/2}$:

$$F_{\Gamma_{i+1/2}} =: F_{i+1/2} \Leftarrow \underbrace{|\Gamma_{i+1/2}| \cdot \left( f_{i+1/2} n_x + g_{i+1/2} n_y \right)}_{\hat F_{i+1/2}}$$

**Define:** $F_i := |\Gamma_{i+1/2}| \left( f_i n_x + g_i n_y \right)$ approx from left

$F_{i+1} := |\Gamma_{i+1/2}| \left( f_{i+1} n_x + g_{i+1} n_y \right)$ approx from right

<u>Central differencing:</u> $\hat F_{i+1/2} \Leftarrow \frac{1}{2} \left( F_i + F_{i+1} \right)$

<u>In general:</u>

$$\boxed{\hat F_{i+\frac{1}{2}} \Leftarrow \frac{1}{2} \left( F_i + F_{i+1} \right) - \frac{1}{2} d\left( \xi_i, \xi_{i+1} \right)}$$

with $d(\xi_i, \xi_{i+1}) = O(\| \xi_i - \xi_{i+1} \|)$

and $\xi = \begin{pmatrix} u \\ v \\ p \end{pmatrix}$

# Flux-difference splitting for convective part

**Def:** Let $P, Q$ be two points in space, $\varphi$ a function

then: $$\Delta\varphi_{Q,P} := \varphi(P) - \varphi(Q)$$
$$\bar{\varphi} = \frac{1}{2}\left(\varphi(P) + \varphi(Q)\right)$$

## Flux-differences

$$n_x \Delta f + n_y \Delta g = \underbrace{\begin{pmatrix} n_x\bar{u} + \bar{w} & n_y\bar{u} & n_x \\ n_x\bar{v} & n_y\bar{v} + \bar{w} & n_y \\ c^2 n_x & c^2 n_y & 0 \end{pmatrix}}_{A} \Delta \vec{z}$$

with $\vec{z} = \begin{pmatrix} u \\ v \\ p \end{pmatrix}$, $\quad \bar{w} := n_x\bar{u} + n_y\bar{v}$ mean velocity in $(n_x n_y)$-direction

## Eigenvalues of A

Let $n_x^2 + n_y^2 = 1$

$$\lambda_1 = \bar{w}, \quad \lambda_{2,3} = \bar{w} \pm a \quad \text{with } a = \sqrt{\bar{w}^2 + c^2}$$

## Positive / Negative parts of A:

$$\Lambda^+ := \operatorname{diag}(\lambda_i^+) \qquad \lambda_i^+ = \max(0, \lambda_i)$$
$$\Lambda^- := \operatorname{diag}(\lambda_i^-) \qquad \lambda_i^- = \min(0, \lambda_i)$$
$$\Lambda := \Lambda^+ + \Lambda^-$$

$L, R$ : left and right eigenvector matrices of $A$

**Then:**

$$A = R \Lambda L$$
$$A^+ := R \Lambda^+ L$$
$$A^- := R \Lambda^- L$$

# The Numerical Flux Function $d$:

Note: Flux difference can be written as:

$$\Delta F_{i,i+1} := F_{i+1} - F_i = |\Gamma_{i+1/2}| \left( k_x \Delta f_{i,i+1} + k_y \Delta g_{i,i+1} \right)$$

$$= |\Gamma_{i+1/2}| \, A_{i,i+1} \, \Delta \zeta_{i,i+1}$$

$$= |\Gamma_{i+1/2}| \left( A^{+}_{i,i+1} + A^{-}_{i,i+1} \right) \Delta \zeta_{i,i+1}$$

**Def:**

$$d(\zeta_i, \zeta_{i+1}) := |\Delta F_{i,i+1}| := |\Gamma_{i+1/2}| \left( A^{+}_{i,i+1} - A^{-}_{i,i+1} \right) \Delta \zeta_{i,i+1}$$

Note: with this definition of $d$

$$F_{i+\frac{1}{2}} = F_i + |\Gamma_{i+\frac{1}{2}}| \, A^{-}_{i,i+1} \, \Delta \zeta_{i,i+1} \quad (*)$$

and

$$F_{i+\frac{1}{2}} = F_{i+1} - |\Gamma_{i+\frac{1}{2}}| \, A^{+}_{i,i+1} \, \Delta \zeta_{i,i+1}$$

Flux balance:

$$F_{i+1/2} + F_{i-1/2} + F_{j+1/2} + F_{j-1/2} =$$

$$|\Gamma_{i+1/2}| \, A^{-}_{i,i+1} (\zeta_{i+1} - \zeta_i) + |\Gamma_{i-1/2}| \, A^{-}_{i,i-1} (\zeta_{i-1} - \zeta_i)$$

$$+ |\Gamma_{j+1/2}| \, A^{-}_{j,j+1} (\zeta_{j+1} - \zeta_j) + |\Gamma_{j-1/2}| \, A^{-}_{j,j-1} (\zeta_{j-1} - \zeta_j)$$