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I.C.T.P., P.O. BOX 586, 34100 TRIESTE, ITALY, CABLE: CENTRATOM TRIESTE

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Multigrid Methods for Partial Differential Equations: Basic Algorithmic Components

J. Linden
Gmd
Institut F1
Germany

Multigrid Methods for Partial Differential Equations: Basic Algorithmic Components

Johannes Linden
GMD - F1
D-5205 St. Augustin 1
P.O. Box 1240
Germany

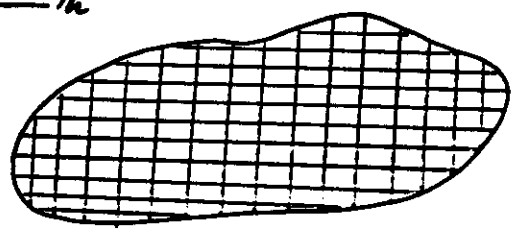
This talk gives a survey on the basic structures of multigrid algorithms for solving partial differential equations. It introduces into the under-lying ideas and the related terminology.

- The basic idea : Smoothing and coarse grid correction
- The 2-grid iteration: linear and nonlinear problems
- The multigrid iteration
- The multigrid components
- The Full - Multigrid Method

Partial Differential Equation: Boundary value problem

$$L u = f \qquad (\Omega)$$
$$B u = g \qquad (\partial\Omega)$$

Grid G_h:



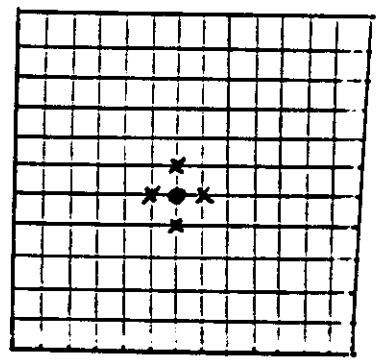
Discrete Formulation:

$$L^h u^h = f^h \quad \text{"interior equ."}$$
$$B^h u^h = g^h \quad \text{"boundary equ."}$$

Example: Laplace equation

$$-\Delta u = f \qquad (\Omega = (0,1)^2)$$
$$u = g \qquad (\partial\Omega)$$

Square cartesian grid $G_h = \{(ih, jh) : 0 \le i,j \le n\}$, $h = 1/n$



Finite Difference or Finite volume Formulation:

$$\frac{1}{h^2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}_{ij} u^h = f(ih, jh) \qquad (G_h \cap \Omega)$$

$$u^h_{ij} = g(ih, jh) \qquad (G_h \cap \partial\Omega)$$

The Basic Observation:

- Conventional relaxation schemes (e.g. Gauss-Seidel, damped Jacobi, SOR, ...) converge only very slowly.

→ Their convergence speed decreases with h (mesh size)

- Conventional relaxation schemes are very efficient in smoothing errors.

  i.e. The amplitudes of high frequency Fourier components of the initial error

  $$v^h = u^h - u^h$$

  are strongly reduced by one (or only a few) relaxation sweep.

→ This smoothing property does usually not depend on h.

Notation: $S^\nu(u^h, L^h, f^h)$: ν steps (or sweeps) of a relaxation scheme for $L^h u^h = f^h$ Starting from initial approximation u^h.

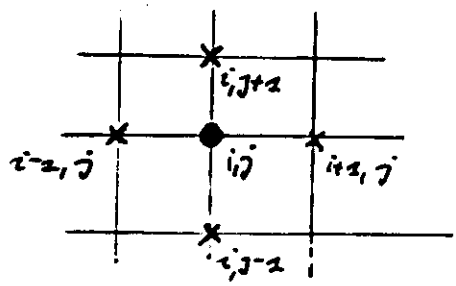
- Quantitative Analysis of smoothing properties of general relaxation schemes by

  Local Mode Analysis

Example: Gauss-Seidel relaxation for the 5-point Laplacian

The discrete problem: (see above)

$$\frac{1}{h^2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}_{ij} u^h = f^h, \quad 1 \le i,j \le n-1$$



The relaxation:

for $i = 1,\dots,n$ and $j = 1,\dots,n$:

$$u_{ij}^h \leftarrow \frac{1}{4}\left(u_{i-1,j}^h + u_{i+1,j}^h + u_{i,j-1}^h + u_{i,j+1}^h + h^2 f_{ij}^h\right)$$

"replace"

or in terms of the error $v_{ij}^h := u_{ij}^h - u_{ij}^h$

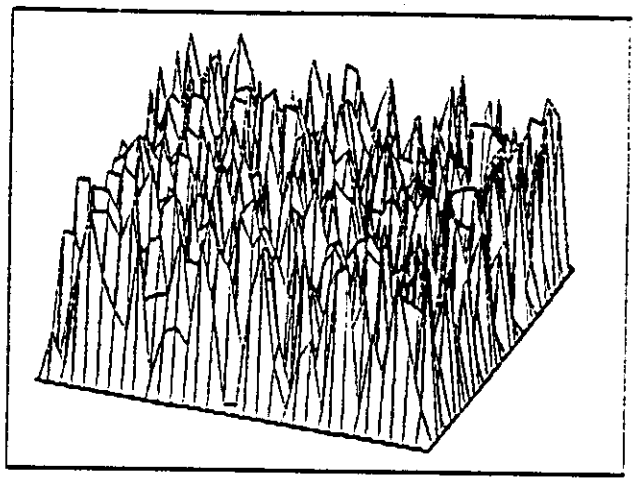
$$\boxed{v_{ij}^h \leftarrow \frac{1}{4}\left(v_{i-1,j}^h + v_{i+1,j}^h + v_{i,j-1}^h + v_{i,j+1}^h\right)}$$

- averaging process $\Rightarrow$ smoothing property

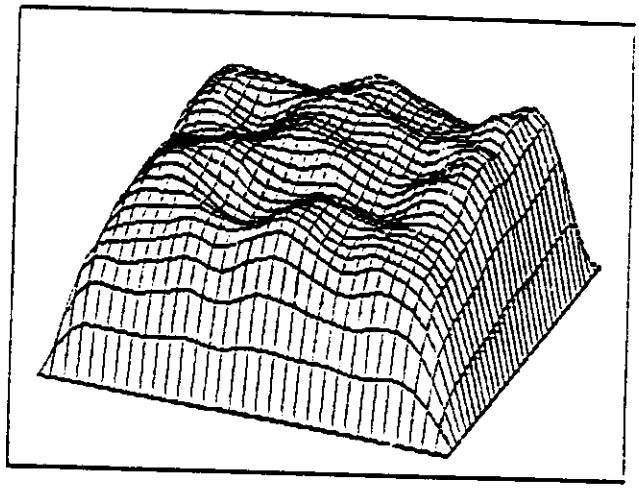
Gauss-Seidel relaxation

5-point Laplacian

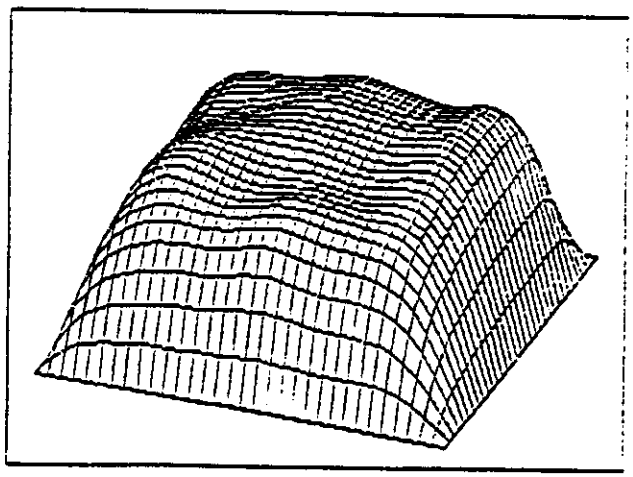
Square cartesian mesh



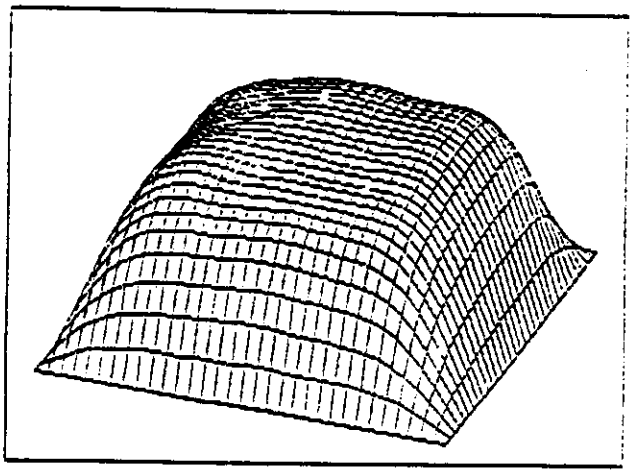
error of initial approximation



error after 5 relaxations



error after 10 relaxations



error after 15 relaxations

The 2-level method

Discrete problem: $L^h u^h = f^h$ Linear

on grid h

initial approximation: u_0^h

initial error: $v_0^h = u^h - u_0^h$

ν relaxation sweeps: $u_\nu^h = S^\nu(u_0^h; L^h, f^h)$

error after ν sweeps: $v_\nu^h = u^h - u_\nu^h$ Smooth

Correction equation: $$\underbrace{L^h v_\nu^h = f^h - L^h u_\nu^h}_{\text{residual or defect}} \quad (*)$$

- Due to the smoothness, $v^h = v_\nu^h$ can be approximated on a coarser grid H (e.g. $H = 2h$).

  This coarse grid approximation is denoted by v^H.

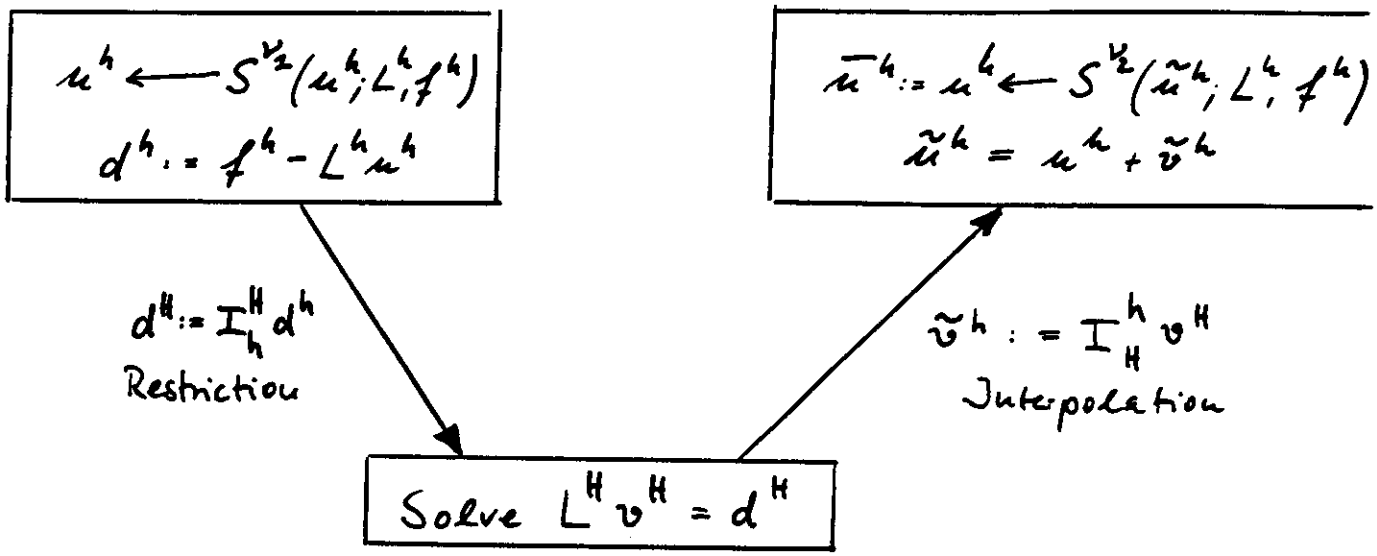
Coarse grid correction equation:

$$L^H v^H = I_h^H (f^h - L^h u_\nu^h) \quad (**)$$

- $(**)$ is a "representation" of $(*)$ on the coarser grid H.
- L^H is the "coarse grid operator"
- I_h^H is a (linear) mapping from grid h to grid H and is called

  restriction or residual transfer

2-level method: $u_h \rightarrow \bar{u}_h$ Linear case



- I_H^h is a mapping from grid H to grid h, called "Interpolation" (or correction transfer or prolongation)
- For the above 2-level method, the assumption of linearity for L^h is essential (Superposition principle used)
- The above scheme is called Correction Scheme (CS), because only corrections are computed on the coarse grid.
- On the way the method has been presented so far, it is a general algorithmic scheme rather than a well defined iterative method. For concrete problems the components of the method have still to be choosen properly (depending on the problem):
  - relaxation S^1
  - restriction I_h^H
  - interpolation I_H^h
  - coarse grid operator L^H

Multigrid Iterations (or Multigrid Cycling)

→ more than 2 levels

- in both versions, CS and FAS, the coarse grid problems have the form $L^H u^H = f^H$

  with the coarse grid operator L^H (linear or nonlinear) and a given right hand side f^H.

- in multigrid iterations, this problem is solved again by (2-level-) multigrid using still coarser grids, ...

→ **Recursive Definition of a Multigrid Cycle:**

(1) A multigrid cycle $C(\gamma, \nu_1, \nu_2)$ over 2 grids is the 2-level method with ν_1 pre-smoothing and ν_2 post-smoothing steps.

(2) A multigrid cycle $C(\gamma, \nu_1, \nu_2)$ over a hierarchy of grids $h = h_1 < h_2 < \dots < h_m$ consists of

(i) ν_1 pre-smoothing steps on grid h_1

(ii) Restriction to next coarser grid h_2

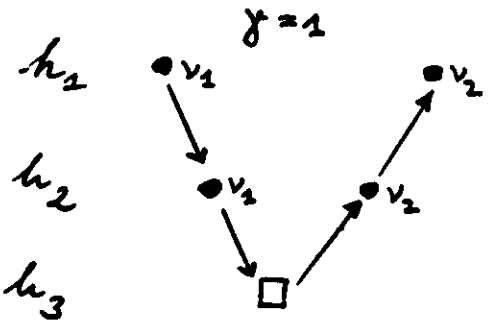
(iii) γ multigrid cycles $C(\gamma, \nu_1, \nu_2)$ over the grids $h_2 < \dots < h_m$

(iv) Interpolation of corrections from grid h_2 to h_1

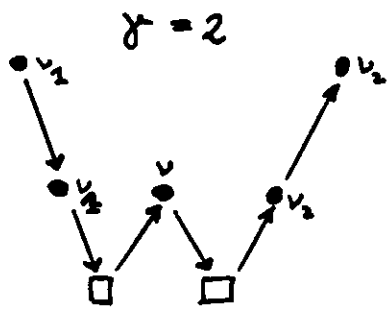
(v) ν_2 post-smoothing steps on grid h_1

- γ is called the cycling parameter
- for large γ the multigrid cycle "approximates" the 2-level method

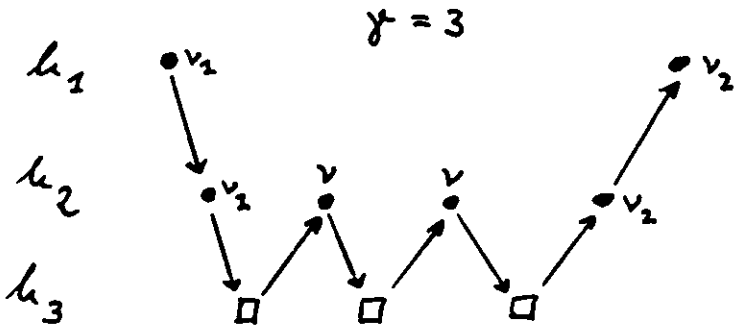
Examples: Cycles over 3 grids; $\nu := \nu_1 + \nu_2$



V-cycle



W-Cycle



- Smoothing steps (relaxations)
- □ (exact) solution on coarsest grid
- ↓ Restriction (fine to coarse)
- ↑ Interpolation (coarse to fine)

$$\sum_{\nu=0}^{\infty} q^{\nu} = \frac{1}{1-q}$$

Computational work of multigrid cycles

- 2D case, Standard coarsening $h, 2h, 4h, \ldots$

$\gamma = 1$: $(\nu_1+\nu_2)\cdot\frac{4}{3}$

$\gamma = 2$: $(\nu_1+\nu_2)\cdot 2$

$\gamma = 3$: $(\nu_1+\nu_2)\cdot 4$

Remark on Computational Work:

- Let $W :=$ computational work of one relaxation sweep over the finest grid "Work unit"
  $W = O(N)$ with $N =$ number of gridpoints
- Consider a hierarchy of grids where the number of gridpoints decreases by a fixed factor k
  e.g. Standard coarsening $h, 2h, 4h, \ldots$
  then in 1D problems: $k=2$
  2D problems: $k=4$
  3D problems: $k=8$
- Then: Neglecting the computational work for transfers between grids and for solving the problem on the very coarsest grid, the asymptotic computational work (i.e. number of grids $\to \infty$) for a $C(\gamma, \nu_1 \nu_2)$ cycle is

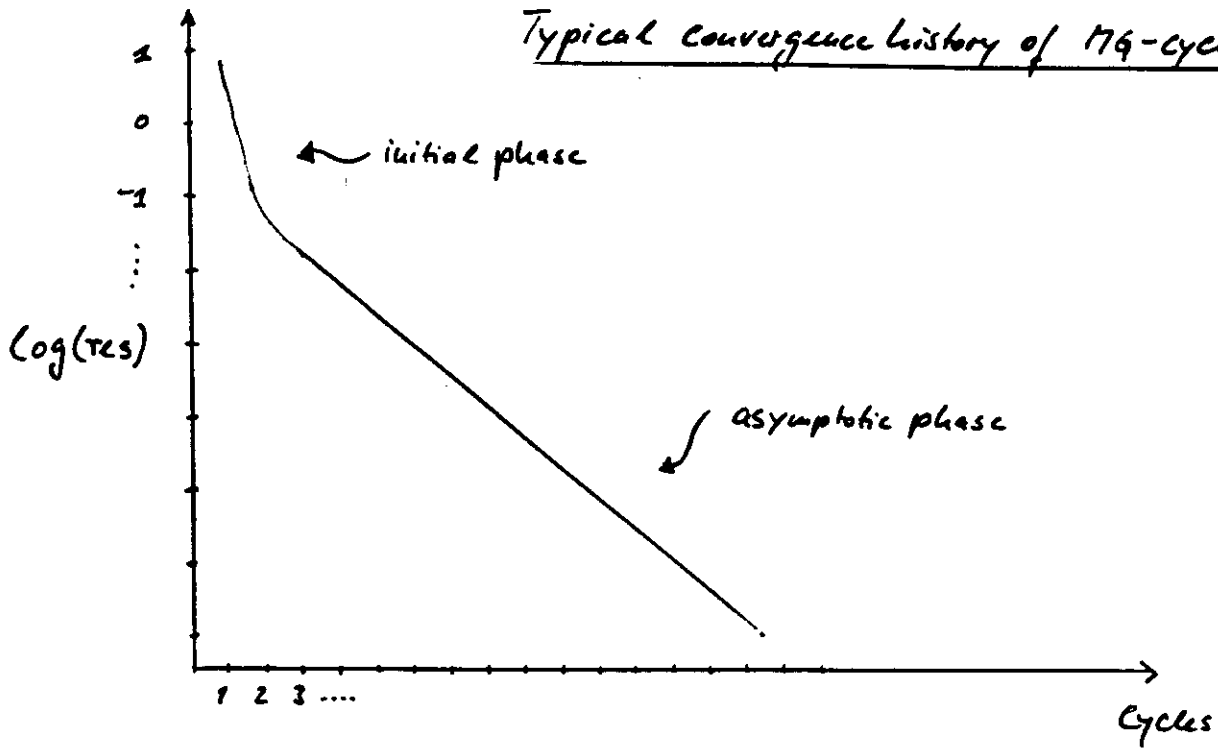
$$W \cdot \nu \sum_{n=0}^{\infty} \left(\frac{\gamma}{k}\right)^n, \qquad \nu = \nu_1 + \nu_2 .$$

$\Rightarrow$ in case of standard coarsening

in 1D	only	$\gamma = 1$	admissible
2D	only	$\gamma = 1,2,3$	"
3D	only	$\gamma = 1, \ldots, 7$	"

- Note: The computational work increases drastically with γ!
- in practice: $\gamma = 1$ or (more robust) $\gamma = 2$

Typical convergence history of MG-cycling

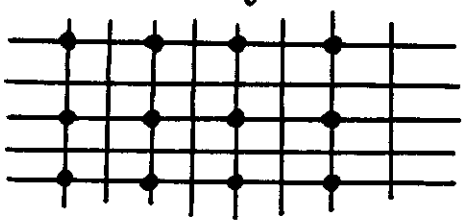


- typically: independent of h !

Some Multigrid Components

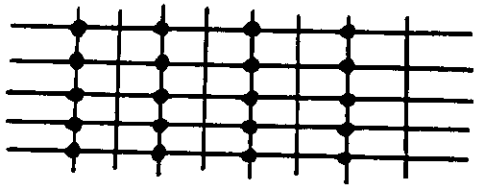
① Coarsening Strategy:

(i) Standard coarsening $h, 2h, 4h, \ldots$



• coarse grid

(ii) semi coarsening (i.e. coarsening only in some directions)



Coarsening in x-direction only

- more computational work on coarser grids
- sometimes preferable in cases of anisotropies

② Relaxations:

The most sensitive component in a multigrid method.
The choice of a proper relaxation depends on the problem. The relaxation must strongly reduce high frequency error components which are not "visible" on the coarse grids.

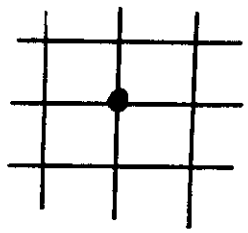
Note: This set of components depends on the coarsening strategy.

More about relaxation in the talk about local mode smoothing analysis.

③ Interpolations and Residual Transfers

- Interpolation of corrections from coarse to fine grid typically by polynomial interpolation
  e.g. linear interpolation

- residual transfers:
  - Simplest possibility: injection
    (if the coarse gridpoints also belong to the fine grid)
  - Standard Full Weighting



$$I_h^{2h} = \frac{1}{16}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

- averaging operator
  provides additional smoothing for the residuals

Remark: Interpolation and residual transfer operators must not be choosen independently. There is a rule on the "orders" of these operators which results from a theoretical Fourier analysis of the 2-level operator. For scalar equations it's roughly the following:

$$\text{Order (interpolation)} + \text{Order (residual transf.)} \geq m$$

where m is the order of the differential equation.

- a definition of the term "order" cannot be given here
(see. A. Brandt, Guide '84; p. 52-54)

But examples:

- Linear interpolation : order 2
- Cubic interpolation : order 4
- Injection : order 0
- Full-Weighting : order 2

⇒ for 2nd order boundary value problems
linear interpolation / injection
is the cheapest correct choice.

(4) A remark on coarse grid operators L^H:

Generally: L^H should approximate L^h quite well.

- standard approach: L^H = proper discretization of the differential operator L on coarse grid H.
- Alternative: "Galerkin Operators": $L^H := I_h^H L^h I_H^h$
Here, L^H is algebraically defined by the (given) fine grid operator L^h and by the transfers between the grids.
Needed in cases where a differential operator L is not available (→ see algebraic multigrid).
Galerkin operators are also considered for theoretical purposes in connection with algebraic convergence proofs. (e.g. McCormick, Nandel, Wesseling, ...)

The Full Multigrid Method:

- the above multigrid cycling is an iterative method which converges on initial approximation given on the finest grid to the exact discrete solution.
- in a Full Multigrid Method, the initial approximation on a fine grid is obtained by interpolation of a solution from the next coarser grid.

Definition of the nFMG (γ, ν_1, ν_2) - Method

Given a hierarchy of grids with meshsizes $h_1 < \dots < h_m$

① on the coarsest grid h_m solve the discretized problem

$$L^{h_m} u^{h_m} = f^{h_m}$$

② for $K = m-1, \dots, 1$ do

(i) $u^{h_K} \leftarrow I_{K+1}^{K} u^{h_{K+1}}$: initial approximation on grid h_K by interpolati

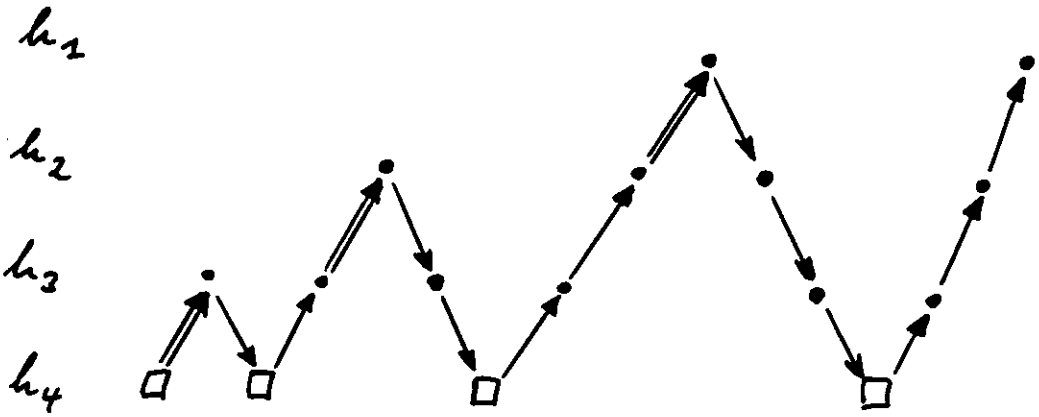
(ii) Starting with this u^{h_K} as initial approximation on grid h_K perform

n multigrid cycles $C(\gamma, \nu_1, \nu_2)$ over the grids $h_K, \dots, h_m$.

$\longrightarrow$ new u^{h_K}

- I_{k+1}^{k} is an interpolation from grid $k+1$ to the next finer grid k (similar to the interpolation within the multigrid cycle, not necessarily the same). It is called "FMG-interpolation"

Example: $1FMG(1, \nu_1, \nu_2)$: i.e. FMG with V-Cycles



⇗ FMG-interpolation

• relaxations

□ Solution on coarsest grid

↘ residual transfer (restriction)

↗ correction transfer (interpolation)

- The main idea behind FMG is to compute approximations with an accuracy of the order of the truncation error.

  i.e. let u be the continuous solution of $Lu = f$ and u^h the discrete solution of $L^h u^h = f^h$ on the finest grid.

  Then: FMG produces an approximation u^h with

  (*) $$\| u^h - u^h \| \leq \underbrace{\| u - u^h \|}_{\text{"Truncation error"}}$$

  (- provided that the multigrid parameters and components are choosen properly).

  e.g: the order of the FMG-interpolation should always be $\geq$ the order of the discretization

  (see A. Brandt, Guide '84, p. 73)

- Because the convergence of the multigrid cycling is independent of h, only a fixed number of cycles per level is sufficient to achieve (*).

- With a fixed number of cycles per level the computational complexity of FMG is optimal.

  i.e. the number of arithmetic operations needed in FMG is proportional to the number of gridpoints on the finest grid.