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Full Approximation Scheme

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Full Approximation Scheme

EXAMPLE: $L^h U^h \equiv U^h_x U^h_{xx} + \cdots = F^h$

$U^h = u^h + V^h$ $\quad$ $u^h =$ a given approximation
$\quad$ $V^h =$ the correction sought

$$\Downarrow$$

$$u^h_x V^h_{xx} + u^h_{xx} V^h_x + V^h_x V^h_{xx} + \cdots = F^h - u^h_x u^h_{xx} - \cdots$$
$$= R^h$$

<u>Coarse-grid equation</u>:

$$(I_h^{2h} u^h)_{\underline{x}} V^{2h}_{\underline{xx}} + (I_h^{2h} u^h)_{\underline{xx}} V^{2h}_{\underline{x}} + V^{2h}_{\underline{x}} V^{2h}_{\underline{xx}} + \cdots = \hat{I}_h^{2h} R^h$$

2h difference-quotients

No linearization: Good even when u^h is no approx., e.g., $u^h = 0$.

$$(\underline{I_h^{2h}} u^h + V^{2h})_{\underline{x}} (\underline{I_h^{2h}} u^h + V^{2h})_{\underline{xx}} - (\underline{I_h^{2h}} u^h)_{\underline{x}} (\underline{I_h^{2h}} u^h)_{\underline{xx}} + \cdots = \hat{I}_h^{2h} R^h$$

$$L^{2h}(\underline{I_h^{2h}} u^h + V^{2h}) - L^{2h}(\underline{I_h^{2h} u^h}) = I_h^{2h} R^h$$

the two $I_h^{2h} u^h$ should be identical.

This $I_h^{2h} u^h$ should also be used as the first approximation on grid 2h

• Frozen-τ Techniques $\quad L^{2h} U^{2h} = F^{2h} + \tau_h^{2h}$

Chains of problems

- Implicit time steps (evolution problems)
- Inverse problems • Design • Optimization
- Optimal control problems
- Continuation techniques (non linear problems with bifurcations).

are solved very inexpensively by solving most of them on coarser grids, with τ_h^{2h} that was obtained at a previous step (thus neglecting for several steps the h-f <u>changes</u>)

Infrequently a full fine-grid solution is made, to update τ_h^{2h}. May be done locally.

- <u>Refinement criteria:</u> $\quad \tau_h^{2h}(x) > \lambda G(x)^{-1} W^h(x)$

- <u>Local refinements:</u> $\quad F^{2h} + \tau_h^{2h}$ where grid h exists

  F^{2h} otherwise

NON-LINEAR EQUATIONS $L^h U^h = F^h$

Approximation u^h, correction sought $V^h = U^h - u^h$

<u>Residual equation</u>:

$$\hat{L}^h V^h \equiv L^h(u^h + V^h) - L^h u^h$$
$$= F^h - L^h u^h$$
$$\equiv R^h \text{, the residual}$$

V^h [illegible] smoothed by relaxation (with $\hat{L}^h$ [illegible]) [illegible] GS (actually not different from [illegible])

<u>[illegible] grid correction equations</u>:

Linear case: $L^{2h} V^{2h} = I_h^{2h} R^h$ [illegible]

Nonlinear : $L^{2h}\left(\underline{I_h^{2h}} u^h + V^{2h}\right) - L^{2h}\left(\underline{I_h^{2h}} u^h\right) =$ [illegible]

<u>FAS form</u>: $L^{2h} \bar{U}^{2h} = f^{2h}$

<u>where</u>: $\bar{U}^{2h} = \underline{I_h^{2h} u^h} + V^{2h}$

$f^{2h} = L^{2h}\left(\underline{I_h^{2h} u^h}\right) + I_h^{2h} R^h$

If u^{2h} approximates $\bar{U}^{2h}$, the fine-grid correction [illegible]

$$u^h_{NEW} \leftarrow u^h_{OLD} + I_{2h}^h\left(u^{2h} - \text{[illegible]}\right)$$

FAS-FMG Solution of $L^h U^h = F^h$

A. Solve*) $L^{2h} U^{2h} = F^{2h}$ approximately: u^{2h}.

B. Interpolate $u^h \leftarrow I^h_{2h} u^{2h}$ smooth comp.

C. Relax $L^h u^h \doteq F^h$ non-smooth comp.

$\Rightarrow$ (Aliasing) smooth error $V^h = U^h - u^h$

Approximable by V^{2h} via residual eq.:

$$L^h(u^h + V^h) - L^h u^h = F^h - L^h_* u^h \equiv R^h$$

$$L^{2h}\underbrace{\left(\bar{I}^{2h}_h u^h + V^{2h}\right)}_{\bar{U}^{2h}} - L^{2h}\bar{I}^{2h}_h u^h = I^{2h}_h R^h$$

FAS: $\bar{U}^{2h}$ $= \bar{I}^{2h}_h u^h$ at convergence

D. Solve*) $L^{2h}\bar{U}^{2h} = \bar{F}^{2h}$

$$\bar{F}^{2h} = I^{2h}_h R^h + L^{2h}\bar{I}^{2h}_h u^h = F^{2h} + \tau^{2h}_h$$

$$\tau^{2h}_h = L^{2h}\bar{I}^{2h}_h u^h - I^{2h}_h L^h u^h$$ = defect correction ~ truncation error

E. Interpolate $u^h_{NEW} \leftarrow u^h_{OLD} + I^h_{2h}\left(\bar{u}^{2h} - \bar{I}^{2h}_h u^h_{OLD}\right)$ aliasing

*) solving L^{2h} — similarly (L^{4h} & relaxation) ...

FULL APPROXIMATION SCHEME : DUAL VIEW

Equations: differential $LU = F$

difference $L^h U^h = F^h \equiv I^h F$

$$L^{2h} U^{2h} = F^{2h} \equiv I^{2h} F \equiv I_h^{2h} F^h$$

Local Truncation Errors (LTE): $L^h I^h U = F^h + \tau^h$

$\tau^h = L^h I^h U - I^h L U.$ $\quad L^{2h} I^{2h} U = F^{2h} + \tau^{2h}$

Relative LTE: $L^{2h} I_h^{2h} U^h = F^{2h} + \tau_h^{2h}$

$$\tau_h^{2h} = L^{2h} I_h^{2h} U^h - I_h^{2h} L^h U^h$$ "defect" correction.

$$\approx L^{2h} I_h^{2h} u^h - I_h^{2h} L^h u^h$$ after smoothing

Coarse-grid corrected equations:

$$L^{2h} \overline{U}^{2h} = F^{2h} + L^{2h} I_h^{2h} u^h - I_h^{2h} L^h u^h$$
$$= L^{2h} I_h^{2h} u^h + I_h^{2h} (F^h - L^h u^h)$$
$$= f^{2h}$$ FAS eq.

Dual multi-grid: $L^{2h} U^{2h} = F^{2h} + \tau_h^{2h}$

fine-to-coarse correctio

At convergence $U^{2h} = I_h^{2h} U^h$

Full Approximation Scheme (FAS)

The coarse-grid (2h) variable: $\bar{U}^{2h} = I_h^{2h} u^h + V^{2h}$

The coarse-grid equations: $L^{2h} \bar{U}^{2h} = F^{2h} + \tau_h^{2h}$

$\tau_h^{2h} = L^{2h}(I_h^{2h} u^h) - I_h^{2h} L^h u^h$ = <u>relative</u> local truncation,

(approximates the local truncation $\tau^{2h} = L^{2h} U - I L^h U$)

= fine-to-coarse defect corrections.

- <u>Nonlinear problems</u> solved as efficiently as linear ($L = L(U)$). No linearization needed. No storage

Steady Euler & Navier-Stokes (DGS relaxation)

Transonic flows: $(K - U_x) U_{xx} + U_{yy} = 0$

Constrained minimization: $\min_{U \geq 0} \int U_x^2 + U_y^2 + 2FU$

Eigenvalue problems: $\Delta u = \lambda u$.

- <u>Needed storage</u> $\ll n$ (τ_h^{2h} computed in segments).
- <u>τ-extrapolation</u>: $\tau^h \approx ch^p \Rightarrow \tau^{2h} \approx \frac{2^p}{2^p - 1} \tau_h^{2h}$

Multiplying τ_h^{2h} by $\frac{2^p}{2^p-1}$ gives higher-order for no cost.

(More general than Richardson extrapolation)