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Smoothing Analysis of Relaxation Schemes

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Smoothing Analysis of Relaxation Schemes

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The most general tool for analyzing a multigrid process is the so-called local mode analysis. In its simplest form it is used to analyze quantitatively the smoothing properties of relaxation schemes. This talk introduces into this technique. It covers the following topics:

- The 1D scalar case: Fourier components
  Difference operators
  Relaxations
  high and low frequencies
  Smoothing rates
  prediction of multigrid performance

- Systems of equations

- Higher dimensions
  2D examples

The one-dimensional case

Let $G^h = \{kh : k \in \mathbb{Z}\}$ an infinite uniform grid with meshsize h.

Usual 1D Fourier components: $e^{i\omega x} = \cos\omega x + i \sin \omega x$

- On grid G^h only Fourier components $e^{i\vartheta x/h}$ with $\vartheta \in (-\pi, \pi]$ are "visible".

Def: A component $e^{i\vartheta x/h}$ is "visible" on G^h if there is no frequency ϑ_0 with $|\vartheta_0| < |\vartheta|$ such that

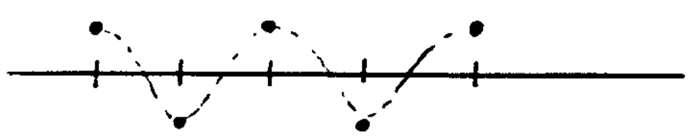
$$e^{i\vartheta_0 x/h} = e^{i\vartheta x/h} \quad \text{for all } x \in G^h.$$

i.e. the component $e^{i\vartheta x/h}$ does not coincide with any lower frequency component on grid G^h.

$\Rightarrow$ on G^h only components $e^{i\vartheta x/h}$, $\vartheta \in (-\pi, \pi]$ have to be considered.

- ϑ "Frequency"
- $\left|\frac{2\pi h}{\vartheta}\right|$ "Period" of the Fourier component
- $\vartheta = 0$: constant component

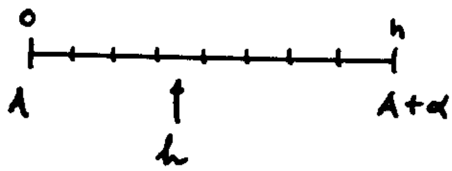
  $\vartheta = \pi$: $x = kh \in G^h$: $e^{i\pi x/h} = (-1)^k$



highly oscillating, Period $2h$

- Consider a finite interval of length α: $[A, A+\alpha]$, $A \in \mathbb{R}$ covered by a grid with meshsize $h = \frac{\alpha}{n}$, $n \in \mathbb{N}$.

$$G^h = \{ x_\ell = A + h\ell ; \ \ell = 0, \dots, n \}.$$



A grid function $u^h = (u^h_i)_{i=0,\dots,n}$ on G^h is called "periodic" if $u^h_0 = u^h_n$.

<u>Then</u>: The Fourier components $e^{i\vartheta_k x/h}$ with

$$\vartheta_k = -\pi + \frac{2\pi}{n} k \quad , \quad k = 1, \dots, n$$

form an orthogonal basis of the linear space of periodic grid functions on G^h

Difference operators with constant coefficients

$$G^h = \{kh,\ k \in \mathbb{Z}\} \qquad \text{1D infinite, uniform grid}$$

$$u^h = u^h(x), \quad x \in G^h \quad : \text{grid function}$$

$$I = \{-\nu, \dots, 0, \dots \nu\} \quad \text{set of indices}$$

Difference operator L^h with constant coefficients a_μ:

$$L^h u^h(x) = \sum_{\mu \in I} a_\mu u^h(x + \mu h) \qquad (x \in G^h)$$

Example:

$$I = \{-1, 0, 1\}$$

$$a_{-1} = \frac{1}{h^2} \qquad a_0 = -\frac{2}{h^2} \qquad a_1 = \frac{1}{h^2}$$

$$\Rightarrow \quad L^h u^h(x) = \frac{1}{h^2}\begin{bmatrix} 1 & -2 & 1 \end{bmatrix}_x u^h$$

$$= \frac{1}{h^2}\left(u^h(x-h) - 2u^h(x) + u^h(x+h) \right)$$

finite difference discretization of $u''(x)$.

On the 1D infinite grid G^h the Fourier components

$$e^{i\vartheta x/h}$$

are eigenfunctions of a constant coefficient operator L^h:

$$L^h e^{i\vartheta x/h} = \underbrace{\left(\sum_{\mu \in I} a_\mu e^{i\vartheta\mu} \right)}_{\tilde{L}^h(\vartheta)} \cdot e^{i\vartheta x/h}$$

$\tilde{L}^h(\vartheta)$ Eigenvalue or "Symbol" of L^h.

<u>Example:</u> $L^h u^h(x) = \frac{1}{h^2}\left[1 \;\; -2 \;\; 1\right]_x u^h$

$$\Rightarrow L^h e^{i\vartheta x/h} = \frac{1}{h^2}(2\cos\vartheta - 2)\, e^{i\vartheta x/h}$$

<u>Remark:</u> When we consider a finite intervall of length α, covered by a uniform mesh of meshsize $h = \alpha/n$ and take a constant coefficient operator with

periodic boundary conditions

then: the Fourier components

$$e^{i\vartheta_k x/h} \quad \text{with } \vartheta_k = -\pi + \frac{2\pi}{n}, \; k = 1, \dots, n$$

are the eigenfunctions with respect to these boundary conditions.

Relaxation schemes:

Example: Gauss-Seidel relaxation for

$$L^h u^h(kh) = \frac{1}{h^2}\left[1 \quad -2 \quad 1\right]_{kh} u^h = f^h(kh)$$

$$\Rightarrow \quad u^h_{k-1} - 2u^h_k + u^h_{k+1} = h^2 f^h_k \qquad k \in \mathbb{Z}$$

Initial approximation: $\mu^h = (\mu^h_k)_{k\in\mathbb{Z}}$

Initial error : $v^h = (v^h_k)_{k\in\mathbb{Z}} := (u^h_k - \mu^h_k)_{k\in\mathbb{Z}}$

Relaxation: $\mu^h \to \bar{\mu}^h$

$$\boxed{\bar{\mu}^h_{k-1} - 2\bar{\mu}^h_k + \mu^h_{k+1} = f^h_k \cdot h^2}$$

or in terms errors: $\bar{v}^h = u^h - \bar{\mu}^h$

$$(*) \quad \boxed{\bar{v}^h_{k-1} - 2\bar{v}^h_k + v^h_{k+1} = 0}$$

Assume the initial error v^h of the form: $v^h(x) = A e^{i\vartheta x/h}$ on G^h, with $A \in \mathbb{R}$ Amplitude.

Then the difference equation $(*)$ for the errors can be solved directly: Take $\bar{v}^h(x) = \bar{A} e^{i\vartheta x/h}$ and put it into $(*)$:

$$\bar{A} e^{i\vartheta x_{k-1}/h} - 2\bar{A} e^{i\vartheta x_k/h} + A e^{i\vartheta x_{k+1}/h} = 0$$

$$\Leftrightarrow \quad e^{i\vartheta x_k/h} \cdot \left(\bar{A} e^{-i\vartheta} - 2\bar{A} + A e^{i\vartheta}\right) = 0$$

$$\Leftrightarrow \quad \bar{A} = \frac{e^{i\vartheta}}{2 - e^{-i\vartheta}} A$$

- $\mu(\vartheta) := \left| \dfrac{e^{i\vartheta}}{2 - e^{-i\vartheta}} \right|$ is the factor by the amplitude of the error component $e^{i\vartheta x/h}$ is damped ("amplification factor")

  e.g. $\vartheta = \pi$, the highest oscillating component, is damped by $\mu(\pi) = 1/3$,
  while for $\vartheta \to 0$: $\mu(\vartheta) \to 1$.

  "Oscillating components have much smaller amplification factor than smooth (small ϑ) components."

<u>More general situation:</u>

Consider a linear difference operator

$$L^h u^h(x) = \sum_{\mu \in I} a_\mu u^h(x + \mu h).$$

Many relaxations can be written in form of a decomposition of the operator $\boxed{L^h = A^h + B^h}$

where A^h, B^h are again difference operators (in the above sense).

→ Relaxation for $L^h \mathcal{U}^h = f^h$, $u^h \to \bar{u}^h$

$$\boxed{A^h u^h(x) + B^h \bar{u}^h(x) = f^h(x)}$$

in error terms: $v^h = \mathcal{U}^h - u^h$, $\bar{v}^h = \mathcal{U}^h - \bar{u}^h$

$$(*) \quad \boxed{A^h v^h(x) + B^h \bar{v}^h(x) = 0}$$

Note: Not all relaxation schemes can be represented in this form: e.g. red-black relaxations.

Amplification Factor:

let $v^h(x) = A\, e^{i\vartheta x/h}$ before relaxation

$\bar{v}^h(x) = \bar{A}\, e^{i\vartheta x/h}$ after relaxation

then from ⊛:

$$\tilde{A}^h(\vartheta)\, A + \tilde{B}^h(\vartheta)\, \bar{A} = 0$$

$$\Rightarrow \quad \bar{A} = \frac{\tilde{A}^h(\vartheta)}{\tilde{B}^h(\vartheta)}\, A$$

$\tilde{A}^h(\vartheta)$ symbol of A^h
$\tilde{B}^h(\vartheta)$ symbol of B^h

Amplification factor of component ϑ

$$\boxed{\mu(\vartheta) := \left| \frac{\tilde{A}^h(\vartheta)}{\tilde{B}^h(\vartheta)} \right|}$$

Question: What is smoothing?

Remember: Within a multigrid cycle, relaxations are used to "smooth" the highly oscillating error components which cannot be approximated on a coarse grid.

$\Rightarrow$ What are the "highly oscillating" components?

Can the smoothing property of a given relaxation scheme be measured?

What are the high frequencies?

- A Fourier component $e^{i\vartheta x/h}$ on grid G^h is called a high frequency component if its restriction (injection) to the coarse grid G^H is not "visible" (see above) there. Otherwise it's called a low frequency.

e.g. $G^h = \{kh : k \in \mathbb{Z}\}$, $G^H = G^{2h} = \{2kh : k \in \mathbb{Z}\}$

$$\boxed{\text{Component } e^{i\vartheta x/h} \text{ on } G^h, \ |\vartheta| \le \pi}$$
$$\Big\downarrow \text{ inject to } G^{2h}$$
$$\boxed{e^{i2\vartheta x/2h} \text{ on } G^{2h}}$$

- this component is "visible" on G^{2h} (see definition above) only if $|2\vartheta| \le \pi$, i.e $|\vartheta| \le \pi/2$
- if $\pi \geqslant |\vartheta| > \pi/2$, on G^{2h} the component $e^{i2\vartheta x/2h}$ coincides with $e^{i2\vartheta_0 x/2h}$

$$\text{where } \vartheta_0 = \begin{cases} \vartheta - \pi, & \vartheta > 0 \\ \vartheta + \pi, & \vartheta \le 0 \end{cases} \qquad (\text{note: } |\vartheta_0| < \pi/2)$$

i.e. The high frequency components on G^h (with respect to G^{2h}) are those with $\pi \geqslant |\vartheta| \geqslant \pi/2$.

Remarks: • The definition of high and low frequencies depends on the coarsening.

- For any component ϑ, $|\vartheta| \le \pi$, $\vartheta_0 := \begin{cases} \vartheta - \pi, & \vartheta > 0 \\ \vartheta + \pi, & \vartheta \le 0 \end{cases}$

is called the harmonic frequency. On the coarse grid G^{2h} the components ϑ and ϑ_0 coincide.

The smoothing rate

For a relaxation scheme with amplification factors $\mu(\vartheta)$

$$\mu := \max \{ \mu(\vartheta) : \vartheta = \text{high frequency} \}$$

Example: $\frac{1}{h^2}[1 \ \ -2 \ \ 1]_x u^h = f^h(x)$

$G^H = G^{2h}$

- Gauss-Seidel relaxation : $\mu(\vartheta) = \left| \frac{e^{i\vartheta}}{2 - e^{-i\vartheta}} \right| = \frac{1}{\sqrt{5 - 4\cos\vartheta}}$

$$\Rightarrow \mu = \max_{\frac{\pi}{2} \le |\vartheta| \le \pi} \mu(\vartheta) = \frac{1}{\sqrt{5}}$$

- Damped Jacobi relaxation: initial approximation $u^h(x)$
  - (i) Compute intermediate value $\tilde{u}^h(x)$:
  $$u^h(x-h) - 2\tilde{u}^h(x) + u^h(x+h) = h^2 f^h(x), \quad x \in G^h$$
  - (ii) Damping: $\bar{u}^h(x) := (1-\omega)\, u^h(x) + \omega\, \tilde{u}^h(x)$

    $\omega > 0$ damping factor

$$\Rightarrow \underbrace{\left[1 \quad -2\frac{\omega-1}{\omega} \quad 1\right]_x}_{A^h} u^h + \underbrace{\left[0 \quad -\frac{2}{\omega} \quad 0\right]_x}_{B^h} \bar{u}^h = h^2 f^h(x)$$

$\Rightarrow \mu(\vartheta) = |\omega\cos\vartheta - \omega + 1|$ amplification factor

$\Rightarrow \mu = \max\limits_{\frac{\pi}{2} \le |\vartheta| \le \pi} = \max(|1-\omega|, |1-2\omega|)$

- $\omega = 1 \Rightarrow \mu = 1$ no smoothing
- $\omega = \frac{1}{2}$ optimal $\Rightarrow \mu = 1/2$ good smoothing

Multigrid Performance

- let μ be the smoothing rate of a relaxation
- assume that by the coarse grid correction cycle all low frequency error components are exactly approximated and there is no interaction between low and high frequencies.

Thm: The (asymptotic) error reduction by one step of a 2-level method is about

$$\boxed{\rho := \mu^{\nu_1+\nu_2}}$$

where ν_1 and ν_2 are numbers of pre- and post-smoothing steps.

- Attention: This is not a rigorous mathematical estimate of 2-level convergence!

But quite useful in practice.

So far: Scalar, constant coefficient operators in 1D.
relaxations based on operator decomposition
definition of smoothing rates
estimate of 2-level performance

basic assumption: infinite grid, i.e. influence of boundary equation neglected.

Remark: With Fourier components of the form $e^{i\vartheta x/h}$ can also be regarded as an exact finite grid analysis for periodic boundary conditions. !

The smoothing analysis can be generalized to

- Systems of equations;
- higher dimensions, of course;
- more general relaxation schemes
  e.g. red-black schemes

see: Stüben, Trottenberg in: "Multigrid Methods", Lecture Notes in Mathematics, vol 960; Springer-Verlag, Heidelberg, 1982.

Systems of equations   here 1D only

$q \times q$ system $L^h = \left(\ell^h_{\kappa,\ell}\right)_{\kappa,\ell = 1,\dots,q}$

with: $\ell^h_{\kappa\ell}\, u^h(x) := \sum_{\mu \in I} a_\mu^{\kappa,\ell}\, u^h(x+\mu h)$

i.e.: L^h is a $q \times q$ matrix of scalar, const. coeff. operators

Problem to be solved:

$$\underline{L}^h \underline{u}^h = \underline{f}^h$$

$$\underline{u}^h = (u_1^h, \dots, u_q^h)^t \;;\; \underline{f}^h = (f_1^h, \dots, f_q^h)^t$$

Consider now Fourier Components: $\mathcal{A} e^{i\vartheta x/h}$ with $\mathcal{A} \in \mathbb{C}^q$

Symbol of L^h: The symbol of the $q \times q$ operator L^h is

$$\boxed{\tilde{L}^h(\vartheta) := \left(\tilde{\ell}^h_{\kappa,\ell}(\vartheta)\right)_{\kappa,\ell = 1,\dots,q} \in \mathbb{C}^{q \times q}}$$

It's the amplification matrix of the above component:

$$\boxed{L^h\left(\mathcal{A} e^{i\vartheta x/h}\right) = \left(\tilde{L}^h(\vartheta) \cdot \mathcal{A}\right) e^{i\vartheta x/h}}$$

Relaxations Let $A^h + B^h = L^h$ be a decomposition of L^h where A^h and B^h are $q\times q$-systems of difference operators

Relaxation: $u^h \to \bar{u}^h$: $A^h u^h + B^h \bar{u}^h = \underline{f}^h$

in error terms: $A^h v^h + B^h \bar{v}^h = 0$

Let $v^h = \mathcal{A} e^{i\vartheta x/h}$

$\bar{v}^h = \bar{\mathcal{A}} e^{i\vartheta x/h}$

$$\Rightarrow \quad \tilde{A}^h(\vartheta)\mathcal{A} + \tilde{B}^h(\vartheta)\bar{\mathcal{A}} = 0$$

$$\Rightarrow \quad \boxed{\bar{\mathcal{A}} = - \underbrace{\tilde{B}^h(\vartheta)^{-1}\tilde{A}^h(\vartheta)}_{\in \mathbb{C}^{q\times q}} \mathcal{A}}$$

- (asymptotic) amplification factor of component ϑ:

$$\mu(\vartheta) := \varrho\left(\tilde{B}^h(\vartheta)^{-1}\tilde{A}^h(\vartheta)\right)$$

$$= \max\{|\lambda| : \det(\lambda B + A) = 0\}$$

- Smoothing rate:

$$\mu := \max\{\mu(\vartheta) : \vartheta \text{ high frequency}\}$$

Higher Dimensions: here only 2D

- scalar operators, const. coefficients

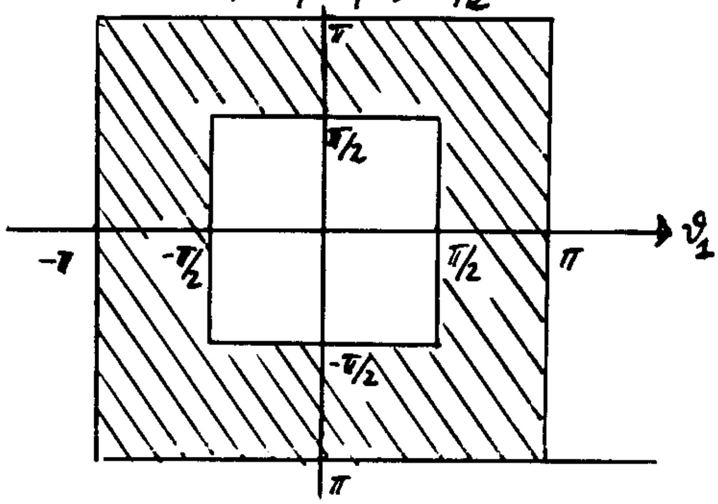
- Fourier components: $e^{i\vartheta_1 x/h} \cdot e^{i\vartheta_2 y/h} =: e^{i\underline{\vartheta}\underline{x}/h}$

$$\vartheta = (\vartheta_1, \vartheta_2), \quad |\vartheta| = \max(|\vartheta_1|, |\vartheta_2|)$$

"visible" on grid $G^h = \{(kh, \ell h) ; k, \ell \in \mathbb{Z}\}$ only if $|\vartheta| \le \pi$.

- High frequencies with respect to standard coarsening:

$$\pi > |\vartheta| > \pi/2$$



- The above 1D-definitions can directly be generalized (applied) to 2D.

2D Examples:

Differential equation $\quad -\varepsilon u_{xx} + u_{yy} = f, \qquad 0 < \varepsilon \leq 1$

2nd order finite difference approximation on a square cartesian grid G^h:

$$\frac{1}{h^2}\begin{bmatrix} & -1 & \\ -\varepsilon & 2+2\varepsilon & -\varepsilon \\ & -1 & \end{bmatrix}_x u^h = f^h(x)$$

① Gauss-Seidel relaxation:

$$\underbrace{\begin{bmatrix} & 0 & \\ -\varepsilon & 2+2\varepsilon & 0 \\ & -1 & \end{bmatrix}_x}_{B^h} \bar{u}^h + \underbrace{\begin{bmatrix} & -1 & \\ 0 & 0 & \varepsilon \\ & 0 & \end{bmatrix}_x}_{A^h} u^h = h^2 f^h(x)$$

Symbols: $\quad \tilde{B}^h(\vartheta_1, \vartheta_2) = 2 + 2\varepsilon - \varepsilon e^{-i\vartheta_1} - e^{-i\vartheta_2}$

$\qquad\qquad \tilde{A}(\vartheta_1, \vartheta_2) = -e^{i\vartheta_2} - \varepsilon e^{i\vartheta_1}$

$$\Rightarrow \quad \mu(\vartheta_1, \vartheta_2) = \left| \frac{e^{i\vartheta_2} + \varepsilon e^{i\vartheta_1}}{2 + 2\varepsilon - \varepsilon e^{-i\vartheta_1} - e^{-i\vartheta_2}} \right|$$

Smoothing rate: $\quad \mu := \max_{\pi \geq |\vartheta| \geq \frac{\pi}{2}} \mu(\vartheta_1, \vartheta_2)$

- for $\vartheta_1 = \pi, \vartheta_2 = 0$: $\mu(\pi, 0) = \dfrac{1-\varepsilon}{1+3\varepsilon}$: for $\varepsilon \to 0$ we have: $\mu \to 1$ <u>Bad smoothing</u>
- but for $\varepsilon = 1$: $\mu = 0.5$ <u>Good smoothing</u>

"The above Gauss-Seidel scheme has good smoothing rates only if ε is close to 1.

What is to do if $\varepsilon \ll 1$:

(2) Line-relaxation:

$$\underbrace{\begin{bmatrix} & -1 & \\ -\varepsilon & 2+2\varepsilon & 0 \\ & -1 & \end{bmatrix}_x}_{B^h} \bar{u}^h + \underbrace{\begin{bmatrix} & 0 & \\ 0 & 0 & -\varepsilon \\ & 0 & \end{bmatrix}_x}_{A^h} u^h = h^2 f^h(x)$$

i.e.: Sweep over all vertical gridlines and solve simultaneously for all unknowns located on such a line.

Symbols: $\tilde{B}^h(\vartheta_1,\vartheta_2) = 2+2\varepsilon - 2\cos\vartheta_2 - \varepsilon e^{-i\vartheta_1}$

$\tilde{A}^h(\vartheta_1,\vartheta_2) = -\varepsilon e^{i\vartheta_1}$

$$\mu(\vartheta_1,\vartheta_2) = \left| \frac{\varepsilon e^{i\vartheta_1}}{2+2\varepsilon - 2\cos\vartheta_2 - \varepsilon e^{-i\vartheta_1}} \right|$$

Smoothing rate $\mu := \max\limits_{\pi \geq |\vartheta| \geq \frac{\pi}{2}} \mu(\underline{\vartheta}) = \frac{1}{\sqrt{5}} = 0.447$

$0 < \varepsilon \leq 1$

Good smoothing!

Rule: In case of anisotropies use line relaxation with lines into the direction of strong coupling.